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Nonlinear Steklov Problems on Nonsymmetric Domains

J. M. CUSHING*

IBM Thomas J. Watson Research Center, Yorktown Heights, New York

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A way of formulating nonlinear Steklov problems on nonsymmetric domains as an operator equation $u = \mu Pu$, where P is completely continuous, is given. Local and global existence theorems then follow from standard techniques; these results extend earlier results for symmetric domains and equations with symmetric coefficients. Some miscellaneous results are given concerning the nature of the solution branches.

1. INTRODUCTION

Consider the nonlinear Steklov problem (P) given by

$$Lu \equiv \sum_{i,j=1}^{m} D_i(a_{ij}(x)\, D_j u) = 0, \qquad x \in D, \tag{1.1}$$

$$u_{\nu(x)} \equiv \sum_{i,j=1}^{m} a_{ij}(x)\, n_i(x)\, D_j u = \mu f(x, u), \qquad x \in \partial D, \tag{1.2}$$

where D is a bounded region in Euclidean m-space E^m whose boundary ∂D is of type $C^{1+\lambda}$, $0 < \lambda < 1$; where $n(x) = (n_1(x),..., n_m(x))$ is the outwardly directed unit normal to ∂D at $x = (x_1 ,..., x_m) \in \partial D$; and where $D_i = \partial/\partial x_i$. Here $a_{ij}(x)$ are given functions satisfying

$$a_{ij} \in C^{1+\lambda}(E^m), \qquad a_{ij}(x) = a_{ji}(x), \qquad x \in E^m,$$

and

$$\sum_{i,j=1}^{m} a_{ij}(x)\, \xi^i \xi^j > 0, \qquad \sum_{i=1}^{m} \xi^i \neq 0, \qquad x \in E^m.$$

The function $f(x, u)$ is assumed to be given in advance and to satisfy $f(x, 0) \equiv 0$, $x \in \partial D$; more will be specifically required of f below. By a

* Present address: Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

solution to (P) we mean an ordered pair (μ, u) where μ is a real number and $u \in C^0(\bar{D}) \cap C^2(D)$, $\bar{D} = D \cup \partial D$, for which u satisfies (1.1) on D and μ, u satisfy (1.2) on ∂D.

Clearly $(\mu, 0)$ is a solution to (P) for all real $\mu \in E^1$; we are interested, however, in conditions under which (P) admits nontrivial solutions (i.e., solutions for which $u \not\equiv 0$ on D). In [1, 2] existence results (global) were obtained for the special case of Laplace's operator $L = \Delta$ on the unit circle D in E^2. The more general problem (P) was studied in [3] under certain symmetry conditions on D and a_{ij}. (The results in [3], although local, can be extended globally by using a recent result of Rabinowitz [4].) The purpose of this note is to extend the existence results of [3], locally and globally, to nonsymmetric domains D; symmetry conditions on a_{ij} and f will also be dropped. The local result below (Corollary 2.2) contains those of [3]; the global result (Theorem 3.1) however requires an added hypothesis on f not made in [3]. We also give a few results on the nodal structure of u for solutions to (P) and on the asymptotic nature of the solution set for (P) in Section 4.

2. Local Results

We make the following hypotheses concerning the nonlinear term f:

(H1) $f(x, z) = za(x) + g(x, z)$ where $a(x) \in C^0(\partial D)$, $g(x, z) \in C^0(\partial D \times I_\delta)$, $I_\delta = (-\delta, \delta)$, $\delta > 0$, and $g(x, z) = o(|z|)$ as $z \to 0$. Further, $g_z(x, z)$ exists and is continuous on $\partial D \times I_\delta$.

(H2) $\int_{\partial D} a(x)\, dx \neq 0$.

We will, in fact, assume that $\alpha = \int_{\partial D} a(x)\, dx = 1$. Under (H2) this is done without any loss of generality for we can always replace f and μ in (P) by $\alpha^{-1} f$ and $\alpha\mu$, respectively.

As is well known in bifurcation theory for nonlinear eigenvalue problems such as (P) the linearized problem (L)

$$Lu = 0, \quad x \in D; \qquad u_{\nu(x)} = \mu a(x)\, u, \quad x \in \partial D,$$

plays an important role. From that theory we expect local branches of nontrivial solutions to bifurcate from at least those eigensolutions (μ, u) of (L) for which μ has odd multiplicity. We will not analyze here in detail the bifurcation phenomena of (P), but will only demonstrate how to reformulate problem (P) so that the standard techniques and results apply.

