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Local Uniqueness for Harmonic Functions under Nonlinear Boundary Conditions*

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1. GENESIS OF THE PROBLEM

Consider the boundary problem

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{on} \quad S, \qquad \frac{\partial u}{\partial n} = h(s) f(u) \quad \text{on} \quad \partial S \tag{1.1}$$

where S is a bounded, simply connected region in the x, y plane, its boundary ∂S is a regular closed curve, and n is the outwardly directed normal to ∂S. Here $h(s)$ is a prescribed continuous function of arc length s along S and $f(u)$ is a function given in advance and assumed analytic in its argument u. We will assume that a solution $u(x, y)$ to (1.1) exists satisfying at least $u \in C^2(S)$, $u \in C^0(S + \partial S)$. The proof of the existence of such a solution is, in general, very difficult and will not be attempted here.

In the simple case $f(u) \equiv 1$ in (1.1), we have the Neumann problem for which it is well known that the solution is unique up to an additive constant. One proof [1] follows from the integral identity valid for two harmonic functions u_1, u_2 on S,

$$\oint_{\partial S} (u_2 - u_1) \left(\frac{\partial u_2}{\partial n} - \frac{\partial u_1}{\partial n} \right) ds = \iint_S [(p_1 - p_2)^2 + (q_1 - q_2)^2] \, dS \tag{1.2}$$

where $p_i = \partial u_i / \partial x$ and $q_i = \partial u_i / \partial y$. If u_1, u_2 are both solutions to the Neumann problem, then $\partial u_1 / \partial n = \partial u_2 / \partial n$ on ∂S and (1.2) clearly implies $p_1 = p_2$, $q_1 = q_2$ in S, which in turn yields $u_2 = u_1 + \text{const.}$

In studying the question of uniqueness for the more general problem (1.1) Martin [2] generalized the integral identity (1.2) to obtain

$$\oint_{\partial S} \tau \left(f_2 \frac{\partial u_1}{\partial n} - f_1 \frac{\partial u_2}{\partial n} \right) ds = \iint_S Q \, dS \tag{1.3}$$

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where

$$Q = ap_1^2 + 2bp_1p_2 + cp_2^2 + aq_1^2 + 2bq_1q_2 + cq_2^2 \tag{1.4}$$

is a quadratic form in the variables $p_i = \partial u_i/\partial x$, $q_i = \partial u_i/\partial y$, $i = 1, 2$, the coefficients of which are

$$a = f_2\tau_{u_1}, \qquad 2b = (f_2\tau)_{u_2} - (f_1\tau)_{u_1}, \qquad c = -f_1\tau_{u_2}. \tag{1.5}$$

The identity (1.3) arises from Gauss' Theorem

$$\oint_S (Ax_n + By_n)\, ds = \iint_S (A_x + B_y)\, dS$$

upon setting

$$A = \tau(f_2p_1 - f_1p_2), \qquad B = \tau(f_2q_1 - f_1q_2).$$

Here we assume that τ, τ_{u_1}, τ_{u_2}, are continuous functions of x, y in S so that Gauss's Theorem is valid [1] on S and the coefficients of Q are continuous on S.

In the uniqueness theorems of Martin [2–6] and Dunninger [7] it is stated that given a nonconstant solution u_1 to problem (1.1) for a specified function $f(u)$, then no other nonconstant solution u_2 exists which satisfies certain explicitly stated hypotheses. These hypotheses sometimes require (among other things) that $u_2 \neq u_1$ in S; that is, the solutions u_1, u_2 must not be "anywhere close to one another". In order to study the uniqueness question without this restriction we formulate a concept of local uniqueness in the following definition.

DEFINITION 1.1. Let $\rho = \rho(x, y)$ be a nonnegative function of x, y defined on S. A solution u_1 to problem (1.1) is ρ-locally unique if no other solution u_2 to (1.1) exists satisfying

$$| u_2 - u_1 | \leqslant \rho(x, y) \qquad \text{on} \quad S. \tag{1.6}$$

A solution u_1 is ρ-locally unique among the functions in a class C if no solution $u_2 \in C$ exists satisfying (1.6).

