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Nonlinear Steklov Problems on the Unit Circle

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1. INTRODUCTION

Several authors have considered the problem for the existence of functions harmonic in a given region of the plane subject to nonlinear Steklov-type boundary conditions [cf. (2.1)]. Solutions with singularities were studied in [2, 14] while unique regular solutions were obtained in [8, 9]. On the other hand, Levi-Civita's theory [12] of periodic progressing water waves requires a regular nontrivial solution to a problem of this type which does not have a unique solution (the zero function being a solution corresponding to uniform flow). More recently the author [6] has proved the existence of nontrivial solutions for problems of this latter type (for which zero is a trivial solution) involving general self-adjoint elliptic equations of second order in n-dimensions using the techniques of bifurcation theory for compact operators on Banach spaces (especially the expansion techniques of Liapunov and Schmidt). In this paper we consider the restricted problem of Laplace's equation on the unit disk in the plane together with nonlinear boundary conditions of the Steklov type, involving an arbitrary parameter λ, for which zero is a solution. Our purpose is twofold: (1) to characterize the local branches known to exist from [6] by the nodal structure of the solutions on them and (2) to prove that these local branches exist globally.

Our main result is contained in Corollary 4.2 which states, roughly speaking, that there exist infinitely many, disjoint branches of solution to our nonlinear Steklov problem completely characterized by their nodal structure and connecting the trivial solution at $\lambda =$ a positive integer to "∞". This work was suggested by the work of Crandall and Rabinowitz [3, 17] in their study of nonlinear Sturm–Liouville problems and several of our techniques and proofs are adaptations of theirs. The main tool in proving Corollary 4.2 is Leray–Schauder degree theory, the main features of which are stated in an Appendix for reference purposes. In Section 2 the problem is explicitly stated and reformulated as an operator equation in suitable Banach spaces. The local branches of solutions are characterized by their nodal structure in

Section 3 and in Section 4 we prove global existence of solutions. Some theorems on the nature of the spectrum are offered in Section 5. Although Levi-Civita's problem is not exactly of the type considered here, in Section 6 we briefly consider a nonlinear approximation to his problem which is of the appropriate type and apply our results. It is planned that another paper will be devoted to global existence for nonlinear Steklov problems which will include Levi-Civita's problem.

We remark here that although all of the work in Sections 2–6 is done in the space of harmonic functions $u(x_1, x_2)$ for which $u(x_1, x_2) = -u(x_1, -x_2)$ the techniques and results are valid on the space of functions satisfying $u(x_1, x_2) = -u(-x_1, x_2)$ as well; i.e., the Banach spaces B_k below could be redefined relative to the nodal structure of the linear eigensolutions $r^k \cos k\theta$ instead of $r^k \sin k\theta$. Thus, the results of this paper imply the global existence of *four* branches of solutions to the nonlinear problem bifurcating from the zero solution at integer values of λ: one pair differing only in sign and having the nodal structure of $r^k \sin k\theta$, the other pair differing only in sign and having the nodal structure of $r^k \cos k\theta$. The results below apply to both pairs of branches individually.

The problems studied in this paper are retricted to the unit circle of the plane mainly because the nodal structure of the eigensolutions to the linearized problem is known (since the eigensolutions are known explicitly). If enough were known about the nodal structure of the eigensolutions for more general problems (say, self-adjoint elliptic equations on arbitrary domains in n dimensions) so as to permit the construction of sets of functions with the necessary topological characteristics as in Sections 2, 3, then the results of this work could be extended to these more general problems.

2. Preliminaries

We consider the following nonlinear problem for $u = u(r, \theta)$ (hereafter referred to as problem N):

$$\Delta u = 0, \qquad r < 1,$$
$$\partial u/\partial r = \lambda f(u, \theta), \qquad -\pi \leqslant \theta \leqslant \pi, \qquad r = 1, \tag{2.1}$$

where r, θ are polar coordinates in the x_1, x_2-plane, Δu is the Laplace operator, f is a given function of its arguments, and λ is a real parameter to be determined as part of the solution. By a solution to problem N we mean the ordered pair (u, λ); the smoothness of the harmonic function u will be brought out below. We assume f satisfies the following hypotheses:

(H1) $f(\xi, \theta)$ is analytic in its arguments for $(\xi, \theta) \in (-\infty, +\infty) \times [-\pi, \pi]$ with $f(\xi, \pi) = f(\xi, -\pi)$ for all $\xi \in (-\infty, +\infty)$;

(H2) $f(0, \theta) = 0$ and $f_\xi(0, \theta) = 1$ for $\theta \in [-\pi, \pi]$;

(H3) $f(\xi, \theta) = -f(-\xi, \theta)$ for $(\xi, \theta) \in (-\infty, +\infty) \times [-\pi, \pi]$.