We begin by remarking that the assumptions made above concerning the domain D and the coefficients a_{ij} insure that the Neumann problem (N)

$$Lu = 0, \quad x \in D; \qquad u_{\nu(x)} = h(x), \quad x \in \partial D,$$

where $h(x) \in C^0(\partial D)$ is a given function, has a solution if and only if the orthogonality condition

$$\int_{\partial D} h(x)\, dx = 0 \tag{2.1}$$

is satisfied. Moreover, solutions to (N) differ by constants and the solution (as always, in $C^2(D) \cap C^0(\bar{D})$) satisfying

$$\int_{\partial D} a(x)\, u(x)\, dx = 0 \tag{2.2}$$

is given by

$$u(x) = \int_{\partial D} N(x, y)\, h(y)\, dy$$

where $N(x, y)$ is the so-called Neumann function for L on D. The linear operator defined by

$$Ah = \int_{\partial D} N(x, y)\, h(y)\, dy, \qquad x \in \partial D,$$

is a compact operator from $C^0(\partial D)$ into $C^0(\partial D)$ under the sup norm

$$\| u \| = \sup_{\partial D} | u(x)|$$

(in fact, A is compact from $C^0(\partial D)$ into $C^1(\partial D)$ under the norm

$$\| u \| = \sum_{i=0}^{1} \sup_{\partial D} | D_i u(x)|$$

although we will not use this fact). Details concerning all of these remarks may be found in [3, 5].

The difficulty in treating (P) arises from (2.1). Clearly, any solution to (P) which satisfies (2.2) must satisfy

$$u(x) = \mu \int_{\partial D} N(x, y) f(y, u(y))\, dy; \tag{2.3}$$

the converse, however, is not true unless

$$\int_{\partial D} f(y, u(y))\, dy = 0 \tag{2.4}$$

holds. In [1, 2, 3], symmetry properties of D, a_{ij}, and f were used to guarantee (2.4) a priori, allowing (P) to be studied via the integral equation (2.3). Here, in order to handle (2.4), we do the following. For a given $u \in C^0(\partial D)$ let $k = k(u)$ be a real constant satisfying the equation

$$\int_{\partial D} f(y, u(y) + k)\, dy = 0. \tag{2.5}$$

In place of (2.3) we then consider the integral equation

$$u(x) = \mu \int_{\partial D} N(x, y) f(y, u(y) + k(u(y)))\, dy. \tag{2.6}$$

Suppose that $u \in C^0(\partial D)$ satisfies (2.6); then $\bar{u}(x) = u(x) + k$ solves (P). In order to justify the use of (2.6) in solving (P) we must first show that the operator $k = k(u)$: $C^0(\partial D) \to E^1$ is well-defined and continuous on a sufficiently small open ball $B_\epsilon \subset C^0(\partial D)$ of radius ϵ centered at 0 and that $k(0) = 0$. Then, since $fu \equiv f(y, u(y))$: $B_\epsilon \to C^0(\partial D)$ is continuous (by (H1)) and A is compact, the operator $Pu \equiv Af(I + k)\, u$ is well-defined and completely continuous on $B_\epsilon \subset C^0(\partial D)$ for $\epsilon > 0$ sufficiently small. Equation (2.6) can then be written as the operator equation

$$u = \mu P u, \qquad (\mu, u) \in E^1 \times B_\epsilon\,. \tag{2.7}$$

If $\mu \neq 0$, then problem (P) is equivalent to Eq. (2.7). Clearly, by the manner in which P is defined, if $(\mu, u) \in E^1 \times C^0(\partial D)$ is a solution to (2.7), then $(\mu, u + k) \in E^1 \times (C^2(D) \cap C^0(\bar{D}))$ solves (P) (u being uniquely defined by its boundary values). Conversely, let (μ, u), $\mu \neq 0$, be a solution (P). Defining $\bar{u} = u - k$, $k = \int_{\partial D} au\, dy$, we find that $\bar{u}$ satisfies equations (1.1) and (2.2) together with the boundary condition

$$\bar{u}_{\nu(x)} = \mu f(x, \bar{u} + k), \qquad x \in \partial D.$$

Thus, by (2.1) it follows, since $\mu \neq 0$, than (2.5) is satisfied for $\bar{u}$, k and, hence, $\bar{u}$ and k satisfy (2.6). In other words, $\bar{u}$ satisfies the operator equation (2.7). Since the solution set of (P) for $\mu = 0$ is completely known: $(0, u)$, $u \equiv$ const (and it will follow below that this is the only possible branch bifurcating from $(0, 0)$), we accordingly lose nothing in considering the operator equation (2.7), to which all of the standard techniques and results for such equations apply, for $\mu \neq 0$.

