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Admissible Operators and Solutions of Perturbed Operator Equations

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The purpose of this note is to prove an abstract operator version of certain existence theorems for differential and Volterra integral equations which deal with stability properties of solutions. We will do this in such a way that the results unify, as well as generalize, many of the basic theorems concerning stability of such equations. Thus, our main result (Theorem 2) is an abstract version of well-known results for differential systems and was obviously motivated by the work begun by Massera and Schäffer and others relating to admissibility [6]. (More precisely, Theorem 2 is appropriately applied to the equivalent Volterra integral system obtained by integrating a differential system.) In the same way, Theorem 2 can by applied to integrodifferential systems to obtain known stability results (e.g., Theorems 1 and 2 in [4], Theorem 1 in [5], and Theorem 10 in [10]). Also as a corollary we can derive a functional analytic theorem of Miller [7] which in turn has many stability theorems of Volterra integral equations as corollaries. These applications are made more explicit in the remarks below.

Let F denote a real Fréchet space and let B_1, B_2 denote two normed linear subspaces of F whose norms $|\cdot|_1, |\cdot|_2$ respectively yield topologies stronger than that induced by the metric on F. We consider here the problem of describing (locally) the set of those elements $f \in B_2$ for which the operator equation

$$(E) \qquad Lx = f + p(x)$$

has a solution $x \in B_1$. Here L is a linear operator defined on F and p is an operator about which more is specified below. Our main result (Theorem 2 (ii)) gives conditions under which the set of $f \in B_2$ yielding solutions of (E) in B_1 is locally homeomorphic to the set yielding solutions in B_1 of the linear problem $Lx = f$. We begin by assuming the following hypothesis:

$H1$: L is a linear, one-one, closed operator from F onto F.

By the closed graph theorem L and L^{-1} are continuous on F. In our motivation for considering (E), L is thought of as a linear Volterra integral operator of the form

$$(V) \qquad Lx = x(t) - \int_a^t K(t,s)x(s)ds.$$

If $F = C^0[a, +\infty)$ with the metrizable topology of uniform convergence on compact subsets, then $H1$ is fulfilled under any conditions on K which guarantee the existence (global), uniqueness, and continuity of solutions with respect to f [8]. For $S \subseteq F$, let $L(S)$ denote the range of L restricted to S. We also need

$$H2: \begin{cases} \text{There exist complementary subspaces } B_2^1, B_2^2 \text{ of } B_2 \text{ such that} \\ \forall h \in B_2^2 \ \exists g \in B_2^1 \text{ for which } g+h \in L(B_1). \end{cases}$$

Lemma. *Assume $H1$ and $H2$. Let C be any subspace of B_2^1 complementary to the subspace $B_2^1 \cap L(B_1)$. Then $\forall h \in B_2^2$ $\exists$ a unique $g \in C$ for which $g+h \in L(B_1)$. Denote $g = Ah$; then A is a linear operator from B_2^2 into C.*

Proof. For $h \in B_2^2$ there exists, by $H2$, a $g^* \in B_2^1$ and a $u^* \in B_1$ such that $Lu^* = g^* + h$. Write $g^* = f + g$ for $f \in B_2^1 \cap L(B_1)$ and $g \in C$. Let $u^{**} \in B_1$ be such that $Lu^{**} = f$ and set $u = u^* - u^{**} \in B_1$. Obviously $Lu = g + h$ or $g + h \in L(B_1)$. As for the uniqueness of the element $g \in C$ suppose $g' \in C$ is such that $Lu' = g' + h$ for some $u' \in B_1$. Then $L(u - u') = g - g'$ and hence $g - g' \in B_2^1 \cap L(B_1)$ as well as $g - g' \in C$. Thus $g - g' = 0$. The linearity of A is easily demonstrated. ∎

Let P denote the projection from B_2^1 onto $B_2^1 \cap L(B_1)$ with respect to C.

