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Extremal tests for scalar functions of several real variables at degenerate critical points

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1. Introduction

It is, of course, well known to students of calculus that the extremal nature of a function f at a critical point is decided if the so-called discriminant (or Hessian) given by the expression $\Delta = f_{xy}^2 - f_{xx} f_{yy}$ evaluated at the critical point is nonzero. It is also known through simple examples that, in the so-called degenerate case when $\Delta = 0$ at the point, f may have either an extremum or a saddle point. Consequently, the extremal nature of f in this case is indeterminate from a knowledge of the second derivatives at the point in question alone and higher order partial derivatives at the point must be considered. (It is interesting that during the last century a certain confusion existed concerning the degenerate case, even apparently in the minds of some renowned mathematicians. For a short account of this history of the degenerate case see [1].) Systematic, yet straightforward and simple methods by which to take into account the higher order derivatives seem, however, difficult to come by. In fact, the only method known to the author which offers an essentially complete account of this case is due to Freedman [2]. (His techniques are concerned with the solution of the equation $f(x, y) = 0$ for $x = x(y)$ but implicitly yield information about extrema as well. He also considers cases other than the degenerate case. Also in a recent paper [3] Butler and Freedman consider the case when the lowest order terms of f are cubic or higher; as stated below, we do not consider this case here.) The purpose of this note is to present a complete method for determining the extremal nature of f on the basis of its derivatives at the point in question under the two assumptions that (i) f possesses the necessary number of partial derivatives and (ii) the lowest order terms in its Taylor expansion with remainder at the point are quadratic. Under these conditions we will show how the extremal nature of f may be decided in the degenerate case through a sequence of tests each involving a discriminant and each having a degenerate case, whose occurence, however, can be followed by the next test of the sequence. Each test has the same format as the standard discriminant test using Δ, which itself may be considered as simply the first test of the sequence. Although they accomplish more or less the same ends, the details

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of our method are for the most part significantly different from those presented by Freedman in [2]. (A few particulars do overlap, however.) In addition, our method seems to be conceptually more concise in that it involves one simple algorithmic principle while Freedman's method consists of a rather long list of technical cases, some of which may loop back upon themselves. (No discussion is given in [2] concerning this looping back nor the possibility of this indefinitely happening; such a possibility is briefly discussed, but not characterized, in [3] for the case, not considered here, that the lowest order terms in f are cubic or higher.)

In Theorem 1 we describe our method as a sequence of discriminant tests. Theorem 2 contains specific information about the existence and behavior of the implicitly defined functions $f(x, y)=0$ in the case of a saddle point, as derived from our tests. Theorem 3 characterizes those analytic functions for which the sequence of tests terminates after a finite number of steps, or equivalently those for which the sequence is indefinitely inconclusive. Finally, Theorem 4 is a stronger version of Theorems 1 and 2 for the case when f is analytic.

2. Main results

Let f be a real valued function of real variables x, y which is defined and possesses at least three continuous partial derivatives in some neighborhood N of a critical point which we assume, without loss of generality, to be the origin. There is also no loss in generality in assuming $f(0, 0)=0$. The function f is said to have a *proper relative minimum* (*maximum*) at the origin if $f(x, y)>0$ (<0) in some deleted neighborhood of the origin and, in either case, is said to have a *proper relative extremum* there. If the values of $f(x, y)$ change sign in every neighborhood of the origin, then we say that f has a *saddle point* at the origin. Finally, if $f(x, y)\geqslant 0$ ($\leqslant 0$) in some neighborhood and $f(x, y)=0$ somewhere in every neighborhood of the origin, then f has an *improper relative minimum* (*maximum*) at the origin. Under assumption (ii) above, the vanishing of both f_{xx} and f_{yy} at the origin would imply $\Delta=f_{xy}^2>0$ and the fact that f has a saddle point. Since we are only interested in the degenerate case, we may assume without loss of generality that $f_{xx}\neq 0$ at the origin. Let $i!j!a_{ij}=\partial^{i+j}f/\partial x^i y^j$ at $x=y=0$. Then, for our purposes, without any loss of generality, we may assume in everything done below that the function f has the form $f(x, y)=x^2+a_{11}xy+a_{02}y^2+o(r^2)$ in N where $r=(x^2+y^2)^{1/2}$.