We have only to justify (2.7) by giving conditions under which $k(u)$ is well-defined and continuous on some ball B_ϵ, $\epsilon > 0$, and $k(0) = 0$. To do this we consider Eq. (2.5) which, by (H1), may be written

$$F(k, u) \equiv k + G(k, u) = 0, \qquad (k, u) \in (-\eta, \eta) \times B_\eta\,,$$

$$G(k, u) \equiv \int_{\partial D} au\, dy + \int_{\partial D} g(y, u + k)\, dy,$$

for $\eta > 0$ sufficiently small (in order that $u + k \in B_\delta$ so that $g(y, u + k)$ is meaningful). By (H1), $F(0, 0) = 0$, and $F(k, u)$ has a continuous Fréchet derivative with respect to k on $(-\eta, \eta) \times B_\eta$. It is easy to show that this derivative at $(k, u) = (0, 0)$ is the invertible linear operator $F_k(0, 0)\, w \equiv w$. Thus, by a well-known implicit function theorem [6] there exists a unique function $k = k(u)$ defined and continuously (Fréchet) differentiable on a sufficiently small neighborhood B_ϵ of $u \equiv 0$ such that $F(k(u), u) = 0$, $u \in B_\epsilon$, and $k(0) = 0$. The operator Pu is, then, well-defined and completely continuous on $B_\epsilon \to C^0(\partial D)$. We have proved the following theorem.

THEOREM 2.1. *Let* (H1) *and* (H2) *be satisfied. Then for* $\mu \neq 0$ *and* $\epsilon > 0$ *sufficiently small problem* (P) *on* B_ϵ *is equivalent to the operator equation* (2.7) *where* P *is completely continuous on* B_ϵ.

The linearized problem $u = \mu Lu$ where L is the Fréchet derivative of P at $u \equiv 0$ plays a fundamental role in the consideration of (2.7). The linear operator L is necessarily compact (since P is completely continuous [7]). Moreover, it is not difficult to show that $u = \mu Lu$ for $\mu \neq 0$ is equivalent to problem (L). To show this we observe that

$$Lw \equiv \int_{\partial D} N(x, y)\, a \left[w - \int_{\partial D} aw\, dz\right] dy; \tag{2.8}$$

for, by (2.8)

$$\begin{aligned} Pw - Lw &= \int_{\partial D} N(x, y) \left[f(y, w + k(w)) - aw + a \int_{\partial D} aw\, dz\right] dy \\ &= \int_{\partial D} N(x, y) \left[ak(w) + a \int_{\partial D} aw\, dz + g(y, w + k(w))\right] dy \\ &= \int_{\partial D} N(x, y) \left[g(y, w + k(w)) - a \int_{\partial D} g(z, w + k(w))\, dz\right] dy. \end{aligned}$$

Since $k(w)$ is continuously (Fréchet) differentiable in B_ϵ, $k(0) = 0$ (implying $k(w) = O(\| w \|)$), and (H1) holds, we have that $Pw - Lw = o(\| w \|)$, $w \in B_\epsilon$; that is, Lw as defined above is the Fréchet derivative of Pu at $u \equiv 0$. Remembering that $\int_{\partial D} a\, dx = 1$, we see that

$$\int_{\partial D} a \left[w - \int_{\partial D} aw\, dz\right] dx = 0$$

and, hence, that any solution to $w = \mu Lw$ yields a solution (μ, u), $u \equiv w - \int_{\partial D} aw\, dx$, to problem (L). Conversely, if (μ, u) is a solution to (L) with $\mu \neq 0$, then $w \equiv u - k$, $k = \int_{\partial D} au\, dx$, satisfies (1.1) and (2.2) and the boundary condition $w_{\nu(x)} = \mu a(w + k)$. Thus, since w satisfies (2.2) and since by (2.1) $k = -\int_{\partial D} aw\, dx$, it follows that (μ, w) satisfies $w = \mu Lw$.

As a result of this, it follows, since L is compact, that problem (L) can have at most a countable number of eigenvalues with no finite accumulation point each of which has finite multiplicity.

