Yet More Elements in the Shafarevich-Tate Group of the Jacobian of a Fermat Curve

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Abstract. For certain irregular primes p we construct non-trivial elements in the Shafarevich-Tate group of the jacobian of a quotient of the Fermat curve $x^p + y^p = 1$. These elements are different in general from elements previously constructed by McCallum and McCallum-Tzermias.

1. Introduction

For an odd prime number p and an integer s with $1 \le s \le p-2$, the complete nonsingular curve F_s with equation

$$y^p = x^s(1-x)$$

is a quotient of the Fermat curve $x^p + y^p = 1$. It has genus $(p-1)/2$ and its jacobian J has complex multiplication by $\mathbb{Z}[\mu_p]$, where μ_p is the group of p-th roots of unity in $\overline{\mathbb{Q}}$. In this paper we construct non-trivial elements in the p-torsion of the Shafarevich-Tate group

$$Ш = Ш(J, \mathbb{Q}(\mu_p))$$

for certain irregular primes p. These are new elements, in the sense that there is not in general any linear dependence relation over $\mathbb{Z}[\mu_p]$ between them and ones previously constructed [**McC88**], [**MT03**]. This result is of interest because a Selmer group calculation shows that, if $\mathfrak{p}$ is the prime ideal in $\mathbb{Z}[\mu_p]$ above p, then

$$\operatorname{rank}_{\mathbb{Z}/p\mathbb{Z}} Ш/\mathfrak{p} + \operatorname{rank}_{\mathbb{Z}/p\mathbb{Z}} J(\mathbb{Q}(\mu_p))/\mathfrak{p}$$

is relatively large: heuristically, it is asymptotic to $p/4$ as p grows [**McC92**, Theorem 1 and introductory discussion]. Thus we should be able to find either an abundance of $\mathbb{Z}[\mu_p]$-independent elements in Ш or an abundance of $\mathbb{Z}[\mu_p]$-independent points in $J(\mathbb{Q}(\mu_p))$. The current result adds to the growing body of circumstantial evidence that it easier to do the former. Indeed, modulo torsion, the only systematically occurring explicitly known infinite $\mathbb{Z}[\mu_p]$-submodule of $J(\mathbb{Q}(\mu_p))$ is the one generated by the Gross-Rohrlich point [**GR78**].

Consider the usual descent sequence

$$0 \to J(\mathbb{Q}(\mu_p))/\lambda^k J(\mathbb{Q}(\mu_p)) \to S_{\lambda^k} \to Ш[\lambda^k] \to 0,$$

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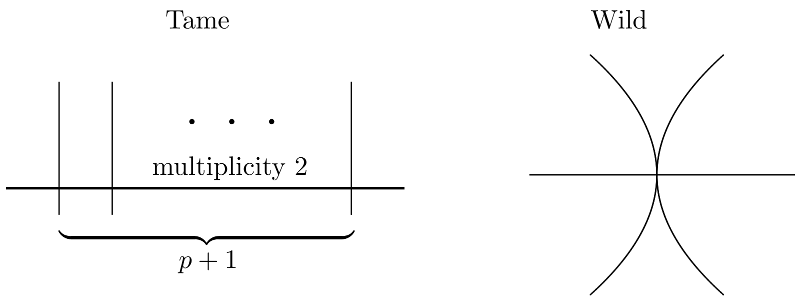
FIGURE 1. Reduction types of F_s



where λ is a generator of $\mathfrak{p}$ and $S_{\lambda^k} = S_{\lambda^k}(J, \mathbb{Q}(\mu_p))$ is the Selmer group associated with a positive integer power λ^k. A choice of group isomorphism between $J[\lambda]$ and the group μ_p of p-th roots of unity enables us to identify $S_\lambda(J, \mathbb{Q}(\mu_p))$ with a subgroup of $\mathbb{Q}(\mu_p)^\times/\mathbb{Q}(\mu_p)^{\times p}$. The two previous non-triviality results and the one we prove in this paper depend on finding a specific element $\eta \in \mathbb{Q}(\mu_p)^\times/\mathbb{Q}(\mu_p)^{\times p}$ which is contained in S_λ under this identification and, for some k, lifts to an element of S_{λ^k} whose image in Ш is non-trivial.