Definition. The linear operator L is called (B_1, B_2)-*admissible* if $H1$ and $H2$ are satisfied and the linear operator A is bounded.

Theorem 1. *If L satisfies $H1$ and $H2$ with B_1, B_2 complete and C, B_2^2 closed, then L is (B_1, B_2)-admissible.*

Proof. We need only show A is bounded. Since B_1, B_2, C, and B_2^2 are all complete, it is not difficult to verify that $D = \{x \in B_1 : Lx \in C \oplus B_2^2\}$ is a Banach subspace of B_1 under the norm $|x|_D = |x|_1 + |Lx|_2$. Also since C, B_2^2 are closed, the projection P' from $C \oplus B_2^2$ onto B_2^2 is bounded. Thus, the linear operator $L^* \equiv P'L : D \to B_2^2$ is bounded. Using the above lemma and the invertibility of L one can easily verify that L^* is one-one and onto B_2^2. Banach's theorem implies $L^{*-1} : B_2^2 \to D$ is bounded. Now $x \in D$ implies $Lx = g + h$ for $g \in C$ and $h \in B_2^2$ and the lemma implies $g = Ah$; thus, $L^{*-1}h = x$ and

$$|Ah + h|_2 = |Lx|_2 \leq |L^{*-1}h|_1 + |Lx|_2 = |L^{*-1}h|_D \leq |L^{*-1}|\,|h|_2$$

or

$$|Ah|_2 \leq (|L^{*-1}| + 1)|h|_2. \quad ∎$$

Remarks. (1) The definition of admissibility given above is a straightforward generalization of that given and extensively studied by Massera and Schäffer [6] for differential equations where in this case L is operator given in (V) with $K=K(s)$. For differential equations in Euclidean space we take $B_2^1=R^n$ and the space $L(B_1) \cap B_2^1$ becomes the set of initial conditions giving rise to solutions of the homogeneous system lying in B_1 while C is any complementary subspace (and hence closed).

(2) It is important to note, however, that the above definition of admissibility is not equivalent to the notion of admissibility of linear operators as frequently defined in the study of Volterra integral equations [2,3,7]. Miller [8, p. 252] defines L to be (B_2, B_1)-admissible if $L(B_2)\subseteq B_1$. Since L is assumed invertible this is equivalent to L^{-1} being (B_1, B_2)-admissible by our definition above with $B_2^1=\{0\}$ and hence $B_2^2=B_2$ and $A\equiv 0$. The advantage in the greater generality ($A\not\equiv 0$) of the definition above is that in applications to integral equations we obtain existence results in the case that L is not necessarily a stable Volterra operator.

We now are ready to consider equation (E). Let $S(r)=\{x \in B_1: |x|_1<r\}$. Concerning the operator p we need the hypothesis

$$H3:\quad \begin{cases} \text{Assume } L \text{ satisfies } H2. \text{ Then } p: B_1\to B_2^2 \text{ in such a way that} \\ |p(x)-p(y)|_2\le\theta\,|x-y|_1 \ \forall x, y \in S(r) \text{ for some } r, \theta>0,\ r\le+\infty. \end{cases}$$

Theorem 2. *Assume p satisfies $H3$ and L is (B_1, B_2)-admissible where B_1 and B_2^2 are complete. Then the following couclusions hold*: (i) $\exists k>0$ *such that if* $\theta<k$ *and* $|p(0)|_2<k$ *then to each* $g \in L(S(kr))$ *there corresponds an* $h \in C$ *for which a unique solution* x *of* (E) *with* $f=g+h$ *exists in* $S(r)$; (ii) *further,* $\exists$ *constants* r^*, k^* *satisfying* $0<r^*<r$, $0<k^*<k$ *such that* $\theta<k$, $|p(0)|_2<k^*$ *imply the existence of a one-one, bicontinuous correspondence* Q *from the set of those* $y \in S(kr)$ *for which* $Ly \in B_2^1$ *onto the set of those* $x \in S(r^*)$ *for which* $Lx-p(x) \in B_2^1$. *This correspondence* $x=Qy$ *is such that* $P(Lx-p(x))=Ly$.