The linearized problem (called the Steklov problem on D, see [19])

$$\partial u/\partial r = \lambda u, \qquad r = 1, \qquad -\pi \leqslant \theta \leqslant \pi \tag{2.2}$$

plays an important role in the theory of problem N. It is well known [1, 2] that the Steklov problem on the unit circle D has nontrivial solutions if and only if $\lambda = k$, $k \in Z^+ \cup \{0\}$ (Z^+ is the set of positive integers) in which case we have the eigensolutions $(Ar^k \sin k\theta, k)$, $(Br^k \cos k\theta, k)$ for $k \in Z^+$ and $(C, 0)$ for $\lambda = 0$, where A, B, C are arbitrary constants. Thus, each positive Steklov eigenvalue is double while $\lambda = 0$ is a simple eigenvalue.

In order to study problem N we will reformulate the problem as an operator equation on an appropriate Banach space. Let $\|\phi\|_{1,T} = \max_T |\phi| + \sum_{i=1,2} \max_T |D_i\phi|$, where $D_i\phi = \partial\phi/\partial x_i$ and ϕ is a differentiable function defined on a set T. We denote by x the point (x_1, x_2) or, in polar coordinates, (r, θ). Let $C^1[-\pi, \pi]$ be the set of all continuously differentiable functions of $\theta \in [-\pi, \pi]$ such that $\phi(\pi) = \phi(-\pi)$ and consider the single layer potential of density ϕ

$$(\bar{L}\phi)(x) \equiv \int_{-\pi}^{\pi} \phi(\theta) \log R^{-1}\, d\theta,$$

where $\phi \in C^1[-\pi, \pi]$ and R is the Euclidean distance between x and the point on the boundary ∂D of D given by θ. It is shown in [18] (for three dimensions, but the arguments are valid in two dimensions also) that $\bar{L}\phi \in C^1(\bar{D})$, $\bar{D} = D + \partial D$, and consequently the linear operator $\bar{L}$ maps $C^1[-\pi, \pi]$ into $C^1(\bar{D})$, both of which are Banach spaces under the respective norms $\|\cdot\|_{1,\partial D}$ $\|\cdot\|_{1,\bar{D}}$. From [18, Lecture 7] we know that

$$\int_{-\pi}^{\pi} |\log R^{-1}|\, d\theta, \qquad \int_{-\pi}^{\pi} |D_i \log R^{-1}|\, d\theta, \qquad i = 1, 2,$$

are continuous functions and, hence, bounded on $\bar{D}$. (Actually the latter two may not converge if $x \in \partial D$, but it is shown that these integrals have, at worst, removable singularities on ∂D and consequently coincide with functions continuous on $\bar{D}$.) From the remarks we conclude that

$$\|\bar{L}\phi\|_{1,\bar{D}} \leqslant K \|\phi\|_{0,\partial D}, \tag{2.3}$$

where $\|\phi\|_{0,\partial D} = \max_{x\in\partial D} |\phi(x)|$ and K is some positive constant. Thus,

$$\|\bar{L}\phi\|_{1,\bar{D}} \leqslant K \|\phi\|_{1,\partial D} \tag{2.4}$$

and $\bar{L}$ is a continuous linear operator. Suppose now that $\{\phi_n\}$ is a sequence of functions from $C^1[-\pi, \pi]$ satisfying $\| \phi_n \|_{1,\partial D} \leqslant M$ for some constant $M < +\infty$. A careful reading of the estimates used in [18] to show that $\bar{L}\phi$, $D_i\bar{L}\phi$ are continuous in $\bar{D}$ will show that these estimates, when applied to ϕ_n, are independent of n. It follows readily from the arguments in [18] that $\{\bar{L}\phi_n\}$, $\{D_i\bar{L}\phi_n\}$ are equicontinuous on $\bar{D}$ and since they are also uniformly bounded [by (2.4)] it further follows by the well-known Ascoli theorem $\{\bar{L}\phi_n\}$ has a subsequence converging to a function in $C^1(\bar{D})$ with respect to the norm $\| \cdot \|_{1,\bar{D}}$. Thus, $\bar{L}$ is a compact operator from $C^1[-\pi, \pi]$ into $C^1(\bar{D})$.

It is well known (see, for example, [6]) that a Neumann function $N(x, y)$ exists for D. Using the compactness of L and the construction of $N(x, y)$ as given in [6], we have that $N(x, y) = (2\pi)^{-1} \log R^{-1} + \psi(x, y)$, where $\psi(x, y)$ is $C^2(D)$, $C^1(\bar{D})$ in both x and y. Consequently the linear operator

$$(L\phi)(x) = \int_{-\pi}^{\pi} N(x, \theta)\, \phi(\theta)\, d\theta$$

is compact as an operator from $C^1[-\pi, \pi]$ to $C^1(\bar{D})$. Moreover, if $\phi \in C^1[-\pi, \pi]$ satisfies the condition

$$\int_{-\pi}^{\pi} \phi(\theta)\, d\theta = 0 \tag{2.5}$$

then $L\phi \in C^2(D)$, $C^1(\bar{D})$ is the unique harmonic function on D satisfying $\partial u/\partial r = \phi$ on ∂D and $\int_{-\pi}^{\pi} L\phi \,|_{r=1}\, d\theta = 0$ (see [6]).