This concept of ρ-local uniqueness asserts that solutions to (1.1) cannot be "everywhere close to one another", but it does not, together with results of the type of Martin and Dunninger, exclude the possibility, which still remains, that solutions may equal one another at some points of S and differ greatly at others.

Note that if a solution u_1 is ρ-locally unique and $0 \leqslant \sigma(x, y) \leqslant \rho(x, y)$ then u_1 is also σ-locally unique. We must point out, however, that for some functions $\rho(x, y)$ a solution u_1 is always ρ-locally unique. For example, such is the case if $\rho(x, y) \to 0$ as $(x, y) \to (x_0, y_0) \in \partial S$ or if $\rho(x, y)$ vanishes on an open subset of S; for the condition (1.6) then implies $u_2 = u_1$ in S. The definition lacks content for such functions $\rho(x, y)$ and they are accordingly excluded from consideration.

In the special case $\rho(x, y) \equiv \epsilon$ where $\epsilon = \text{const.} > 0$, the concept of ϵ-local uniqueness implies that a solution to (1.1) is ϵ-locally unique if and only if it cannot be uniformly approximated to within "distance" ϵ by another solution.

As an example, let u_1 be a solution to the Neumann problem $f \equiv 1$. Then $u_2 = u_1 + k$, $k = \text{const.}$, is also a solution and since $|k|$ can be made arbitrarily small we see that u_1 is not ϵ-locally unique for any constant $\epsilon > 0$; however, if $\rho(x, y)$ is any nonnegative function with a zero in S, then since any solution is of the form $u_2 = u_1 + k$ and since (1.6) implies $k = 0$, u_1 is seen to be ρ-locally unique.

We now ask for what functions $f(u)$ and under what hypotheses will solutions to problem (1.1) be ρ-locally unique for a suitable function ρ.

2. Fundamental Lemmas

If in identity (1.3) we choose $f_1 = f(u_1)$, $f_2 = f(u_2)$ where u_1, u_2 are two nonconstant analytic functions which solve (1.1), then identity (1.3) becomes

$$\iint_S Q\, dS = 0 \tag{2.1}$$

where Q is the quadratic form (1.4) in the variables p_1, p_2, q_1, q_2 with coefficients (1.5) which are functions of u_1, u_2 and τ is a function which is at our disposal.

Let D denote the set of points in two dimensional Euclidean space E_2 at which Q is positive definite. D clearly depends on the choice of τ. If for the two nonconstant solutions u_1, u_2 the manifold

$$M_1 : u_1 = u_1(x, y), \qquad u_2 = u_2(x, y), \qquad (x, y) \in S$$

is contained in D, then (2.1) implies $Q \equiv 0$ in S which in turn implies $p_i = q_i = 0$ $(i = 1, 2)$ in S, i.e., $u_1 = \text{const.}$, $u_2 = \text{const.}$, a contradiction.

As is well known [8], the quadratic form Q will be positive definite if and only if the descending principal minors of its associated symmetric matrix are all positive. A routine calculation shows these are equal to

$$a, \qquad ac - b^2, \qquad a(ac - b^2), \qquad (ac - b^2)^2,$$

and hence Q will be positive definite if and only if

$$a > 0, \qquad \Delta = b^2 - ac < 0.$$

These are partial differential inequalities for the unknown function τ which can be written as

$$a = f_2\tau_1 > 0, \qquad \Delta = U^2 + W < 0 \tag{2.2}$$

where

$$U = \tfrac{1}{2}(f_1\tau_1 + f_2\tau_2 + (f_2' - f_1')\tau), \qquad W = f_1(f_1' - f_2')\,\tau\tau_1 . \tag{2.3}$$