The theory of nonlinear, completely continuous operators P may now be applied to problem (P). Since the results of this theory are numerous and the application now straightforward we only state the following basic corollary of Theorem 2.1 which follows from Corollary 1.12 in [4]. For a detailed study of the solution branches of problem (P) we refer to the many results in [4].

We denote the closure of the set of nontrivial solutions to (P) by S.

COROLLARY 2.2. *Let* (H1) *and* (H2) *be satisfied and let* $r(L)$ *be the spectrum of problem* (L); $r(L)$ *is countable and has no finite accumulation point. If* $\mu \in r(L)$ *is of odd multiplicity then* S *contains a continuum of solutions to problem* (P) *which contains* $(\mu, 0)$ *and which either meets* $\partial(B_\epsilon \times R)$ *or contains* $(\mu^*, 0)$ *for some* $\mu^* \in r(L)$, $\mu^* \neq \mu$.

Note that $0 \neq r(L)$ and consequently $(0, 0)$ is not a bifurcation point of (2.7) [7]. Thus, problem (P) has only the solution set $(0, u)$, $u \equiv$ constant, branching from $(0, 0)$.

If $\mu \in r(L)$ is simple, then the structure of the continuum is known in more detail; see [4]. In particular, if $g(x, z)$ is analytic in z (as in [1, 2, 3]) and μ is simple, then u and μ can be represented in power series expansions of a small parameter; this can be carried out exactly as in [3].

3. A Global Theorem

In order to show that $k = k(u)$ was well-defined and had the necessary properties to make Pu well-defined and completely continuous, an implicit function theorem was invoked in Section 2, which because of the local nature of such theorems only allowed the local consideration of (2.7) and, hence, of problem (P) on B_ϵ for $\epsilon > 0$ sufficiently small. In this section we give a simple condition on g which insures that $k(u)$ is well-defined globally so that problem (P) may be treated globally. We use the following hypotheses.

(H3) $g_z(x, z) > -s^{-1}$, $(x, z) \in \partial D \times (-\infty, +\infty)$.

(H4) There exist constants $c_1 \geqslant 0$, $c_2 \geqslant 0$ such that for all $x \in \partial D$

$$g(x, z) \leqslant c_1, \quad z \leqslant -c_2 \qquad \text{and} \qquad g(x, z) \geqslant -c_1, \quad z \geqslant c_2.$$

Assuming these hypotheses on g (in addition to (H2) and to (H1) with $\delta = +\infty$) we once again consider Eq. (2.6) for k which we again write as

$F(k, u) \equiv k + G(k, u) = 0$, G as in (2.8). We wish to show that this equation has a unique solution $k(u)$, continuous in u, for each $u \in C^0(\partial D)$.

For a fixed $u \in C^0(\partial D)$, it follows from (H1) and (H3) that

$$\frac{d}{dk} F(k, u) = 1 + \int_{\partial D} g_z(x, u + k)\, dx > 0;$$

that is, $F(k, u)$ is strictly increasing in k. Thus, if $F(k, u) = 0$ has a solution k for $u \in C^0(\partial D)$ it is necessarily unique.

For fixed $u \in C^0(\partial D)$ and for all k sufficiently large, $u(x) + k \geqslant c_2$, $x \in \partial D$, and hence by (H4), $g(x, u + k) \geqslant -c_1$, $x \in \partial D$. Thus,

$$\int_{\partial D} g(x, u + k)\, dx \geqslant -c_1 s > -\infty$$

for k sufficiently large and it follows that $\lim_{k \to +\infty} F(k, u) = +\infty$. Similarly, using (H4) we can show that $\lim_{k \to -\infty} F(k, u) = -\infty$. Since $F(k, u)$ is continuous in k it follows that for each $u \in C^0(\partial D)$ there exists a $k = k(u)$ for which $F(k, u) = 0$. Clearly $k(0) = 0$.