We now introduce the Banach spaces B_k, $k \geqslant 1$, of functions harmonic on D for which $\| u \|_{1,\bar{D}} < +\infty$, which vanish on the lines $\theta = n\pi/k$, $n = 0, \pm 1, \ldots, \pm k$, and whose boundary values $\in C^1[-\pi, \pi]$ are odd and periodic with period $2\pi/k$. We may consider L as an operator defined on each B_k; i.e.,

$$(Lu)(x) = \int_{-\pi}^{\pi} N(x, \theta)\, u(1, \theta)\, d\theta$$

for $u = u(r, \theta) \in B_k$.

LEMMA 2.1. *L is a linear compact operator from B_k into itself for all $k \geqslant 1$.*

Proof. Let $u(r, \theta) \in B_k$ for some fixed $k \geqslant 1$. Then since $\phi(\theta) = u(1, \theta)$ is an odd function of $\theta \in [-\pi, \pi]$ it follows that (2.5) holds and, hence, $(Lu)(x)$ is the unique harmonic function $\in C^1(\bar{D})$ satisfying $\partial Lu/\partial r = \phi$ on $r = 1$ and $\int_{-\pi}^{\pi} Lu \,|_{r=1}\, d\theta = 0$. Define $v(r, \theta) \equiv (Lu)(r, -\theta)$. Certainly v is harmonic and $\in C^1(\bar{D})$; moreover, $\partial v(1, \theta)/\partial r = \phi(-\theta) = -\phi(\theta)$ since $u \in B_k$ and, hence, is odd in θ. But $-Lu$ satisfies the same Neumann problem and consequently $v = -Lu + C$, $C = \text{const}$. However, we also have

$$\int_{-\pi}^{\pi} v(1, \theta)\, d\theta = \int_{-\pi}^{\pi} Lu \,|_{r=1}\, d\theta = 0$$

so that $C = 0$ and, thus, $(Lu)(r, -\theta) = -(Lu)(r, \theta)$; i.e., Lu is "odd" with respect to points symmetric with respect to the x_1 axis. The same argument applies to the lines $\theta = n\pi/k$, $n = \pm 1, \ldots, \pm(k-1)$ in as much as $u \in B_k$ implies $\phi(\theta)$ is odd with respect to all lines $\theta = n\pi/k$; this is easily seen by rotating the axis $n\pi/k$ radians and repeating the argument. It follows that $Lu\,|_{r=1}$ is odd in θ and periodic of period $2\pi/k$ and consequently $Lu \in B_k$ or L maps B_k into itself.

Since $\| u \|_{1,\partial D} = \| u \|_{1,\bar{D}}$ for $u \in B_k$ the compactness of L follows from the remarks concerning L above.

It is clear now that the Steklov problem (2.2) may be formulated as the operator equation $\lambda Lu = u$ on any of the Banach spaces B_k.

To write problem N as an operator equation we must consider the nonlinear operator $\bar{f}u \equiv f(u(1, \theta), \theta)$ defined on B_k. We will need

(H4)$_k$ $f(\xi, \theta)$ is an even periodic function of $\theta \in [-\pi, \pi]$ of period $2\pi/k$ for fixed $\xi \in (-\infty, +\infty)$.

If f satisfies (H3), (H4)$_k$ for $k \in Z^+$ then it is clear that for $u \in B_k$ the function $f(u(1, \theta), \theta)$ is odd and periodic with period $2\pi/k$ as a function of $\theta \in [-\pi, \pi]$. As a result the nonlinear operator defined by $Nu \equiv L\bar{f}u$ maps B_k into itself. By (H1) $\bar{f}$ is a continuous operator and, hence, so is N. Moreover, since $\bar{f}$ clearly takes bounded sets into bounded sets, it follows that N is compact. We may now state

LEMMA 2.2. *Under hypotheses* (H1)–(H4)$_k$ *the nonlinear operator N is a completely continuous* (*i.e., continuous and compact*) *operator mapping B_k into itself whose Fréchet derivative at $u = 0$ is Lu.*

We need only prove that L is the Fréchet derivative of N. Let $h \in B_k$. From now on we denote $\| u \| = \| u \|_{1,\bar{D}}$. Then

$$\| Nh - Lh \| / \| h \| = \left\| \int_{-\pi}^{\pi} N(x, \theta)[f(h, \theta) - h]\, d\theta \right\| \Big/ \| h \| \to 0$$

as $\| h \| \to 0$ since $f_u(0, \theta) \equiv 1$ by (H2), and the proof of the lemma is complete.

It is now clear that problem N on B_k is equivalent to the equation $\Phi(u, \lambda) = 0$, $\Phi(u, \lambda) \equiv u - \lambda Nu$. We note also that the first eigenvalue of L on B_k is $\lambda = k$, that all eigenvalues $\lambda = k$ are *simple* on B_k, and that $\lambda = 0$ is not an eigenvalue on any space B_k.

Finally, we observe that $f \equiv f(u)$ satisfies (H4)$_k$ for all $k \geqslant 1$.