Here we have set $\tau_i = \partial\tau/\partial u_i$ $(i = 1, 2)$ and

$$f_1' = \frac{\partial f_1}{\partial u_1}, \qquad f_2' = \frac{\partial f_2}{\partial u_2}.$$

Once a function τ has been selected the inequalities (2.2) serve to define D. Letting a zero superscript denote evaluation on the one dimensional manifold

$$S_1 : u_2 = u_1$$

of E_2 one observes that

$$\mathring{\Delta} = \mathring{U}^2 \geqslant 0$$

inasmuch as $\mathring{W} = 0$. Consequently D can never be the whole space E_2 and (2.2) cannot possibly be fulfilled unless u_1, u_2 are solutions for which S_1 is avoided. We can, however, still obtain uniqueness and local uniqueness without excluding S_1 by utilizing

LEMMA 2.1. *If u_1 is a nonconstant solution to* (1.1), *then no other solution u_2 exists for which $M_1 \subseteq D + S_1$.*

Proof. Given two solutions u_1, u_2, satisfying $M_1 \subseteq D + S_1$, then Q is positive definite in S except on the nodal lines $u_2 - u_1 = 0$ and, hence, $Q \geqslant 0$ in S. Identity (2.2) implies $Q \equiv 0$ in S which, in turn, implies the contradiction $u_1 = \text{const}$.

We note that in order for $M_1 \subseteq D + S_1$ it is necessary that τ be chosen so that $\mathring{\Delta} = 0$, i.e., so that

$$\mathring{U} = 0. \tag{2.4}$$

We will also need the following lemma.

LEMMA 2.2. *Let $g(t, s) = \sum_{m=2}^{\infty} a_m(t)(s - t)^m$ where the $a_m(t)$ are continuous functions of $t \in [a, b]$, $b > a$. Assume $g(t, s)$ converges for all $t \in [a, b]$, $0 \leqslant | s - t | \leqslant r$ for some constant $r > 0$.*

(a) *If* $a_2(t) \leqslant k < 0$ *for* $t \in [a, b]$ *and some constant* k, *then there exists a constant* $\epsilon > 0$ *such that* $g(s, t) < 0$ *for* $0 < | s - t | \leqslant \epsilon$.

(b) *If* $a_2(t) \leqslant 0$ *for* $t \in [a, b]$ *where* $a_2(t) = 0$ *if and only if* $t = t_0 \in [a, b]$, *then there exists a function* $\rho(t) \geqslant 0$ (*which may be taken as continuous in* t) *defined on* $[a, b]$ *where* $\rho(t) = 0$ *if and only if* $t = t_0$ *such that* $g(s, t) < 0$ *for* $0 < | t - s | \leqslant \rho(t)$, $t \in [a, b]$, $t \neq t_0$.

To prove part (a) we set $g(t, s)(s - t)^{-2} = a_2(t) + (s - t)\, h(s, t)$ where $h(s, t)$ is a continuous and, hence, bounded function on the closed set $t \in [a, b]$, $0 \leqslant | s - t | \leqslant r$. Let $| h(s, t)| \leqslant B$ for $B > 0$. Since $a_2(t) \leqslant k < 0$ on $[a, b]$, we have $a_2(t) + (s - t)\, h(s, t) < 0$ if $0 \leqslant | s - t | \leqslant \epsilon = k/2B$, $t \in [a, b]$ and part (a) follows.