Finally we show that $k(u)$ is continuous in u (with respect to the sup norm $\| u \|$). Suppose this were not the case and we could find a u_0 and a sequence $u_n \to u_0$ such that $k_n = k(u_n) \not\to k(u_0)$. Extracting a subsequence if necessary, we assume $k_n \to k_0$ where k_0 may be finite and $\neq k(u_0)$ or $k_0 = \pm\infty$. First, if k_0 is finite then from $F(k_n, u_n) = 0$ we obtain in the limit (using the obvious continuity of F insured by (H1)) the equation $F(k_0, u_0) = 0$. Since $k_0 \neq k(u_0)$ we have a contradiction to the uniqueness proved above. Finally, suppose $k_0 = +\infty$, the case $k_0 = -\infty$ being similar. Since $u_n \to u_0$ in sup norm we know that for n large all of the functions u_n are uniformly bounded on ∂D. Thus, $u_n + k_n \to +\infty$ in sup norm and as a result of (H4) we have, for large enough n, that

$$\int_{\partial D} g(x, u_n + k_n)\, dx \geqslant -c_1 s.$$

Consequently, $F(k_n, u_n) \to +\infty$ which contradicts $F(k_n, u_n) = 0$ for all n.

To sum: under (H1)–(H4) (with $\delta = +\infty$ in (H1)) the operator $k = k(u)$ defined by Eq. (2.6) is well-defined and continuous on $C^0(\partial D) \to E^1$ with $k(0) = 0$.

THEOREM 3.1. *Let* (H1) *be satisfied with* $\delta = +\infty$ *together with* (H2), (H3), *and* (H4). *Then problem* (P) *on* $C^0(\partial D)$ *is equivalent to the operator equation* (2.7) *where* P *is completely continuous on* $C^0(\partial D)$.

This result together with the global bifurcation results in [4] leads to global existence theorems for problem (P). For example, the following, theorem follows from the basic results in [4].

COROLLARY 3.2. *Let* (H1) *with* $\delta = +\infty$ *and* (H2), (H3), (H4) *be satisfied. If* $\mu \in r(L)$ *is of odd multiplicity then* S *contains a continuum of solutions to problem* (P) *which contains* $(\mu, 0)$ *and which either meets* ∞ *or contains* $(\mu^*, 0)$ *for some* $\mu^* \in r(L)$, $\mu^* \neq \mu$.

Once again more can be said if μ is simple (see [4]).

Remark. One can also prove the existence and uniqueness of the root k to $F(k, u) = 0$ for $u \in C^0(\partial D)$ by using the contraction mapping principle on $G(k, u)$. This, however, entails a Lipschitz condition of g in z with sufficiently small Lipschitz constant for all $z \in (-\infty, +\infty)$, which is a stronger, less desirable assumption to make than (H3) and (H4) above. For example, the above results handle the problem $g(x, z) = z^3$ ((H4) is satisfied for $c_1 = c_2 = 0$).

In fact, since by (H1) $g(x, 0) \equiv 0$, hypotheses (H3) and (H4) are both satisfied (with $c_1 = c_2 = 0$) by any function nondecreasing in z for all $x \in \partial D$.

4. Miscellaneous Results

We begin this section with the following theorem concerning the nodal structure of certain solutions to problem (P).

THEOREM 4.1. *Suppose* (H1) *holds and* $a(x) \geqslant 0$ (*or* $\leqslant 0$, *but* $\not\equiv 0$). *If* (μ, u) *is a nontrivial solution to problem* (P) *which is connected to* $(\mu^*, 0)$ *by a continuum of nontrivial solutions to problem* (P) *where* $\mu^* \in r(L)$, $\mu^* \neq 0$, *then* u *changes sign on* ∂D (*and, hence, in* D).

In particular then, u changes sign in ∂D for all solutions (μ, u) lying on the continua whose existence is insured by Corollaries 2.2 and 3.2 above. For an example of the importance of the nodal structure of solutions in certain applications see [1, 2].

The proof will follow easily from a sequence of lemmas. Denote $V = \{u \in C^0(\partial D)$: u changes sign on $\partial D\}$.

LEMMA 4.2. *Suppose* (μ^*, w^*), $\mu^* \neq 0$, $w^* \neq 0$, *solves the linear problem* $w = \mu L w$ *where* $a \geqslant 0$ (*or* $\leqslant 0$, *but* $\not\equiv 0$). *Then* $w^* \in V$.

Proof. As noted in Section 2, for $\mu \neq 0$, the operator equation $w = \mu L w$ is equivalent to problem (L) and, hence, (μ^*, w^*) is also a solution to this problem. But then (2.1) implies $\int_{\partial D} aw^* \, dx = 0$ which, together with the assumptions made concerning a, implies $u^* \in V$.