To motivate briefly our method we consider Peano's well-known example (see [1]) $f(x, y)=(x-py^2)(x-qy^2)$, where p, q are constants, which is an illustration of the possibility of f having, at a degenerate critical point, either an extremum ($p=q$) or a saddle point ($p\neq q$). Although the nature of f at the origin is for this example quite easy to determine by inspection, one way of looking at Peano's example is to view f as a quadratic form, not in x and y, but in x and y^2. This quadratic form has discriminant $(p-q)^2$ and, hence, is indefinite if and only if $p \neq q$ and semi-definite if and only if $p=q$. For a general function f with a degenerate critical point at the origin, our method takes a hint from this example and, after changing variables so as to complete the square on its second order terms, investigates the x^2, xy^2, and y^4 terms (in the new variables) as a quadratic form. If this form is semi-definite the process in repeated, only this time to consider the x^2, xy^3, and y^6 terms; etc. To make this process precise we make the following

DEFINITION. A function f as described above is said to be *one-fold degenerate* at the origin if $\Delta_1 \equiv a_{11}^2 - 4a_{02} = 0$ and *n-fold degenerate* for $n \geqslant 2$ if the following three conditions are met:

(i) f possesses $2n$ continuous partial derivatives in N;

(ii) its Taylor expansion with remainder has the form $f(x, y) = x^2 + a_{1n}xy^n + a_{0\,2n}y^{2n} + m(x, y) + o(r^{2n})$ where $m(x, y)$ consists of all other terms of order three through $2n$ and has the form

$$m(x, y) = \sum_{\substack{i+j=3,\ldots,n+1 \\ i \neq 0,1}} a_{ij}x^i y^j + \sum_{\substack{i+j=n+2,\ldots,2n \\ i \neq 0}} a_{ij}x^i y^j; \tag{1}$$

(iii) $\Delta_n \equiv a_{1n}^2 - 4a_{0\,2n} = 0$.

Since $\Delta_1 = \Delta$ we see that the classical degenerate case $\Delta = 0$ in the standard extremal test corresponds to f being one-fold degenerate.

It is not difficult to see that if f is n-fold degenerate for some $n \geqslant 1$ while possessing $2n+2$ continuous partial derivatives in N and if we make the change of variables

$$\bar{x} = x + (a_{1n}/2)\, y^n, \qquad \bar{y} = y, \tag{2}$$

then the function f takes the form

$$\begin{aligned} f(\bar{x}, \bar{y}) &= \bar{x}^2 + \bar{a}_{1\,n+1}\bar{x}\bar{y}^{n+1} + \bar{a}_{0\,2n+2}\bar{y}^{2n+2} + \bar{a}_{0\,2n+1}\bar{y}^{2n+1} + \bar{m}(\bar{x}, \bar{y}) + o(\bar{r}^{2n+2}) \\ \bar{a}_{ij} &= \sum_{k=i}^{[i+(j/n)]} (-a_{1n}/2)^{k-i} \binom{k}{i} a_{k\,j+n(i-k)} \end{aligned} \tag{3}$$

(here $[p]$ is the largest integer less than p) where $\bar{m}$ has the form (1) with n replaced by $n+1$. This is done to complete the square on the term $x^2 + a_{1n}xy^n + a_{0\,2n}y^{2n}$ which is possible since $\Delta_n = 0$.

Although the form of the function f in the definition above looks rather formidable, it is nonetheless exactly the type which arises from an arbitrary function after n degenerate extremum tests, beginning with the familiar classical discriminant test, as described in the following theorem.