The elements η come from cyclotomic units, as follows. Let $\Delta = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, let $\omega : \Delta \to \mathbb{Z}_p^\times$ be the Teichmüller character, and define for any integer i the usual idempotent

$$\epsilon_i = \frac{1}{p-1} \sum_{\sigma \in \Delta} \omega^{-i}(\sigma)\sigma.$$

Fix a primitive p-th root of unity ζ, and define $\eta_i \in \mathbb{Q}(\mu_p)^\times/\mathbb{Q}(\mu_p)^{\times p}$ by[1]

$$\eta_i = (1 - \zeta)^{\epsilon_i}. \tag{1.1}$$

If i is even, $2 \le i \le p-3$, and p does not divide the Bernoulli number B_i, then η_i is locally non-trivial at p (that is, its image in $\mathbb{Q}_p(\mu_p)^\times/\mathbb{Q}_p(\mu_p)^{\times p}$ is nonzero) [**Was97**, Chapter 8]. Furthermore, by eigenvalue considerations the nontrivial elements among the η_i with $2 \le i \le p-1$ are linearly independent.

We first recall the nontriviality results of McCallum and McCallum-Tzermias. These depend on computing the Cassels pairing of certain elements. The Cassels pairing

$$Ш \times Ш \to \mathbb{Q}/\mathbb{Z}.$$

is skew-symmetric and its kernel is the infinitely divisible subgroup of Ш (a subgroup which is conjectured to be trivial). Its definition is reviewed in [**McC88**]. The computation depends on local p-adic analytic approximations of functions on F_s, and these approximations depend on a minimal regular model for F_s over $\mathbb{Z}_p[\mu_p]$. The special fiber for such a model is a curve over the finite field $\mathbb{F}_p$ with p elements, and has two possible geometric types, wild and tame, shown in Figure 1. The terminology corresponds to the ramification type of a field of good reduction for F_s. The wild type is further divided into split and non-split, according to whether the two

[1]Note that the notation is different from that used in [**MS03**]. The element η_i defined there is equal to the element η_{p-i} defined here.

tangent components are defined over $\mathbb{F}_p$ or conjugate over a quadratic extension. The reduction type can be computed as follows. For a rational number x relatively prime to p let $q(x) = (x^{p-1} - 1)/p$, and let ϵ be the Legendre symbol

$$\epsilon = \left(\frac{2s(s+1)q(s^s/(s+1)^{s+1})}{p}\right).$$

Then the reduction type of F_s is

tame	if $\epsilon = 0$
wild split	if $\epsilon = 1$
wild non-split	if $\epsilon = -1$.

The first nontriviality theorem that we want to recall here is

THEOREM 1.1 (McCallum [**MT03**]). *Suppose p and F_s satisfy the following conditions:*

(1) $p \equiv 1 \pmod 4$
(2) $p \nmid B_{(p-1)/2}B_{(p+3)/2}$
(3) *F_s has wild split reduction at $\mathfrak{p}$.*

Then the image in Ш of the subgroup of S_λ generated by $\eta_{(p-1)/2}$ and $\eta_{(p+3)/2}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

An example of a curve satisfying the conditions of Theorem 1.1 is $y^{17} = x(1-x)$. The proof of the theorem is a calculation of the Cassels pairing between the images of $\eta_{(p-1)/2}$ and $\eta_{(p+3)/2}$ in Ш$[\lambda]$.

The next theorem rests on an extension of the methods in the previous theorem to a calculation of the Cassels pairing between Ш$[\lambda]$ and Ш$[\lambda^3]$.

THEOREM 1.2 (McCallum-Tzermias [**MT03**]). *Suppose that p and s satisfy the following conditions:*

(1) *$p \geq 19$ is regular and $p \equiv 3 \pmod 4$*
(2) *F_s has tame or wild non-split reduction at $\mathfrak{p}$*
(3) *s satisfies the congruence*

$$q(s^s/(s+1)^{s+1}))^3 - s(s+1)B_{p-3} \not\equiv 0 \pmod p,$$

where $q(x) = (x^{p-1} - 1)/p$.