Suppose now that $a_2(t) \leqslant 0$ on $[a, b]$ with $a_2(t) = 0$ if and only if $t = t_0 \in [a, b]$. If $t_0 \neq b$, divide $[t_0, b]$ into the subintervals

$$I_i = [t_0 + (b - t_0)\, 2^{-i}, t_0 + (b - t_0)\, 2^{-i+1}], \qquad i = 1, 2,\dots .$$

Since $a_2(t) \leqslant k_i < 0$ on I_i for some $k_i > 0$ part (a) implies $g(s, t) < 0$ for $0 < | s - t | \leqslant \epsilon_i'$, $t \in I_i$ for some $\epsilon_i' > 0$; clearly we can choose $\epsilon_i' \geqslant \epsilon_{i+1}'$. If $t_0 \neq a$ a similar argument for subintervals $J_i = [a + (t_0 - a)\, 2^{-i}, a + (t_0 - a)\, 2^{-i+1}]$, $i = 1, 2,\dots$, of $[a, t_0]$ yields a sequence $\epsilon_i'' > 0$ such that $\epsilon_i'' \geqslant \epsilon_{i+1}''$ and $g(s, t) < 0$ for $0 < | s - t | \leqslant \epsilon_i''$, $t \in J_i$. Let $\epsilon_i = \min(\epsilon_i', \epsilon_i'')$ $i = 1, 2,\dots$. If $\inf_i \epsilon_i = \epsilon > 0$ then clearly $g(s, t) < 0$ for $0 < | s - t | \leqslant \epsilon$, $t \in [a, b]$ and part (b) holds for any $0 \leqslant \rho(t) \leqslant \epsilon$ for which $\rho(t) = 0$ if and only if $t = t_0$. If $\inf_i \epsilon_i = 0$, we define

$$\rho(t) = \begin{cases} \epsilon_{i+1} + 2^i(\epsilon_i - \epsilon_{i+1})(b - t_0)^{-1}\, [t - t_0 - (b - t_0)\, 2^{-i}], & \\ \qquad\qquad t \in I_i, \quad i = 1, 2,\dots & \\ \epsilon_{i+1} + 2^i(\epsilon_i - \epsilon_{i+1})(t_0 - a)^{-1}\, [t - a - (t_0 - a)\, 2^{-i+1}], & \\ \qquad\qquad t \in J_i, \quad i = 1, 2,\dots & \\ 0, \qquad t = t_0 & \end{cases}$$

which is continuous for $t \in [a, b]$ and vanishes if and only if $t = t_0$. Moreover, since $0 < \rho(t) < \epsilon_i$ on I_i and J_i we have $g(s, t) < 0$ if $0 < | s - t | \leqslant \rho(t)$, $t \in [a, b]$, $t \neq t_0$ which proves part (b).

That we must in general take $\rho(t)$ to vanish at $t = t_0$ in part (b) follows from the example $g(s, t) = -t^2(s - t)^2 - t(s - t)^3 = -st(s - t)^2$ which is nonpositive in the first and third quadrants and nonnegative in the second and fourth; here $t_0 = 0$ and $g(s, t) > 0$ at points in any neighborhood of $s = t = 0$.

3. Local Uniqueness Theorems

If we can find a function $\tau = \tau(u_1, u_2)$ such that the identity (1.3) is valid and inequalities (2.2) hold whenever two nonconstant solutions u_1, u_2 to (1.1) satisfy $0 < | u_2 - u_1 | \leqslant \rho(x, y)$ for some suitable function $\rho(x, y) \geqslant 0$ then by evoking Lemma 2.1 we can conclude that any nonconstant solution to (1.1) is ρ-locally unique. To this end assume that τ permits the expansion

$$\tau = \tau(u_1, u_2) = \sum_{m=0}^{\infty} \frac{\alpha_m}{m!} (u_2 - u_1)^m \tag{3.1}$$

where the coefficients $\alpha_m = \alpha_m(u_1)$ are as yet unspecified. Since f is assumed analytic we may write

$$f_2 = f(u_2) = \sum_{m=0}^{\infty} \frac{f_1^{(m)}}{m!} (u_2 - u_1)^m, \qquad f_1^{(m)} = \left. \frac{d^m f}{du^m} \right|_{u=u_1}$$

and assume the expansion is valid for $0 \leqslant | u_2 - u_1 | < r$ for some constant $r > 0$. Substitution of these series expansions into the expression a, U, and W yields series expansions for a and Δ given by

$$a = \sum_{m=0}^{\infty} \frac{a^{(m)}}{m!} (u_2 - u_1)^m, \qquad \Delta = U^2 + W = \sum_{m=0}^{\infty} \frac{\Delta^{(m)}}{m!} (u_2 - u_1)^m$$

where the coefficients $a^{(m)}$ and $\Delta^{(m)}$ are expressions involving the α_m and $f_1^{(m)}$ which can be computed.