THEOREM 1. *Suppose that the function f is n-fold degenerate at the origin for some* $n \geqslant 1$ *and possesses* $2n+2$ *continuous partial derivatives in N. Suppose the change of variables* (2) *is made.*

(i) *If* $\bar{a}_{0\,2n+1} \neq 0$, *then f has a saddle point at the origin.*

(ii) *Suppose* $\bar{a}_{0\,2n+1} = 0$. *Then*

(a) *f has a proper relative minimum if* $\Delta_{n+1} \equiv \bar{a}_{1\,n+1}^2 - 4\bar{a}_{0\,2n+2} < 0$;

(b) *f has a saddle point* if $\Delta_{n+1} > 0$;

(c) *the extremal nature of f is undecided if* $\Delta_{n+1} = 0$, *but in this event f is* $(n+1)$*-fold degenerate.*

Notice that in the degenerate and inconclusive case (c), the fact that f is then $(n+1)$-fold degenerate allows one to reapply the theorem with n replaced by $n+1$ provided f has enough continuous partial derivatives (viz., $2n+4$). Thus, this theorem provides a systematic manner in which to continue investigating the nature of the critical point (on the basis of higher order partial derivatives at the origin) in the event of any number of degenerate cases. The possibility of indefinitely obtaining the degenerate case is discussed and characterized (for analytic functions) below.

Proof of Theorem 1. From (3), $f(0, \bar{y}) = \bar{a}_{0\,2n+1}\bar{y}^{2n+1} + o(\bar{y}^{2n+1})$ and hence, if $\bar{a}_{0\,2n+1} \neq 0$ then $f(0, \bar{y})$ changes sign with $\bar{y}$ near the origin. This proves (i). Let $\bar{x} = \bar{r}^{n+1}\cos\bar{\theta}$, $\bar{y} = \bar{r}\sin\bar{\theta}$ (where $\bar{r}$, $\bar{\theta}$ are polar coordinates in the $\bar{x}$, $\bar{y}$ plane) in (3). Because $\bar{m}$ has the form (1) with n replaced by $n+1$, this yields

$$f = \bar{r}^{2n+2}[Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) + o(1)]$$

where $Q(s, t) = s^2 + \bar{a}_{1\,n+1}st + \bar{a}_{0\,2n+2}t^2$. If $\Delta_{n+1} < 0$, then the quadratic form Q is positive definite and, hence, there is a constant q such that $Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) \geqslant q > 0$ for all $0 \leqslant \bar{\theta} \leqslant 2\pi$. Thus, for all $\bar{r} \neq 0$ small enough, $f(\bar{x}, \bar{y}) > 0$ and (a) is proved. Suppose now that $\Delta_{n+1} > 0$. We will first show that there exist values $\bar{\theta}_+$ and $\bar{\theta}_-$ for which $Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) > 0$ and < 0 respectively. Clearly, we may take $\bar{\theta}_+ = 0$. Since $Q(s, t) = (s - r_+ t)(s - r_- t)$ for $r_\pm = (1/2)(-\bar{a}_{1\,n+1} \pm \sqrt{\Delta_{n+1}})$ we see that $Q < 0$ for all s, t lying in an infinite sector S formed by the distinct lines $s = r_+ t$ and $s = r_- t$. Inasmuch as for any $n \geqslant 1$ the curve $s = \cos\bar{\theta}$, $t = \sin^{n+1}\bar{\theta}$, $0 \leqslant \bar{\theta} \leqslant \pi$ forms a continuous closed curve connecting $(1, 0)$ and $(-1, 0)$ in the upper half plane $t \geqslant 0$, it must intersect S for some $\bar{\theta}_-$. Now clearly, for all $\bar{r} \neq 0$ sufficiently small, $f(\bar{x}, \bar{y}) > 0$ at $\bar{\theta} = \bar{\theta}_+$ and $f(\bar{x}, \bar{y}) < 0$ at $\bar{\theta} = \bar{\theta}_-$; this proves (b). From (3) and the definition above, it is obvious that if $\bar{a}_{0\,2n+1} = \Delta_{n+1} = 0$ then f is $(n+1)$-fold degenerate. The examples $f = x^2 + xy^{n+2} + y^{2n+4}$ and $f = (x - y^{n+2})(x - 2y^{n+2})$ which have a relative proper minimum and a saddle point respectively and which both have $\Delta_{n+1} = 0$ proves (c). □