3. THE LOCAL THEORY

Theorem 3.1 below on local bifurcation for problem N was proved by the author in [6], at least for solutions which are in $C^0(\bar{D})$. However, the standard existence theorems in bifurcation theory apply to problem N in each B_k and assert (see [15]) that exactly two branches of solutions in B_k, differing only in sign, bifurcate from $\lambda = k$. These solutions are *a priori* in $C^0(\bar{D})$ and, hence, coincide with the solutions found in [6]. The full theorem is stated below.

THEOREM 3.1. *Under hypotheses* (H1)–(H4)$_k$ *problem N has solutions* $u_k \in B_k$ *of the form*

$$u_k = \sum_{n=0}^{\infty} u_{nk}\mu^{2n+1}, \qquad \lambda = k + \sum_{n=1}^{\infty} \lambda_{2n,k}\mu^{2n} \tag{3.1}$$

for each $|\mu|$ *sufficiently small where* $u_{nk} \in B_k$ *and where the convergence is absolute and uniform on* $\bar{D}$. *Let* $\lambda_{2m,k}$ *be the first nonzero coefficient in the series expansion of* λ. *Clearly if* $\lambda_{2m,k} < 0(>0)$ *then* $\lambda < k$ $(>k)$ *for* $|\mu| \leqslant \mu_0$, μ_0 *sufficiently small, in which case bifurcation at k is said to be to the left* (*right*); *in this case there exist reals* $\delta, \rho > 0$ *such that* (3.1) *is the only solution* $\| u \| < \rho$ *for* $\lambda_k - \delta < \lambda < \lambda_k$ $(\lambda_k < \lambda < \lambda_k + \delta)$ *and there exists no solution for* $\lambda_k < \lambda < \lambda_k + \delta$ $(\lambda_k - \delta < \lambda < \lambda_k)$.

The solutions u_k in (3.1) vanish on the lines $\theta = n\pi/k$, $n = 0, \pm 1, ..., \pm k$ by virtue of the fact that $u_k \in B_k$. We wish to show now that the local branches near λ_k vanish only on these lines in $\bar{D}$; i.e., that they have the nodal structure of the k-th eigensolution $r^k \sin k\theta$.

Let $\mathscr{B}_k(\rho) = \{u \in B_k : \| u \| < \rho\}$, $\mathscr{N}_k^+ = \{u \in B_k : u = 0$ only on $\theta = n\pi/k$, $n = 0, \pm 1, ..., \pm k$ in $\bar{D}$, $\partial u/\partial\theta \neq 0$ at $r = 1$, $\theta = n\pi/k$ and $\partial u/\partial\theta > 0$ at $r = 1, \theta = 0\}$, $\mathscr{N}_k^- = -\mathscr{N}_k^+$, and $\mathscr{S}_k(\lambda) = \{u \in B_k : \Phi(u, \lambda) = 0, \| u \| \neq 0\}$.

LEMMA 3.1. (a) *Any* $u \in \mathscr{S}_k$ *can be analytically extended as a harmonic function to a region* $D^* \supset \bar{D}$.

(b) *For each* $k \geqslant 1$, $\mathscr{N}_k^\nu (\nu = +$ *or* $-)$ *is an open set in* B_k.

(c) $\partial\mathscr{N}_k^\nu \cap \mathscr{S}_k(\lambda) = \varnothing$, $\nu = +$ *or* $-$, *all* λ.

Proof. Part (a) follows immediately from (H1) and a theorem of H. Lewy [13].

Suppose $u \in \mathscr{N}_k^+$, $u_n \in B_k$, $n = 1, 2, ...$, and $u_n \to u$. By definition of the norm on B_k it follows that the boundary values of u_n converge to those of u under $\| \cdot \|_{1,\partial D}$ and, hence, for $n \geqslant N$, N sufficiently large, $u_n(1, \theta)$ has simple zeros at and only at $\theta = n\pi/k$, $n = 0, \pm 1, ..., \pm n$ with $\partial u_n/\partial\theta > 0$ at $\theta = 0$. It follows that $u_n \in \mathscr{N}_k^+$ for $n \geqslant N$ and, thus, $\mathscr{N}_k^+$ is open. Similarly for $\nu = -$.

Finally, let $u \in \partial\mathscr{N}_k^+ \cap \mathscr{S}_k(\lambda)$ for some λ. Clearly $u \in \partial\mathscr{N}_k^+$ implies either $u \equiv 0$ or u vanishes exactly on $\theta = n\pi/k$ but poses a critical point at $r = 1$,

$\theta = n_0\pi/k$ for some n_0. But $u \not\equiv 0$ if $u \in \mathscr{S}_k(\lambda)$ so we assume, without loss of generality, that u has a critical point at $r = 1$, $\theta = 0$. Since $u(1, \theta)$ is odd, $u_{\theta\theta}(1, 0) = 0$ and since $u \in \mathscr{S}_k(\lambda)$, u satisfies (2.1) upon differentiation of which with respect to θ yields $u_{r\theta}(1, 0) = 0$. Laplace's equation now implies that $u_{rr}(1, 0) = 0$ and, hence, u is at least third order at $r = 1$, $\theta = 0$, which implies [20] that u has at least three analytic nodal arcs passing through $r = 1$, $\theta = 0$ at equal angles. At least one of these arcs must enter D^+ and as a result u is negative in every neighborhood of $r = 1$, $\theta = 0$ intersected with D^-, which disallows its uniform approximation by functions from $\mathscr{N}_k^+$. This contradicts $u \in \partial\mathscr{N}_k^+$ and, hence, $\partial\mathscr{N}_k^+ \cap \mathscr{S}_k(\lambda) = \varnothing$. The case $\nu = -$ is proved similarly.