Treating Δ as a function of u_2 expanded about the "point" u_1, we must require that Δ be negative in a deleted neighborhood of u_1 in order to utilize Lemma 2.1 for local uniqueness results. Moreover, referring to (2.4) we must require the condition

$$\mathring{U} = \tfrac{1}{2} f_1(\mathring{\tau}_1 + \mathring{\tau}_2) = \tfrac{1}{2} f_1 \alpha_0' = 0$$

on τ, i.e., $\alpha_0 = \text{const}$. Thus, we must require that τ be chosen so that Δ, regarded as a function of u_2, has a local maximum at $u_2 = u_1$, for which it is necessary that

$$\mathring{\Delta}_2 = \mathring{W}_2 = f_1 f_1'' \alpha_0 \alpha_1 = 0$$

where as a matter of notation $\partial\Delta/\partial u_2 = \Delta_2$, $\partial^2\Delta/\partial u_2^2 = \Delta_{22}$, etc. Ignoring the linear case $f'' = 0$ we accordingly choose either α_0 or α_1 to be zero.

(i) *The case* $\alpha_0 = 0$. As we have just seen $\mathring{\Delta} = \mathring{\Delta}_2 = 0$ and, hence, if Δ is to have a local maximum at $u_2 = u_1$ we require

$$\mathring{\Delta}_{22} = \tfrac{1}{2}[(f_1 \alpha_1)']^2 + 2 f_1 f_1'' \alpha_1^2 \leqslant 0. \tag{3.2}$$

On inspection of (3.2) we are led to set $\alpha_1 = -1/f_1$, $\alpha_m = 0$ ($m = 2, 3,\ldots$) in the expansion (3.1) which amounts to setting $\tau = (u_1 - u_2)/f_1$. From (1.5) we find

$$a = \lambda(1 - f_1'\mu), \qquad 2b = -1 - \lambda + f_2'\mu, \qquad c = 1$$

where

$$\lambda = f_2/f_1, \qquad \mu = (u_1 - u_2)/f_1 \tag{3.3}$$

and hence

$$a = \left(\frac{1}{f_1}\right)^2 (f_1^2 + \cdots), \qquad \Delta = \mu^2(f_1 f_1'' + \cdots)$$

where the dots denote (as always in the sequel) terms of the first order in $u_2 - u_1$ with continuous coefficients as functions of u_1. In order to insure the validity of the identity (1.3) we assume λ, μ are continuously differentiable in $S + \partial S$ or equivalently, as can be seen by the expansion

$$1 - \lambda = \mu(f_1' + \cdots)$$

for $| u_2 - u_1 |$ sufficiently small, that λ is continuously differentiable in $S + \partial S$ and f_1, f_1' do not vanish simultaneously. Lemmas 2.1 and 2.2 now yield

THEOREM 2.1. (a) *If the function $f = f(u)$ in* (1.1) *meets the condition $f_1 f_1'' \leqslant k < 0$ for some constant k on the range of a nonconstant solution u_1 to* (1.1), *then u_1 is ϵ-locally unique for all constants $\epsilon > 0$ sufficiently small.*