Then $\eta_{(p+5)/2}$ lifts to an element of S_{λ^3}, and this element and the element $\eta_{(p+1)/2} \in S_\lambda$ have $\mathbb{Z}$-independent nontrivial images in Ш.

An example of a curve satisfying the conditions of Theorem 1.2 is

$$y^{19} = x^2(1-x).$$

In this paper we prove a new non-triviality result. Since J has good reduction outside p and $\deg \lambda = p$, S_{λ^k} is contained in $H^1(G, J[\lambda^k])$, where G is the Galois group of the maximal extension of $\mathbb{Q}(\mu_p)$ unramified outside p. Our result makes use of the cup product pairing

$$H^1(G, \mu_p) \times H^1(G, \mu_p) \to H^2(G, \mu_p) \otimes \mu_p,$$

which gives rise to a pairing

$$\langle\ ,\ \rangle : E/E^p \times E/E^p \to H^2(G, \mu_p) \otimes \mu_p = C/pC \otimes \mu_p, \tag{1.2}$$

where E is the group of p-units and C is the ideal class group of $\mathbb{Z}[\mu_p]$. This pairing was studied in [**MS03**] and shown to be nontrivial for $p = 37$. Sharifi subsequently showed the non-triviality of the pairing for $p \leq 1{,}000$ [**Sha07**].

THEOREM 1.3. *Suppose that p, s, and r satisfy the following conditions*

(1) r *is even and* $2 \leq r \leq (p+1)/2$
(2) F_s *has wild non-split or tame reduction at* $\mathfrak{p}$
(3) $\langle \eta_{p-r+3}, \eta_{p-3} \rangle \neq 0$ *(which implies* $p | B_r$*).*

Then η_{p-r+3} *lifts to an element of* S_{λ^3} *whose image in* Ш *is nontrivial.*

Note that condition (3) implies that p must be an irregular prime for Theorem 1.3 to apply. An example of a curve satisfying the conditions of Theorem 1.3 is $y^{691} = x(1-x)$. It has wild non-split reduction, and 691 divides both B_{12} and B_{200}. Both $r = 12$ and $r = 200$ satisfy condition (2), and $\langle \eta_{682}, \eta_{688} \rangle$ and $\langle \eta_{494}, \eta_{688} \rangle$ are both non-zero [**MS03, Sha07**]. This gives us two independent elements of order 691 in Ш.

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2. Galois structure of the λ^4-torsion

The divisor $(0,0) - \infty$ on F_s is fixed by ζ and therefore represents a nontrivial $\mathbb{Q}$-rational λ-torsion point. Let $K = \mathbb{Q}(\mu_p)$. Greenberg determined the field of definition of the higher λ-torsion:

THEOREM 2.1 (Greenberg [**Gre81**]). *We have*

$$K(J[\lambda^3]) = K$$

and

$$K(J[\lambda^4]) = K(\eta_{p-3}^{1/p}).$$

The first part is Theorem 1 of the cited reference. Although the second part is not explicitly stated, the proof is contained in the first paragraph of Section 5. (The condition $\omega^i(a+1) = \omega^i(a)+1$ stated in the reference boils down to $(s+1)^3 \equiv s^3+1 \pmod{p}$, which is trivially satisfied.)

Theorem 2.1 enables us to determine explicitly the structure of $J[\lambda^4]$ as a Galois module over K. Following [**MT03**], for $i = 1, 2, 3, 4$, we choose a point P_i of exact order λ^i on J so that $\lambda P_i = P_{i-1}$ for $i = 2, 3, 4$. These points form a basis for $J[\lambda^4]$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Furthermore, P_3 is a λ^3-torsion point, and therefore defined over K, and λ itself is defined over K. Therefore, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$

$$\sigma(P_4) = P_4 - \chi(\sigma) P_1 \tag{2.1}$$

for some isomorphism $\chi : \mathrm{Gal}(K(\eta_{p-3}^{1/p})/K) \to \mathbb{Z}/p\mathbb{Z}$.