If R is a subset of B_k let $C_k R$ denote the complement of R with respect to B_k.

LEMMA 3.2. *Let* $\inf \varnothing = +\infty$. *Define* $\mathscr{N}_k = \mathscr{N}_k^+ \cup \mathscr{N}_k^-$ *and*

$$r_k(\lambda) = \inf\{\| u \| : u \in \mathscr{S}_k(\lambda) \cap C\bar{\mathscr{N}}_k\};$$
$$\rho_k(\lambda) = \inf\{\| u \| : u \in \mathscr{S}_k(\lambda)\}.$$

Then $r_k(\lambda)$, $\rho_k(\lambda)$ *are positive and lower semicontinuous on* $(\mathscr{R} - Z^+) \cup \{k\}$, $\mathscr{R} - Z^+$, *respectively.* ($\mathscr{R}$ *is the set of real numbers.*)

Proof. First we show $r_k(\lambda)$ is lower semicontinuous; i.e., if

$$\mu_n \to \lambda \in (\mathscr{R} - Z^+) \cup \{k\}$$

then

$$r_k(\lambda) \leqslant \liminf_{n\to\infty} r_k(\mu_n).$$

If $\liminf_{n\to\infty} r_k(\mu_n) = +\infty$, there is nothing to prove so we assume (passing to a subsequence) that $\lim_{n\to\infty} r_k(\mu_n)$ exists and is finite. Let $u_n \in \mathscr{S}_k(\mu_n) \cap C\bar{\mathscr{N}}_k$ be such that

$$0 < \| u_n \| \leqslant r_k(\mu_n) + (1/n). \tag{3.1}$$

The equation $\Phi(u_n, \mu_n) = 0$ can be written

$$u_n/\| u_n \| = \mu_n L[f(u_n, \theta)/\| u_n \|]. \tag{3.2}$$

Since $\lim_{n\to\infty} r_k(\mu_n)$ exists, (3.1) implies $\{u_n\}$ is a bounded set in B_k and, hence, so is $\mu_n f(u_n, \theta)/\| u_n \|$. Thus, $\{u_n/\| u_n \|\}$ is a precompact set since L is compact on B_k and (passing to a subsequence)

$$(u_n/\| u_n \|) \to v, \qquad \| u_n \| \to \alpha$$

for some $v \in B_k$ and $\alpha \in \mathscr{R}$, $\alpha \geqslant 0$. By (H2), (H3) we may write $f(u, \theta) = u(1 + g(u, \theta))$, where $g(u, \theta) = o(\| u \|^2)$ and we conclude, by passing $n \to \infty$ in (3.2), that

$$v = \lambda L(v[1 + g(\alpha v, \theta)]). \tag{3.3}$$

Now if $\alpha = 0$, then v is a Steklov eigensolution and, hence, $\lambda = k$ and $v \in \mathscr{N}_k$. But $u_n/\| u_n \| \in C_k\mathscr{N}_k$ so that in addition $v \in \overline{C_k\mathscr{N}_k}$. Thus, $v \in \partial\mathscr{N}_k$, contrary to Lemma 3.1(c) and from this contradiction we conclude $\alpha \neq 0$. Noting that $\partial\overline{C_k\mathscr{N}_k} = \partial\mathscr{N}_k$, $\| v \| = 1$, and $v \notin \partial\mathscr{N}_k$ [by Lemma 3.1(c)] we see that $v(\neq 0) \in C_k\mathscr{N}_k$. Moreover, this implies $\alpha v \in C_k\mathscr{N}_k$ and since $\| \alpha v \| = \alpha$, $\Phi(\alpha v, \lambda) = 0$ [multiply (3.3) by α] we find

$$\begin{aligned} r_k(\lambda) &\leqslant \| \alpha v \| = \alpha \\ &= \lim_{n\to\infty} \| u_n \| \leqslant \lim_{n\to\infty} \left(r_k(\mu_n) + \frac{1}{n}\right) = \lim_{n\to\infty} r_k(\mu_n) \end{aligned} \tag{3.4}$$

and the lower semicontinuity of $r_k(\lambda)$ for $\lambda \in (\mathscr{R} - Z^+) \cup \{k\}$ is established. Taking $\mu_n = \lambda$ for all n we find from (3.4) that

$$0 < \alpha \leqslant \lim_{n\to\infty} r_k(\mu_n) = r_k(\lambda)$$

and this completes the proof of the lemma for $r_k(\lambda)$. The proof for $\rho_k(\lambda)$ is similar and is omitted.