(b) *If the function $f = f(u)$ in* (1.1) *meets the condition $f_1 f_1'' \leqslant 0$ and f_1, f_1' do not vanish simultaneously on the range of a nonconstant solution u_1 to* (1.1), *then u_1 is ρ-locally unique among the class of nonconstant solutions u_2 for which $f_2/f_1 \in C'(S + \partial S)$ for all sufficiently small functions $\rho = \rho(u_1) \geqslant 0$ which vanish on and only on the level lines $u_1 = k$ where k is a constant such that $f(k) f''(k) = 0$.*

In particular this theorem applies to the boundary problem $f = \sin u$ studied by Martin [6] and Dunninger [7] inasmuch as $ff'' = -\sin^2 u \leqslant 0$ and $f^2 + (f')^2 = 1$. For example Martin's strongest result [6, Theorem 7.1] for this problem states that two nonconstant solutions u_1, u_2 cannot exist satisfying $0 < u_1 < \pi$, $0 < u_2 < \pi$ on $S + \partial S$. This result can be restated in terms of local uniqueness to assert that any nonconstant solution u_1 satisfying $0 < u_1 < \pi$ on $S + \partial S$ is ρ-locally unique for

$$\rho = \rho(u_1) = \begin{cases} \sqrt{2}\, u_1, & 0 \leqslant u_1 \leqslant \pi/2, \\ \sqrt{2}(\pi - u_1), & \pi/2 \leqslant u_1 \leqslant \pi. \end{cases}$$

(That $0 < u_1 < \pi$ is needed on $S + \partial S$ and not just on S is not explicitly stated in [6] but nonetheless is needed to insure the validity of the identity used.) This is stronger than the result of part (a), Theorem 2.1, but part (b) asserts local uniqueness for this problem without ruling out the possibility that $u_1 = n\pi$, $n =$ integer, in $S + \partial S$.

Theorem 2.1 is also valid for the mixed boundary problem

$$\Delta u = 0 \quad \text{on} \quad S, \qquad u = g(s) \quad \text{on} \quad C_1, \qquad \frac{\partial u}{\partial n} = h(s) f(u) \quad \text{on} \quad C_2 \tag{3.4}$$

where $g(s)$ is a prescribed function of arc length along C_1 and $\partial S = C_1 + C_2$, since the integral identity (1.3) still reduces to (2.1) for two solutions u_1, u_2 to (3.4) and $\tau = \mu$.

(ii) *The case* $\alpha_1 = 0$, $\alpha_0 \neq 0$. In this case, we require, in place of (3.2),

$$\mathring{\Delta}_{22} = \tfrac{1}{2}\alpha_0 f_1''(\alpha_0 f_1'' + 4\alpha_2 f_1) \leqslant 0,$$

an inequality we try to satisfy by placing $\alpha_2 = -1/f_1$, $\alpha_m = 0$ $(m = 3, 4, \ldots)$ in the expansion (3.1) whereupon

$$\mathring{\Delta}_{22} = \tfrac{1}{2}\alpha_0 f_1''(\alpha_0 f_1'' - 4) \tag{3.5}$$

and $\tau = \alpha_0 + \frac{1}{2}\mu(u_2 - u_1)$ so that from (1.5)

$$\begin{aligned} a &= \lambda(u_2 - u_1 - \tfrac{1}{2}\mu f_1'(u_2 - u_1)), \\ 2b &= (f_2' - f_1')\,\alpha_0 + (u_2 - u_1)(-1 - \lambda + f_2'\mu), \\ c &= u_2 - u_1 \end{aligned}$$

where λ, μ are defined by (3.3).