LEMMA 4.3. *V is an open subset of $C^0(\partial D)$ (under the* sup norm). *Let* (H1) *hold; if (μ, u) is a solution to problem (P), then $u \notin \partial V$.*

Proof. That V is open is obvious. Let $u \in \partial V$. Then $u \geqslant 0$ or $u \leqslant 0$ on ∂D with $u(x_0) = 0$ for at least one point $x_0 \in \partial D$; u thus has, by the minimum–maximum principle for solutions to elliptic equations and as a function on $\bar{D}$, a minimum or maximum at x_0. But (1.2) and (H1) imply that $u_{\nu(x_0)} = 0$ in contradiction to the fact that at its extrema, $u_\nu \neq 0$ (see [8, p. 65]). We conclude $u \notin \partial V$.

LEMMA 4.4. *Suppose* (H1) *holds. To each $\mu^* \in r(L)$ there exists a constant $\epsilon > 0$ such that if (μ, u) is any nontrivial solution to problem (P) for which $(\mu, u) \in (\mu^* - \epsilon, \mu^* + \epsilon) \times B_\epsilon$ then $u \in V$.*

Proof. Suppose that no such $\epsilon > 0$ exists. Then we can find a sequence of solutions (μ_n, u_n) such that $\mu_n \to \mu^*$, $u_n \to 0$ and $u_n \in C^0(\partial D) - \bar{V}$. Solutions to problem (P) necessarily satisfy $u_n = \mu_n P u_n$; i.e., $u_n = A\mu_n \bar{f} u_n$ or

$$u_n / \| u_n \| = A\mu_n (\bar{f} u_n / \| u_n \|). \tag{4.1}$$

By (H1) the sequence $\{\mu_n \bar{f} u_n / \| u_n \|\}$ is bounded and consequently, since A is a compact operator, $\{u_n / \| u_n \|\}$ is precompact. Extracting a subsequence if necessary we assume $u_n / \| u_n \| \to u^* \in C^0(\partial D)$, $\| u^* \| = 1$. Now (H1) implies $\bar{f} u_n / \| u_n \| \to au^*$ so that (4.1) implies $u^* = \mu^* A u^*$ or that (μ^*, u^*) solves problem (L). By Lemma 4.2, $u^* \in V$. But V is open by Lemma 4.3 and, hence, $u_n / \| u_n \|$ or u_n is in V for large n. This contradicts $u_n \in C^0(\partial D) - \bar{V}$ and proves the lemma.

Proof of Theorem 4.1. Since the continuum of solutions from the solution (μ, u) meets $(\mu^*, 0)$ it follows that on this continuum there exists a solution $(\mu^{**}, u^{**}) \in (\mu^* - \epsilon, \mu^* + \epsilon) \times B_\epsilon$ where $\epsilon > 0$ is as in Lemma 4.4; thus, $u^{**} \in V$. Let C be a continuum joining (μ, u) to (μ^{**}, u^{**}). Suppose that $u \notin V$ and hence $u \in C^0(\partial D) - \bar{V} = V^*$ by Lemma 4.3. Then $(\mu, u) \in C \cap (E^1 \times V^*)$ and $(\mu^{**}, u^{**}) \in C \cap (E^1 \times V)$. Since C cannot equal the union of the two disjoint, relatively open, nonempty sets $C \cap (E^1 \times V)$ and $C \cap (E^1 \times V^*)$, there exists a $(\mu', u') \in C$ for which $(\mu', u') \in E^1 \times \partial V$. But C is a continuum of solutions so that $u' \in \partial V$ contradicts Lemma 4.3. This contradiction implies that $u \in V$ as asserted. This proves the theorem.

One can also easily show, using (2.1), that if $zf(x, z) \geqslant 0$ ($\leqslant 0$) for all $(x, z) \in \partial D \times (-\infty, +\infty)$ then every solution of problem (P) for $\mu \neq 0$

must change sign in D, regardless of whether or not it is connected to $(\mu^*, 0)$, $\mu^* \in r(L)$, by a continuum.