THEOREM 3.2. *Assume* (H1)–(H4)$_k$. *Then to each real* λ' *there correspond constants* $\delta = \delta(\lambda') > 0$, $\epsilon = \epsilon(\lambda') > 0$ *such that for* $\lambda \in [\lambda' - \delta, \lambda' + \delta]$

(a) $\lambda' = k \Rightarrow \mathscr{S}_k(\lambda) \cap \overline{\mathscr{B}_k(\epsilon)} \subset \mathscr{N}_k$,

(b) $\lambda' \notin Z^+ \Rightarrow \mathscr{S}_k(\lambda) \cap \overline{\mathscr{B}_k(\epsilon)} = \varnothing$,

(c) $\lambda' \in Z^+ - \{k\} \Rightarrow \mathscr{S}_k(\lambda) \cap \mathscr{N}_k \cap \overline{\mathscr{B}_k(\epsilon)} = \varnothing$.

Proof. (a) Let $\delta \in [0, 1)$ and

$$\epsilon = \tfrac{1}{2} \inf_{[k-\delta, k+\delta]} r_k(\lambda).$$

We know from Lemma 3.2 that $\epsilon > 0$. By the definition of $r_k(\lambda)$ if $u \in \overline{\mathscr{B}_k(\epsilon)} \cap \mathscr{S}_k(\lambda)$ then $u \notin C\mathscr{N}_k$. Since $u \notin \partial\mathscr{N}_k$ it follows that $u \in \mathscr{N}_k$ and (a) is established.

(b) Suppose $\lambda' \notin Z^+$. Then set $\delta(\lambda') = \frac{1}{2}\min\{\lambda' - [\lambda'], [\lambda' + 1] - \lambda'\}$ where $[\alpha]$ is the greatest integer in α and set

$$\epsilon = \epsilon(\lambda') = \tfrac{1}{2} \min_{[\lambda'-\delta, \lambda'+\delta]} \rho_k(\lambda) \tag{3.5}$$

which, by Lemma 3.2, is positive; (b) follows immediately.

(c) We need only note that $\mathcal{N}_k^\nu \cap \mathcal{N}_l^\nu = \emptyset$ for $k \neq l$. Consequently, (c) follows from (a).

THEOREM 3.3. *Let* (H1)–(H4)$_k$ *be satisfied. Then* $\delta = \delta(k) > 0$, $\epsilon = \epsilon(k) > 0$ *can be chosen in Theorem* (3.2) *such that* $\mathscr{S}_k(\lambda) \cap \partial\mathscr{B}_k(\epsilon) = \emptyset$,

$$\mathscr{S}_k(\lambda) \cap \mathscr{B}_k(\epsilon) \subset \mathcal{N}_k^\nu \qquad \textit{for all} \quad \lambda \in [k - \delta, k + \delta].$$

Proof. Let $\epsilon(k) > 0$ and $0 < \delta^* < 1$ be so small that

$$u \in \mathscr{S}_k(\lambda) \cap \mathscr{B}_k(\epsilon(k)) \subset \mathcal{N}_k^\nu, \qquad \lambda \in [k - \delta^*, k + \delta^*],$$

is given by (3.1) for $0 \leqslant |\mu| \leqslant \mu_0$. Since $\lambda - \lambda_k$ is analytic as a function of μ it has at most a finite number of critical points on $0 \leqslant |\mu| \leqslant \mu_0$. Let $\mu' \neq 0$ be the critical point for which $|\mu'|$ is smallest. Then $\lambda - k$ has no zero on $0 < |\mu| \leqslant |\mu'|$ and if we let λ' be given by (3.1) for $\mu = \mu'$ then $\delta^* > \delta(k) = \frac{1}{2}\inf_{0 \leqslant |\mu| \leqslant |\mu'|}\{|\lambda - k| : \|u(\lambda)\| = \epsilon(k)\} > 0$; here $u(\lambda)$ is the solution (3.1) corresponding to λ. It follows that for $\lambda \in [k - \delta(k), k + \delta(k)]$ we have $\mathscr{S}_k(\lambda) \cap \partial\mathscr{B}_k(\epsilon(k)) = \emptyset$ and, of course, $\mathscr{S}_k(\lambda) \cap \mathscr{B}_k(\epsilon(k)) \subset \mathcal{N}_k^\nu$.

We conclude this section with a degree calculation which will be needed in the next section. See the Appendix for notation.