Assume that $f(u)$ has a nonvanishing second derivative. Then there exist constants M, m such that $0 < m \leqslant | f_1'' | \leqslant M$ on the range of u_1 and if the constant α_0 is chosen to have the same sign as f_1'' such that $M \mid \alpha_0 \mid < 4$ then from (3.5) we have

$$\mathring{\Delta}_{22} \leqslant \tfrac{1}{2} m \mid \alpha_0 \mid (M \mid \alpha_0 \mid - 4) = k < 0.$$

Since now

$$a = (u_2 - u_1)\left(\frac{1}{f_1}\right)^2 (f_1^2 + \cdots), \qquad \Delta = \mu^2(\tfrac{1}{2} f_1^2 \mathring{\Delta}_{22} + \cdots)$$

Lemmas 2.1 and 2.2 yield

THEOREM 2.2. *Assume the function $f = f(u)$ in* (1.1) *has a nonvanishing second derivative on the range of a nonconstant solution u_1 to* (1.1). *If f_1, f_1' do* *not vanish simultaneously on the range of* u_1, *then* u_1 *is* ρ*-locally unique among the nonconstant solutions* u_2 *for which* $\lambda = f_2/f_1 \in C'(S + \partial S)$ *and* $u_2 \geqslant u_1$ *for all sufficiently small functions* $\rho = \rho(u_1) \geqslant 0$ *which vanish on and only on the level lines* $u_1 = k$ *where* k *is a constant such that* $f(k) = 0$.

In this theorem the condition $u_2 \geqslant u_1$ may be replaced by $u_2 \leqslant u_1$ by choosing $\tau = -\alpha_0 - \frac{1}{2}\mu(u_2 - u_1)$ and arguing as above.

4. Semidefinite Forms Q

In this section we obtain some local uniqueness theorems from integral identities derived from (1.3) other than (2.1). Assume that the subset

$$S^* = \{(x, y) \in S : f'(u_1(x, y)) < 0\} \tag{4.1}$$

of S is nonempty. Clearly S^* is an open set in S and $f_1' \leqslant 0$ on $S^* + \partial S^*$. Now

$$I \equiv (u_2 - u_1)[f(u_2) - f(u_1)] = f_1'(u_2 - u_1)^2 + \cdots,$$

and by Lemma 2.2b there exists a function $\rho = \rho(u_1) \geqslant 0$ which vanishes on and only on the level lines $u_1 = k$ where k is a constant for which $f'(k) = 0$ such that $I \leqslant 0$ on S^* provided

$$| u_2 - u_1 | \leqslant \rho(u_1) \qquad \text{on} \quad S^*. \tag{4.2}$$

Under assumption (4.2), ∂S^* consists of points on ∂S and on the nodal lines $u_2 - u_1 = 0$ and, consequently, if $u_1 \in C(R)$ for some region $R \supset S + \partial S$, then S^* is a regular subregion of S. This follows from the fact that nodal lines of harmonic functions in the plane are regular analytic curves which can intersect only at critical points (see [9, p. 269]). The number of critical points is finite under the assumption that $u_2 - u_1$ is harmonic in $R \supset S + \partial S$, and, thus, the subregion S^* on which $u_2 - u_1 > 0$ (or <0) has a boundary consisting of a finite number of regular analytic arcs either on ∂S or on nodal lines $u_2 - u_1 = 0$ arranged in order such that the terminal point of each arc is the initial point of the next arc. We conclude that this subregion is a regular region [1] and that Gauss' theorem and, consequently, identity (1.2) are valid on S^* [1, p. 118]. However, since the integrand of the boundary integral in (1.2) vanishes when $u_2 - u_1 = 0$ we have a nonzero contribution to this integral only on that portion of ∂S^* coinciding with ∂S. This leads us to the identity

$$\oint_{\partial S^* \cap \partial S} h(s) I \, ds = \iint_{S^*} [(p_1 - p_2)^2 + (q_1 - q_2)^2] \, dS^* \tag{4.3}$$

from which we conclude $u_2 = u_1 + k$, $k = \text{const.}$, if $h(s) \geqslant 0$. Since f_1' cannot be negative everywhere in S unless $u_1 = \text{const.}$ [10], ρ vanishes somewhere in S and, hence, $k = 0$ or $u_2 \equiv u_1$.