Proof. (i) Consider the linear operator $L^*: B_2^2\to B_1$ defined by $L^*\equiv L^{-1}(A+I)$; L^* is closed, for if $h_n\to h_0 \in B_2^2$ and $Lh_n\to h^* \in B_1$ then $Ah_n+h_n\to Ah_0+h_0$ in B_2^2 and, hence, in F. The continuity of L^{-1} on F implies $L^*h_n\to L^*h_0$ and consequently $L^*h_0=h^*$. The closed graph theorem implies L^* is bounded (B_2^2 is assumed complete). Given $g \in L(S(kr))$ consider the operator $T: S(r)\to B_1$ defined by $Tx=y+L^*p(x)$ where $y=L^{-1}g$. If we choose $k<\min(|L^*|^{-1}, r(r+|L^*|(r+1))^{-1})$, then the estimates $|L^*p(x)-L^*p(z)|_1\le k\,|L^*|\,|x-z|_1$ $\forall x, z \in S(r)$ and

$$|Tx|_1\le|y|_1+|L^*|(|p(x)-p(0)|_2+|p(0)|_2)\le kr+|L^*|(k\,|x|_1+k)<r$$

$\forall x \in S(r)$ (which follow from $H3$ and the assumptions on θ, $|p(0)|_2$) show T is a contraction from $S(r)$ into itself. Thus, B_1 complete implies $Tx=x$ for some $x \in S(r)$. But then $LTx=Lx$, which reduces to (E) with $h=Ap(x)$.

(ii) Given $y \in S(kr)$ for which $Ly \in B_2^1$ we have, by the proof above, a unique fixed point x of T in $S(r)$. Denote $x=Qy$. Since x satisfies $x=y+L^*p(x)$ it follows easily that $Qy_1=Qy_2$ implies $y_1=y_2$. Moreover, for y_1, y_2 in the domain of Q we have

$$Qy_2-Qy_1=x_2-x_1=y_2-y_1+L^*(p(Qy_2)-p(Qy_1)))$$

and consequently (by $H3$ and the above choice of $k>\theta$) the estimate $|Qy_2-Qy_1|_1 \leq (1-k|L^*|)|y_2-y_1|_1$. Hence Q is one-one and continuous. Further,

$$Q^{-1}x_2-Q^{-1}x_1=y_2-y_1=x_2-x_1-L^*(p(x_2)-p(x_1))$$

and hence $|Q^{-1}x_2-Q^{-1}x_1|_1 \leq (1+|L^*|k)|x_2-x_1|_1$; that is, Q^{-1} is continuous (on the range of Q).

Now choose $k^*<\min(k, 1/2kr|L^*|^{-1})$ and $r^*<\min(r, 1/2kr(1+k|L^*|))$. To show Q is onto the set described in the theorem we assume $x \in S(r^*)$ is such that $Lx-p(x) \in B_2^1$ and define $y=x-L^*p(x) \in B_1$. Then we claim y is in the domain of Q and $Qy=x$ provided $|p(0)|_2 \leq k^*$. By definition of L^* we have $Ly=Lx-p(x)-Ap(x)$ which belongs to B_2^1. Further, by $H3$, $|y|_1 \leq |x|_1+|L^*|(k|x|_1+|p(0)|_2) \leq kr$ by the way r^* and k^* were chosen. Thus, $y \in S(kr)$ and $Ly \in B_2^1$ so $Qy=x^* \in S(r)$ is defined. Thus, by the definition of Q, $Tx^*=x^*$ or $x^*=y+L^*p(x^*)$ or $x^*=x-L^*p(x)+L^*p(x^*)$ from which we have $|x^*-x|_1 \leq k|L^*||x^*-x|_1$. Since $k|L^*|<1$ we conclude $x^*=x$ and that Q is onto. The last assertion in (ii) follows immediately from the definition of Q. ∎