THEOREM 3.4. *Let* $\epsilon(\lambda)$, $\delta(\lambda)$ *be as in Theorems* 3.2 *and* 3.3,

$$\epsilon'(\lambda) = \tfrac{1}{2}\min(\epsilon(k), \epsilon(\lambda)) \qquad \textit{and} \qquad \operatorname{sgn} \lambda_{2m,k} = \lambda_{2m,k}/|\lambda_{2m,k}|.$$

Then

$$d(\Phi(\lambda), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda))}] \cap \mathcal{N}_k^\nu, 0) = \operatorname{sgn} \lambda_{2m,k} \tag{3.6}$$

($\nu = +$ *or* $-$) *for* $\lambda \in (k - \delta(k), k)$ *if* $\operatorname{sgn} \lambda_{2m,k} = -1$ *and for* $\lambda \in (k, k + \delta(k))$ *if* $\operatorname{sgn} \lambda_{2m,k} = +1$. *Here we have set* $\Phi(\lambda) \equiv \Phi(u, \lambda)$.

Proof. Suppose $\operatorname{sgn} \lambda_{2m,k} = -1$, the other case being similar. That the degree in (3.6) is well defined follows from Lemma 3.1, Theorem 3.3, and the definition of $\epsilon(\lambda)$. Let $\lambda' \in (k - \delta(k), k)$. By P3 of the Appendix (the zero is hereafter dropped from the degree notation),

$$d(\Phi(k + \delta(k)), \mathscr{B}_k(\epsilon(k))) = d(\Phi(\lambda), \mathscr{B}_k(\epsilon(k))), \quad \lambda \in [k - \delta(k), k + \delta(k)],$$

and by P4 and Theorem 3.1,

$$d(\Phi(k + \delta(k)), \mathscr{B}_k(\epsilon(k))) = i(\Phi(k + \delta(k)), 0, 0) = -1;$$

hence, for $\lambda = \lambda'$,

$$d(\Phi(\lambda'), \mathscr{B}_k(\epsilon(k))) = -1. \tag{3.7}$$

On the other hand, P2 and P5 imply

$$d(\Phi(\lambda'), \mathscr{B}_k(\epsilon(k))) = d(\Phi(\lambda'), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda'))}] \cap \mathscr{N}_k) + d(\Phi(\lambda'), \mathscr{B}_k(\epsilon'(\lambda'))). \tag{3.8}$$

Whereas by P4,

$$d(\phi(\lambda'), \mathscr{B}_k(\epsilon'(\lambda'))) = i(\Phi(\lambda'), 0, 0) = 1,$$

we have by combining (3.7) and (3.8),

$$d(\Phi(\lambda'), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda'))}] \cap \mathscr{N}_k) = -2. \tag{3.9}$$

It follows from the definition of degree and the facts that $N(-u) = -N(u)$, $\mathscr{N}_k^+ = -\mathscr{N}_k^-$, that

$$d(\Phi(\lambda'), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda'))}] \cap \mathscr{N}_k^+)$$
$$= d(\Phi(\lambda'), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda'))}] \cap \mathscr{N}_k^-),$$

and, consequently, from (3.9), $\mathscr{N}_k = \mathscr{N}_k^+ \cup \mathscr{N}_k^-$, $\mathscr{N}_k^+ \cap \mathscr{N}_k^- = \varnothing$, and P2 we have (3.6).

4. The Global Theory

Let $B_k \times \mathscr{R}$ be endowed with the product topology and let $\mathscr{O}_k$ be any bounded open set in $B_k \times \mathscr{R}$ for which $(0, k) \in \mathscr{O}_k$. Let $\mathscr{S}_k^\nu \subset B_k \times \mathscr{R}$, $\nu = +$ or $-$, be defined by $\mathscr{S}_k^\nu = \{(u, \lambda) : u \in \mathscr{S}_k(\lambda) \cap \mathscr{N}_k^\nu$ for some $\lambda \in \mathscr{R}\}$. By Theorems 3.1, 3.2, $(0, k) \in \partial\mathscr{S}_k^\nu$, $\nu = +$ or $-$.

Lemma 4.1. (a) *$C_k^\nu = \overline{\mathscr{S}_k^\nu \cap \mathscr{O}_k}$ is compact in $B_k \times \mathscr{R}$, $\nu = +$ or $-$.*

(b) $\mathscr{S}_k^\nu \cap \partial\mathscr{O}_k = \varnothing \Rightarrow C_k^\nu \subset \mathscr{O}_k$.

Proof. $\mathscr{S}_k^\nu \cap \mathscr{O}_k$ is bounded and consequently (a) follows from the compactness of N on B_k.

Clearly $C_k^\nu \subset \bar{\mathscr{O}}_k$. Suppose $(u, \lambda) \in C_k \cap \partial\mathscr{O}_k$. Then there exists a sequence $(w_n, \mu_n) \in \mathscr{S}_k^\nu \cap \mathscr{O}_k$ such that $(w_n, \mu_n) \to (u, \lambda)$ and as a result (u, λ) is a solution to $\Phi(u, \lambda) = 0$. Since $(0, k) \notin \partial\mathscr{O}_k$ we have [by Theorem 3.2(a)] that $u \not\equiv 0$ and hence $u \in \mathscr{S}_k(\lambda) \cap \mathscr{N}_k^\nu$; i.e., $(u, \lambda) \in \mathscr{S}_k^\nu$. But $(u, \lambda) \in \partial\mathscr{O}_k$ also and we have a contradiction to the assumption made in (b). Consequently $C_k^\nu \cap \partial\mathscr{O}_k = \varnothing$ and (b) is established.