As is well known (see for example [4, §53]) the positivity of $\Delta = \Delta_1$ has a certain geometric significance relating to the two implicitly defined curves $f(x, y) = 0$;

namely, these curves have unequal tangents of slopes $(1/2)(-a_{11}\pm\sqrt{\Delta_1})$. In the degenerate case this feature is also present in an extended form which we describe in the next theorem. First we point out that if f is n-fold degenerate and the hypotheses of Theorem 1 part (ii) and (b) hold, then the relation $f(x, y)=0$ defines, for y sufficiently small, two distinct functions $x_\pm=(-a_{1n}/2)\,y^n+r_\pm y^{n+1}+y^{n+1}u_\pm(y)$ where $r_\pm=(1/2)(-\bar{a}_{1\,n+1}\pm\sqrt{\Delta_{n+1}})$ and $u_\pm$ are continuous functions in a neighborhood of $y=0$ satisfying $u_\pm(0)=0$. A proof of this fact can be constructed by setting $t=\bar{x}/\bar{y}^{n+1}$ and repeating verbatim the proof of the one-fold degenerate case as given, for example, by Goursat [4, p. 111]. It is not difficult to see what this fact says about a function f upon which the test described in Theorem 1 has been applied $m-1\geqslant 0$ times with degenerate results and an mth time with case (b) as a result. The function has, of course, a saddle point by Theorem 1, but moreover, the relation $f(x, y)=0$ (in its original variables) defines, in some neighborhood of $y=0$, two $m+1$ continuously differentiable functions $x_\pm$ intersecting at $y=0$ which have equal derivatives of all orders $1\leqslant i\leqslant m$ (given by $-\tfrac{1}{2}i!a_{1i}$) and distinct $(m+1)$st derivatives (given by $(m+1)!r_\pm$) at $y=0$. More specifically, if we follow the m changes of variables given by (2) which were performed in the process of performing the $m-1$ degenerate tests described by Theorem 1 then the two intersecting arcs become, in the original variables

$$x_\pm=\sum_{k=1}^{m}(-a_{1k}/2)\,y^k+r_\pm y^{m+1}+y^{m+1}u_\pm(y). \tag{4}$$

As far as saddle points are concerned, the remaining possibility is that Theorem 1 has been applied $m-1$ times with degenerate results and an mth time with the result that $\bar{a}_{0\,2m+1}\neq 0$. In this event, as pointed out by Goursat for one-fold degenerate critical points [4, p. 113] the relation $f(x, y)=0$ may define in a neighborhood of the origin either a cusp or again two intersecting curves with no other peculiarities. It can easily be shown that in this case the two branches have m equal derivatives (in the case of a cusp, one-sided derivatives) at $y=0$. Thus, we have

THEOREM 2. *Suppose f is any function with a degenerate critical point at the origin to which the test described in Theorem 1 has been applied $m-1\geqslant 0$ times with degenerate results and an* m*th time with the result that f has a saddle point at the origin*: (a) *if $\bar{a}_{0\,2m+1}\neq 0$ then $f(x, y)=0$ defines either a cusp or two intersecting curves $x_\pm=x_\pm(y)$ with m equal derivatives (one sided, in the case of a cusp) at $y=0$; or* (b) *if $\bar{a}_{0\,2m+1}=0$ and $\Delta_{m+1}>0$ then $f(x, y)=0$ defines two $m+1$ continuously differentiable functions $x_\pm(y)$ given by* (4) *which have m equal derivatives and different* $(m+1)$st *derivatives at $y=0$.*

(The results in this theorem are also proved in [2], if one looks hard enough, but not in the algorithmic format above.)

3. Further results

A natural question arises concerning the repeated application of Theorem 1: are there functions for which the repeated use of Theorem 1 always results in the degenerate case (c) and, hence, for which no decision about the extremal nature of the function can be made on the basis of this procedure? We will denote such functions *infinitely degenerate.* That infinitely degenerate functions exist can easily be observed by noting that, with the exception of part (c), Theorem 1 results in either a saddle point or a proper extremum for f; hence, any function possessing continuous partial derivatives of all orders which has an improper extremum at the origin is necessarily infinitely degenerate. For example, the function x^2 is infinitely degenerate. But then so is $x(x-\exp(-y^{-2}))$, which shows that an infinitely degenerate function may have a saddle point as well as an extremum. (The function $x(x-\exp(-y^{-2}))$ is infinitely degenerate because its Taylor expansion with remainder, of any order, is identical to that of x^2.) It is interesting, however, that within the class of functions analytic in N the set of infinitely degenerate functions is exactly the set of functions with improper extrema at the origin. This fact is contained in the next theorem.