LEMMA 2.2. *Let* χ *be defined by* (2.1). *We have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J[\lambda] & \longrightarrow & J[\lambda^4] & \xrightarrow{\lambda} & J[\lambda^3] & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & \mu_p & \xrightarrow{\iota} & \mu_p^4 & \xrightarrow{\pi} & \mu_p^3 & \longrightarrow & 0
\end{array}$$

in which ι is the embedding via the fourth coordinate and π is projection onto the first three coordinates. Furthermore, if we let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ act on μ_p^4 by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \chi(\sigma) & 0 & 0 & 1 \end{pmatrix},$$

and on all other modules in the natural way, then the diagram commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

PROOF. For $i \leq 4$, let $e_{\lambda^i}(P, Q)$ be the λ^i Weil pairing on $J[\lambda^i]$, as defined, for example, in [**McC88**]. For any two commuting isogenies ϕ and ψ of J, the Weil pairing satisfies

$$e_\phi(\psi P, Q) = e_\phi(P, \hat{\psi} Q),$$

where $\hat{\psi}$ is the dual isogeny[**McC88**, (1.7) and (1.8)]. Define an isomorphism $J[\lambda^i] \simeq \mu_p^i$ by

$$Q \mapsto (e_{\lambda^i}(Q, P_1), \ldots, e_{\lambda^i}(Q, P_i)). \tag{2.2}$$

Our choice of P_i and χ mean that

$$\sigma Q = Q - \chi(\sigma)\lambda^3 Q, \quad Q \in J[\lambda^4].$$

Therefore, if $1 \leq j \leq i \leq 4$,

$$\begin{aligned} e_{\lambda^i}(\sigma Q, P_j) &= e_{\lambda^i}(Q, P_j) - \chi(\sigma) e_{\lambda^i}(\lambda^3 Q, P_j) \\ &= e_{\lambda^i}(Q, P_j) - \chi(\sigma) e_{\lambda^i}(Q, \hat{\lambda}^3 P_j) \\ &= e_{\lambda^i}(Q, P_j) + \chi(\sigma) e_{\lambda^i}(Q, \lambda^3 P_j) \\ &= \begin{cases} e_{\lambda^i}(Q, P_j) + \chi(\sigma) e_{\lambda^i}(Q, P_1) & j = 4 \\ e_{\lambda^i}(Q, P_j) & j < 4. \end{cases} \end{aligned}$$

Here we have used the facts that $\hat{\lambda} = \overline{\lambda}$, the complex conjugate of λ, and that $\overline{\lambda}^3 \equiv -\lambda^3 \pmod{\lambda^4}$. □

3. Selmer Groups

The Selmer group S_{λ^i} is defined by exactness of

$$0 \to S_{\lambda^i} \to H^1(K, J[\lambda^i]) \to \sum_v H^1(K_v, J),$$

where the sum is over a complete set of valuations of K. We summarize here the basic facts about these Selmer groups, and refer the reader to [**McC88**] and [**MT03**] for details.

The isomorphism

$$J[\lambda] \simeq \mu_p$$

chosen in Lemma 2.2 identifies S_λ with a subgroup of $K^\times/K^{\times p}$ defined by local conditions at each valuation of K. For every valuation except the unique p-adic one, the local condition on $x \in K^\times/K^{\times p}$ is simply that it be a local unit modulo p-th powers. For the valuation corresponding to the unique prime $\mathfrak{p}$ of K above p, Faddeev calculated the local condition, and found that in the wild non-split and tame cases it is

$$x_\mathfrak{p} \in 1 + \mathfrak{p}^{(p+3)/2}\mathcal{O}_\mathfrak{p} \qquad (\mathrm{mod}\ K_\mathfrak{p}^{\times p}). \tag{3.1}$$

(See [**Fad61**] or [**McC88**] for the calculation.) Let $\eta_i \in K^\times/K^{\times p}$ be the element defined by (1.1). By construction it is an eigenvector for the action of $\mathrm{Gal}(K/\mathbb{Q})$ with character ω^i. On the other hand, for $k \geq 1$, the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $(1+\mathfrak{p}^k\mathcal{O}_\mathfrak{p})/(1+\mathfrak{p}^{k+1}\mathcal{O}_\mathfrak{p})$ is through the character ω^k. Thus

$$\eta_i \in 1+\mathfrak{p}^i\mathcal{O}_\mathfrak{p} \pmod{K_\mathfrak{p}^{\times p}}, \quad 2 \leq i \leq p-3. \tag{3.2}$$