Remarks. (3) From Theorem 2(ii) one can derive many well known results concerning the conditional stability (as well as stability and instability) of perturbed systems of differential equations. Here L is given by (V) with $K=K(s)$ and $F=C^0[a, +\infty)$ with $B_1=BC$ the Banach space of all bounded continuous functions under the sup norm, and $B_2^1=R^n$. If $B \subseteq F$ is a Banach space of forcing functions for the differential system we take $B_2^2=\left\{h: h=\int_0^t m(s)ds, m \in B\right\}$ and $B_2=B_2^1 \oplus B_2^2$ where $|f|_2=f(0)+|m|_0$, $f(t)=|f(0)|+g(t) \in B_2$. The (B_1, B_2)-admissibility of L means that to each forcing term $m \in B$ the linear nonhomogeneous differential system has at least one solution in B_1. For example, Theorems 11, 12 in Coppel [1, Chap. 3] follow by taking B to be L^1 and BC respectively.

In the same way, the generalizations of these results to integrodifferential systems found in [4], [5] and [10] as mentioned above (but proved there in a completely different way) can be derived from Theorem 2(ii).

(4) One can replace the Lipschitz condition on p in $H3$ by the condition $|p(x)|_2 \leq \theta |x|_1$ and derive a theorem similar to Theorem 2 using the Schauder Theorem in place of the contraction principle. This is at the expense of assuming L^{-1}

is compact, however, which turns out to be the case for many applications to differential systems (see, for example, [1, Theorem 13, Chap. 3]).

(5) Frequently in applications involving Volterra integral equations the systems concerned have the form

$$(E') \qquad x = f + K(x + q(x))$$

where K is a linear operator. If $K: F \to F$ is continuous and if $I - K: F \to F$ is one-one and onto, then (E') is equivalent to (E) with $L = I$, $y = (I-K)^{-1}f$, and $p = (I-K)^{-1}Kq$. The identity operator I is (B_1, B_1)-admissible (with $C = \{0\}$ and $A \equiv 0$) for any Banach space $B_1 \subseteq F$. If $B \subseteq F$ is any Banach subspace of F with a stronger topology than F; if $q: B_1 \to B$; and if $R = (I-K)^{-1}K$ satisfies $R(B) \subseteq B_1$ (hence R is continuous, by the closed graph theorem [7, Lemma 2]); then p maps B_1 into itself. From Theorem 2(i) we obtain the following

Corollary (Miller, [7, Theorem 1]). *If K is a continuous, linear operator form F into F such that $I-K$ is one-one and onto; if B_1 and B are Banach subspaces of F (with stronger topologies than that of F) for which $R(B) \subseteq B_1$, $R = (I-K)^{-1}K$; if q maps $B_1 \subseteq B$ such that $|q(x) - q(y)|_B \leq \theta |x - y|_1$ $\forall x, y \in S(r)$ and some $\theta \geqslant 0$; then there exists a constant $k > 0$ such that for $|y|_1 \leq kr$, $|p(0)|_2 < k$, and $\theta < k$ equation (E') has a unique solution in $S(r)$.*

The application of Theorem 2(ii) gives a more detailed account of the solutions of (E') in B_1 at least for smaller $|p(0)|_2$. As pointed out by Miller [7] many results dealing with the existence of B_1 solutions and the stability of systems of perturbed Volterra equations follow from this corollary (see, for example, results in [9]). The conditions placed on the resolvent kernel of the linear equation which appear in the literature can usually be interpreted as sufficient conditions to assure the admissibility of the resolvent operator R. One can also apply Theorem 2 to (E') directly with $L = I - K$ and $p = Kq$; this would yield results appropriate to proving the existence of B_1 solutions to integral equations when the linear operator $I - K$ is not stable.

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