THEOREM 4.1. *If $h(s) \geqslant 0$ in (1.1) and $u_1 \in C^2(R)$ for some region $R \supset S + \partial S$ is a nonconstant solution for which S^* is nonempty, then u_1 is ρ-locally unique for all sufficiently small functions $\rho = \rho(u_1) \geqslant 0$ which vanish on and only on the level lines $u_1 = k$ where k is a constant such that $f'(k) = 0$.*

Note that the restriction (4.2) on u_2 need only be made on S^* and, hence, in Theorem 4.1 u_2 can be considered unrestricted on $S - (S^* + \partial S^*)$.

Assume u_1, u_2 are two nonconstant solutions to (1.1). Then one of the sets

$$S_+ = \{(x, y) \in S : u_1 > 0\}, \qquad S_- = \{(x, y) \in S : u_1 < 0\}$$

is nonempty; assume for definiteness that S_+ is nonempty. Just as for S^*, we know S_+ is a regular region if u_1 is harmonic in $R \supset S + \partial S$. If we replace f_1, f_2 by u_1, u_2 respectively in (1.3), set $\tau = \lambda - 1$, $\lambda = u_2/u_1$ and assume $\lambda \in C'(S + \partial S)$ in order to insure the validity of (1.3) which we apply over the regular S_-, we obtain

$$\oint_{\partial S_+ \cap \partial S} h(s) I \, ds = \iint_{S_+} [(\lambda p_1 - p_2)^2 + (\lambda q_1 - q_2)^2] \, dS_+ \tag{4.4}$$

where

$$\begin{aligned} I &\equiv (1 - \lambda)[u_2 f(u_1) - u_1 f(u_2)] \\ &= -\frac{1}{u_1}(u_2 - u_1)^2 [f(u_1) - u_1 f'(u_1) + \cdots]. \end{aligned} \tag{4.5}$$

Note that the integrand of the boundary integral in (1.3) for f_1, f_2 replaced by u_1, u_2 respectively vanishes when $u_1 = 0$ and $\lambda \in C'(S + \partial S)$, since $u_1 = 0$ implies $u_2 = 0$ under this assumption, and consequently we have nonzero contributions to the boundary integral in (4.4) only on $\partial S_+ \cap \partial S$. Suppose $h(s) \geqslant 0$ and $f = f(u)$ is a function satisfying

$$f(u) - uf'(u) \geqslant k > 0, \qquad u \geqslant 0 \tag{4.6}$$

for some constant k. Then by Lemma 2.2, $I \leqslant 0$ on S_+ provided $0 \leqslant | u_2 - u_1 | < \epsilon$ for $\epsilon = \text{const.} > 0$ sufficiently small. Identity (4.4) now yields $I = 0$ on S_+ which implies $u_2 = u_1$ on S_+ and hence S as can be seen from the power series expansion (4.5).

THEOREM 4.2. *Suppose $u_1 \in C'(R)$ for some region $R \supset S + \partial S$ is a nonconstant solution to* (1.1) *for which $u_1 > 0$ at some point in S. If $h(s) \geqslant 0$*

and $f = f(u)$ meets condition (4.6) *for some* k = const. > 0, *then for any* ϵ = const. > 0 *sufficiently small u_1 is ϵ-locally unique among those nonconstant solutions u_2 for which $\lambda = u_2/u_1 \in C'(S + \partial S)$.*

Theorem 4.2 remains valid if $u_1 < 0$ at some point in S provided the hypothesis (4.6) on $f(u)$ is replaced by

$$f(u) - uf'(u) \leqslant k < 0, \qquad u \leqslant 0$$

for some k = const. The proof remains the same as above except identity (4.4) is now applied over the regular subregion S_-.

Both Theorems 4.1 and 4.2 remain valid for the mixed boundary problem (3.4). The proofs carry over exactly as given where we need only notice that the integrands of the boundary integrals appearing in (4.3) and (4.4) vanish identically on C_1 for two solutions u_1, u_2 to (3.4).

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