It follows from (3.1) that $\eta_i \in S_\lambda$ if $(p+3)/2 \leq i \leq p-3$. We next show that these elements lift to S_{λ^3}. As before, the local condition at every valuation other than the p-adic one is that an element be a unit mod p-th powers in each component. The local condition at $\mathfrak{p}$ is harder to determine.

For $i = 1, 2, 3$, the Galois isomorphisms $J[\lambda^i] \simeq \mu_p^i$ chosen in Lemma 2.2 identify $H^1(K, J[\lambda^i])$ with a subgroup of $(K^\times/K^{\times p})^i$. It is shown in [**MT03**, Section 2] that the local descent maps

$$d_k : J(K_\mathfrak{p})/pJ(K_\mathfrak{p}) \to H^1(K_\mathfrak{p}, J[\lambda^i]) = \left(K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times p}\right)^i$$

can be written as

$$d_k = \prod_{j=1}^{k} \iota_{P_j},$$

where the maps

$$\iota_{P_j} : J(K_\mathfrak{p})/pJ(K_\mathfrak{p}) \to K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times p}$$

are defined by evaluating certain functions f_j on the curve at divisors representing points on the Jacobian. (The divisor of the function f_j is p times a divisor representing the point P_j.) We have

$$\iota_{P_j}(\lambda x) = \iota_{P_{j-1}}(x) \quad \text{for } j = 2, 3. \tag{3.3}$$

Lemma 3.1. *If C has wild non-split or tame reduction and $\frac{p+5}{2} \leq i \leq p-3$ then the element*

$$(\eta_i, 1, 1) \in (K^\times/K^{\times p})^3$$

is contained in the Selmer group S_{λ^3}.

Proof. Since η_i defines a cocycle which is unramified outside the primes above p, $(\eta_i, 1, 1)$ satisfies all the local conditions for membership in S_{λ^3} except the one at $\mathfrak{p}$. Thus membership in S_{λ^3} is equivalent to being in the image of the local descent map at $\mathfrak{p}$

$$d_3 : J(K_\mathfrak{p}) \to H^1(K_\mathfrak{p}, J[\lambda^3]) = \left(K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times p}\right)^3.$$

By [**MT03**, Proposition 4.1] we can choose $a \in J(K_\mathfrak{p})/pJ(K_\mathfrak{p})$ such that $\iota_{P_1}(a) = \eta_i$, and by applying the necessary idempotent we can suppose that a is an eigenvector for the action of $\mathrm{Gal}(K/\mathbb{Q})$, in which case it is in the ω^i eigenspace by [**MT03**, (4.2)]

Consider $d_3(a) = (\eta_i, \iota_{P_2}(a), \iota_{P_3}(a))$. To prove the lemma, we will find $b \in J(K_\mathfrak{p})/pJ(K_\mathfrak{p})$ such that $d_2(b) = (\iota_{P_2}(a)^{-1}, \iota_{P_3}(a)^{-1})$. Then, by (3.3),

$$d_3(a+\lambda b) = d_3(a)\cdot(1, d_2(b)) = (\eta_i, 1, 1),$$

showing that $(\eta_i, 1, 1)$ satisfies the local condition at $\mathfrak{p}$. Now, Proposition 4.1 and equation (4.2) of [**MT03**] show that if $i \geq (p+5)/2$ then

$$\iota_{P_2}(a), \iota_{P_3}(a) \in 1+\mathfrak{p}^{(p+1)/2}\mathcal{O}_\mathfrak{p}.$$

It follows from [**MT03**, Proposition 4.2] that we can choose $b \in J(K_\mathfrak{p})$ so that $d_2(b) = (\iota_{P_2}(a)^{-1}, \iota_{P_3}(a)^{-1})$, as we wanted. □

4. Proof of Theorem 1.3

Suppose now that we are given p, r, and s satisfying the conditions of Theorem 1.3. Conditions (1) and (2) imply $(\eta_{p-r+3}, 1, 1) \in S_{\lambda^3}$, by Lemma 3.1.