THEOREM 3. *Suppose f is analytic in N (and as always is quadratic in lowest terms at the origin). Then f is infinitely degenerate if and only if it can be written as $f(x,y)=(x-\sum_1^\infty d_i y^i)^2 g(x,y)$ where g is a function analytic in N with $g(0,0)=1$. Thus, f is infinitely degenerate if and only if it has an improper minimum at the origin. As a result, for analytic functions whose lowest order terms are quadratic, saddle points and proper extrema are always found by our procedure within some finite number of applications of Theorem* 1.

Note that if f is analytic, its lowest order terms being quadratic, and infinitely degenerate (i.e., if f has an improper extremum at the origin) then $f(x,y)=0$ defines a single analytic function $x=\sum_1^\infty d_i y^i$ near $y=0$. Here the d_i are the coefficients generated by repeated use of the change of variables (2): $d_1=-\frac{1}{2}a_{11}$, $d_2=-\frac{1}{2}\bar{a}_{12}$, etc. In proving Theorem 3 we will also obtain stronger results than those contained in Theorem 2 for analytic f; namely we can show that the cases (a) or (b) in Theorem 3 distinguish respectively the cases that $f(x,y)=0$ defines a cusp or two intersecting, analytic arcs at the origin. Thus, we will prove

THEOREM 4. *Suppose f is analytic at the origin (its lowest order terms being quadratic) and has a saddle point there. Then there exists an integer $m\geqslant 1$ such that the first $m-1\geqslant 0$ extremal tests described in Theorem* 1 *fail, but such that at the* mth *test either $\bar{a}_{0\,2m+1}\neq 0$ or $\bar{a}_{0\,2m+1}=0$ and $\Delta_{m+1}>0$. Furthermore,*

(a) *if $\bar{a}_{0\,2m+1}\neq 0$ then $f(x,y)=0$ defines a cusp at the origin consisting of two arcs both analytic only for either small $y>0$ (if $\bar{a}_{0\,2m+1}<0$) or else small $y<0$ (if $\bar{a}_{0\,2m+1}>0$)*

and both terminating at the origin with m equal one-sided derivatives c_i, $1 \leqslant i \leqslant m$, *at* $y=0$. *On the other hand,*

(b) *if* $\bar{a}_{0\,2m+1}=0$ *and* $\Delta_{m+1}>0$, *then* $f(x, y)=0$ *defines two analytic arcs for small* $|y|$ *which intersect one another at the origin and possess m equal derivatives* c_i, $1 \leqslant i \leqslant m$, *but unequal* $(m+1)$st *derivatives at* $y=0$.

The proofs of Theorems 3 and 4 depend on a theorem from the theory of functions of several complex variables (the Weierstrass preparation theorem [5, p. 68]) from which we may conclude, since f is analytic, that $f(x, y)=[x^2+a(y)\,x+b(y)]\,g(x, y)$ where $g(x, y)$ is analytic at the origin with $g(0, 0)=1$ and where a and b are analytic functions of y. The following facts are easily established (the proofs are omitted for brevity): the function f is infinitely degenerate at the origin if and only if $x^2+a(y)x+b(y)$ is; and $x^2+a(y)\,x+b(y)$ is infinitely degenerate at the origin if and only if $a^2(y)=4b(y)$ for all y in their common domain of definition.

Proof of Theorem 3. If f has an improper extremum at the origin, then as already pointed out f necessarily is infinitely degenerate; suppose conversely that f is infinitely degenerate. Then by the remarks above $f(x, y)=[y+\frac{1}{2}a(y)]^2\,g(x, y)$ where $g(0, 0)=1$, and it becomes clear that f has an improper relative minimum at the origin. ▯