From Theorem 3.1 we have conditions under which a global continuum C of solutions to problem (P), bifurcating from $(\mu, 0)$, $\mu \in r(L)$, exists. The possibility that C joins $(\mu^*, 0)$ for some other $\mu^* \in r(L)$, $\mu \neq \mu^*$, shows that the graph $\Gamma = \{(\mu, \| u \|): (\mu, u) \in C\}$ may be bounded in E^2. In the case of the second alternative that C joins ∞, Γ is unbounded in E^2. To conclude this section we briefly consider the question of in what manner Γ may be unbounded. Denote by I, J the projections $I = \{\mu: (\mu, u) \in C$ for some $u \in C^0(\partial D)\}$ $J = \{\| u \|: (\mu, u) \in C$ for some $\mu \in E^1\}$. Since C is a continuum both I, J are intervals (possibly closed, open, or half open and closed). If C joins ∞ then either I or J (or both) is infinite. The next theorem concerns the projection J.

THEOREM 4.5. *Let f satisfy* (H1) *with* $\delta = +\infty$. *If there exists a real* $z_1 \neq 0$ *such that* $f(x, z_1) = 0$ *for all* $x \in \partial D$, *then there can exist no solution* (μ, u), $\mu \neq 0$, *to problem* (P) *for which* $\max_{\partial D} u(x) = z_1$ *nor for which* $\min_{\partial D} u(x) = z_1$.

COROLLARY 4.6. *Suppose* (H1) *holds with* $\delta = +\infty$ *and that* $a(x) \geqslant 0$ $(\leqslant 0)$. *If C is a continuum of solutions to problem (P) containing $(\mu, 0)$ for some* $\mu \in r(L)$ *and if* $z_1 < 0 < z_2$ *are reals for which* $f(x, z_1) \equiv f(x, z_2) \equiv 0$, $x \in \partial D$, *then* $J = [0, c]$ *or* $[0, c)$ *where c is some constant,* $c < z_3 = \min(-z_1, z_2)$. *If C meets ∞, then I is unbounded and* $0 \notin I$.

Proof of Theorem 4.5. Suppose (μ, u), $\mu \neq 0$, solves problem (P) and $\max_{\partial D} u(x) = z_1$. Then there exists $x_0 \in \partial D$ such that $u(x_0) = z_1$ and inasmuch as u satisfies (1.2) with $\mu \neq 0$ it follows that $u_{\nu(x_0)} = 0$ in contradiction to the strong maximum–minimum principle mentioned above [8]. A similar argument can be made to rule out $\min_{\partial D} u(x) = z_1$.

Proof of Corollary 4.6. Define the open set

$$T = \{u \in C^0(\partial D): z_1 < u(x) < z_2, \, x \in \partial D\}.$$

By Theorem 4.5 no solution (μ, u), $\mu \neq 0$, exists for which $u \in \partial T$. By Theorem 4.1 no solution $(\mu, u) \in C$ exists for which $\mu = 0$; that is, $0 \notin I$. Arguing just as in the proof of Theorem 4.1, we conclude that $C \subset T$. Thus, for $(\mu, u) \in C$, $z_1 < u < z_2$ which implies $\| u \| < z_3$. Since J is connected, bounded, and contains 0 (C contains $(\mu^*, 0)$), we have $J = [0, c]$ or $[0, c)$. Finally, if C meets ∞, then since J is bounded, I must be unbounded. It was already noted that $0 \notin I$.

As a final result, we note that if C meets ∞ and $(\mu, u) \in C$ is known to satisfy an a priori estimate $\| u \| \leqslant M(\mu)$, where $M(\mu)$ is bounded on finite μ

intervals, then clearly I must be unbounded. From this follows easily another result.

THEOREM 4.7. *Let* (H1) *hold* (*with* $\delta = +\infty$) *and suppose that there exist constants* $K > 0$, $0 \leqslant p < 1$, *such that* $|f(x, z)| \leqslant K |z|^p$ *for all* $(x, z) \in \partial D \times (-\infty, +\infty)$. *If* (μ, u) *is a solution to problem* (P) *then* $\| u \| \leqslant \bar{K}\mu^{1/(1-p)}$ *for some constant* $\bar{K} > 0$.

Proof. From (2.3) and the hypothesis on f we find $\| u \| \leqslant M\mu \| u \|^p$ for $M > 0$ depending on $N(x, y)$ but independent of μ and u. Thus, $\bar{K} = M^{1/(1-p)}$.

COROLLARY 4.8. *Suppose* (H1) (*with* $\delta = +\infty$), (H2), (H3), *and* (H4) *hold and that* f *satisfies the hypothesis of Theorem* 4.7. *Let* C *be a continuum of solutions to* (P) *containing* $(\mu, 0)$, $\mu \in r(L)$, μ *of odd multiplicity. Then* I *is unbounded*, $0 \notin I$.

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