Since J has good reduction outside $\mathfrak{p}$ and $\deg \lambda = p$, we can regard the Selmer groups S_{λ^i} as subgroups of $H^1(G, J[\lambda^i])$, where G is the Galois group of the maximal extension of K unramified outside $\mathfrak{p}$. Regarding $(\eta_{p-r+3}, 1, 1)$ as an element of $H^1(G, J[\lambda^3])$, we claim that its coboundary in the G-cohomology of

$$0 \to J[\lambda] \to J[\lambda^4] \to J[\lambda^3] \to 0$$

is equal to a non-zero multiple of $\langle \eta_{p-r+3}, \eta_{p-3} \rangle$ under the identification $H^2(G, J[\lambda]) \simeq H^2(G, \mu_p)$. This can be seen as follows.

In this paragraph we use additive notation for the abelian group μ_p, and we use the diagram and notation of Lemma 2.2. Since μ_p^3 is fixed by G, our element $(\eta_{p-r+3}, 1, 1) \in H^1(G, \mu_p^3)$ is represented by a homomorphism $(x, 0, 0) : G \to \mu_p^3$, which we lift to the cochain

$$\bar{x} = (x, 0, 0, 0) : G \to \mu_p^4.$$

Then

$$\begin{aligned} \delta(\overline{x})(\sigma, \tau) &= \begin{pmatrix} x(\sigma\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} x(\sigma) \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \chi(\sigma) & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= -\begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi(\sigma)x(\tau) \end{pmatrix} = \iota_*(-(\chi \cup x)(\sigma, \tau)). \end{aligned}$$

Thus the coboundary is $-(\chi \cup x)$, which is equal to $x \cup \chi$ by skew-symmetry of of the cup product on H^1. Theorem 2.1 implies that χ is a non-zero multiple of the Kummer character associated with η_{p-3}. Furthermore, x is the Kummer character associated with η_{p-r+3}. Hence, under condition (3), the coboundary is nontrivial.

Finally, non-triviality of the coboundary implies non-triviality in Ш by virtue of the commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} J(K)/\lambda^4 J(K) & \longrightarrow & J(K)/\lambda^3 J(K) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ S_{\lambda^4} & \xrightarrow{\lambda_*} & S_{\lambda^3} & \xrightarrow{\delta} & H^2(G, J[\lambda]) \\ & & \downarrow & & \\ & & Ш & & \end{array}$$

This concludes the proof of the theorem.

5. Concluding remarks

From consideration of the action of $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, one does not expect often any overlap between the non-trivial elements of Ш produced by Theorems 1.1, 1.2 and 1.3. Furthermore, in many cases we can increase a lower bound on the order of Ш using the following result, which depends on showing the triviality in many cases of the Cassels pairing between $Ш[\lambda^2]$ and $Ш[\lambda]$.

THEOREM 5.1. [**MT03**, Theorem 1.2] *Suppose that F_s is wild non-split or tame and either $p \equiv 1 \pmod 4$ and $p \nmid B_{(p-1)/2}$, or $p \equiv 3 \pmod 4$ and $p \nmid B_{(p-3)/2}$. Then $Ш[\lambda^2]/\lambda Ш[\lambda^3] = 0$, that is, $Ш[\lambda^3]$ is a free module over $\mathbb{Z}[\zeta]/(\lambda^3)$.*

Thus, for example, in the case of the curve $y^{691} = x(1-x)$ mentioned in Section 1, where we found that Ш contains a subgroup isomorphic to $\mathbb{Z}/691\mathbb{Z}$, Theorem 5.1 implies, combined with eigenspace considerations, implies that it contains a subgroup isomorphic to $(\mathbb{Z}/691\mathbb{Z})^3$. A survey of small irregular primes produces many more examples with large subgroups of Ш. For the curves studied here it seems to be much easier to find elements in Ш than elements of the Mordell-Weil group.

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