Proof of Theorem 4. We can say immediately from Theorem 3 that if f has a saddle point at the origin then necessarily the sequence of extremal tests terminates at some finite step. Thus, either case (a) or (b) of Theorem 4 holds; the assertion of (b) follows immediately from the proof of Theorem 2. (We need only note in addition that the analyticity of f implies that the functions $u_{\pm}(y)$ are analytic at $y=0$, as well known implicit function theorems tell us [4, p. 399].) For part (a) we observe that the given hypotheses on f together with the remarks made above can readily be shown to yield

$$f(x, y)=\left\{[x+a(y)/2]^2+\sum_{i=2n+1}^{\infty}\left(b_i-\tfrac{1}{4}\sum_{j=1}^{i-1}a_j a_{i-j}\right)y^i\right\}g(x, y). \qquad (5)$$

where $a(y)=\sum_1^\infty a_i y^i$ and $b(y)=\sum_2^\infty b_i y^i$. Since the right hand side vanishes for small $|x|$, $|y|$ if and only if the bracketed expression vanishes the implicitly defined curves are identical for these two expressions. Now the condition $\bar{a}_{0\,2n+1}\neq 0$ means that upon setting $\bar{x}=0$ at the mth test (that is, in the original variables, $x+\frac{1}{2}\sum_1^{n+1}a_i y^i=0$) the resulting power series in $\bar{y}$ has $\bar{y}^{2n+1}$ as its lowest order term. From (5) this condition is precisely $\bar{a}_{0\,2n+1}=b_{2n+1}-\frac{1}{4}\sum_{j=1}^{2n}a_j a_{2n+1-j}\neq 0$. Since $\bar{y}^{2n+1}$ is an odd power of $\bar{y}$, it is clear that $f(x, y)$ vanishes only for small $\bar{y}=y>0$ if $\bar{a}_{0\,2n+1}<0$ or only for small $y<0$ if $\bar{a}_{0\,2n+1}>0$. In fact the branches are found by setting the bracketed term in (5) equal to zero and solving for x. ▯

Remark. All of the above results can be extended in a straightforward manner to functions f of $n \geqslant 3$ variables $x_1, \ldots x_n$ provided the sum of the second order terms is a semi-definite quadratic form of deficiency one; i.e., in canonical variables, $f=\sum_1^{n-1} e_i x_i^2 + o(r^2)$, $e_i = \text{const.} > 0$. If the deficiency d is two or more, then the method fails in that the behavior of the quadratic form in the variables $x_1, \ldots, x_{n-d}$, $x_{n-d+1}^2, \ldots, x_n^2$ does not determine that of f near the origin. This can be shown by the following two examples (with $n=3$): $f=(x_1+x_2x_3)^2+x_2^4+(\frac{3}{4})\,x_3^4$ and $f=x_1^2+x_2^4+x_3^4-4x_1x_2x_3$ which have a proper minimum and a saddle point at the origin respectively, while the quadratic forms (in x_1, x_2^2, and x_3^2) given by $x_1^2+x_2^4+(\frac{3}{4})\,x_3^4$ and $x_1^2+x_2^4+x_3^4$ are both positive definite.

EXAMPLE. To illustrate the repeated use of Theorem 1 consider the polynomial $f=x^2-2xy+y^2-2xy^2+2y^3+y^4-y^6+xy^6$. Since $\Delta_1=0$, the origin is a degenerate critical point. To apply Theorem 1 with $n=1$ we make the change of variables (2) with $c_1=1$ and obtain $\bar{x}^2-2\bar{x}\bar{y}^2+\bar{y}^4-\bar{y}^6+\bar{x}\bar{y}^6+\bar{y}^7$. Inasmuch as $\bar{a}_{03}=0$ and $\Delta_2=\bar{a}_{12}^2-4\bar{a}_{14}=(-2)^2-4(1)=0$, f is two-fold degenerate. To apply the theorem again with $n=2$ we make a second change of variables given by (2) (after replacing the coordinates $\bar{x}$, $\bar{y}$ with x, y for simplicity to avoid complicating the notation): $\bar{x}=x-y^2$, $\bar{y}=y$. This yields $\bar{x}^2-\bar{y}^6+\bar{x}\bar{y}^6+\bar{y}^7+\bar{y}^8$ and $\bar{a}_{05}=0$, $\Delta_2=\bar{a}_{13}^2-4\bar{a}_{06}=0-4(-1)=4>0$. From part (b) of Theorem 1 we find that this polynomial has a saddle point at the origin. Furthermore, by Theorem 4, $f(x,y)=0$ defines two analytic functions of the form $x_+=y+y^2+y^3+y^3u_+(y)$ and $x_-=y+y^2-y^3+y^3u_-(y)$ for $|y|$ sufficiently small.

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