We now are ready to state and prove our main result.

THEOREM 4.1. *Let $\mathcal{O}_k$ be an arbitrary bounded open set in $B_k \times \mathcal{R}$ with $(0, k) \in \mathcal{O}_k$ and let* (H1)–(H4)$_k$ *be satisfied. Then $\mathcal{S}_k^\nu \cap \partial\mathcal{O}_k \neq \varnothing$ for all $k \geqslant 1$ and $\nu = +$ or $-$.*

Proof. Assume $\mathcal{S}_k^\nu \cap \partial\mathcal{O}_k = \varnothing$. Let $\epsilon'(\lambda) > 0$ be as in Theorem 3.4. Then $\lambda \neq k$, $u \in \mathcal{B}_k(\epsilon'(\lambda)) \Rightarrow u \notin \mathcal{S}_k(\lambda) \cap \mathcal{N}_k^\nu$, $\nu = +$ or $-$. Set $U_\lambda{}^k = \{u \in B_k : (u, \lambda) \in \mathcal{O}_k\}$, an open set in B_k as is $U_\lambda - \overline{\mathcal{B}_k(\epsilon'(\lambda))}$. By Theorem 3.3 $u \in \mathcal{S}_k^\nu$ implies $u \notin \partial\mathcal{B}_k(\epsilon'(\lambda))$, $\lambda \neq k$. Since also $u \notin \partial\mathcal{O}_k$ by assumption we have $u \notin \partial(U_\lambda{}^k - \overline{\mathcal{B}_k(\epsilon'(\lambda))})$ and, hence,

$$d(\Phi(\lambda), [U_\lambda{}^k - \overline{\mathcal{B}_k(\epsilon'(\lambda))}] \cap \mathcal{N}_k^\nu, 0)$$

is well defined. From now on 0 is dropped from this notation. The proof is given for the case $\operatorname{sgn} \lambda_{2m,k} = -1$ (i.e., of bifurcation to the left, the case $\operatorname{sgn} \lambda_{2m,k} = +1$ being similar and, hence, omitted) and is divided into two parts: (i) we first show that this degree is 0 for all $\lambda \neq k$ and secondly that (ii) this is in fact impossible for all $\lambda \neq k$ close to k. This contradiction will establish the theorem.

(i) Let $\lambda > k$ be such that $U_\lambda{}^k \neq \varnothing$. Since C_k^ν is a compact subset of $\mathcal{O}_k$ (by Lemma 4.1), we can find a $\lambda^* > \lambda$ such that $U_{\lambda^*}^k \neq \varnothing$ and $U_{\lambda^*}^k \cap C_k^\nu = \varnothing$. Consider

$$\epsilon^* = \tfrac{1}{2} \inf_{\alpha \in [\lambda, \lambda^*]} \epsilon'(\alpha).$$

If $\epsilon^* = 0$, then $\epsilon(\alpha_n) \to 0$ for $\alpha_n \to$ some $\alpha \in [\lambda, \lambda^*]$ and, hence, by (3.5) $\rho_k(\alpha_n') \to 0$ for a sequence of $\alpha_n' \in [\lambda - \delta(\lambda), \lambda^* + \delta(\lambda^*)]$ which implies, by the lower semicontinuity of ρ_k that ρ_k vanishes at some point in this interval contrary to Lemma 3.2. Thus $\epsilon^* > 0$. Since $\epsilon^* < \epsilon'(\alpha)$, $\alpha \in [\lambda, \lambda^*]$, we know that no solution u lies on $\partial([U_\lambda{}^k - \overline{\mathcal{B}_k(\epsilon^*)}] \cap \mathcal{N}_k^\nu)$ and, hence, by the homotopy property P3,

$$d(\Phi(\alpha), [U_\alpha{}^k - \overline{\mathcal{B}_k(\epsilon^*)}] \cap \mathcal{N}_k^\nu) = c = \text{constant} \tag{4.3}$$

for all $\alpha \in [\lambda, \lambda^*]$ and, in particular, for $\alpha = \lambda$. Since there is no solution in $[U_{\lambda^*}^k - \overline{\mathcal{B}_k(\epsilon^*)}] \cap \mathcal{N}_k^\nu$ nor in $[\mathcal{B}_k(\epsilon'(\lambda) - \overline{\mathcal{B}_k(\epsilon^*)}] \cap \mathcal{N}_k^\nu$ we have by P1 that $c = 0$ and

$$d(\Phi(\lambda), [\mathcal{B}_k(\epsilon'(\lambda)) - \overline{\mathcal{B}_k(\epsilon^*)}] \cap \mathcal{N}_k^\nu) = 0,$$

which together with (4.3) at $\alpha = \lambda$ and the additivity property of degree P2 implies

$$d(\Phi(\lambda), [U_\lambda{}^k - \overline{\mathcal{B}_k(\epsilon'(\lambda))}] \cap \mathcal{N}_k^\nu) = 0. \tag{4.4}$$

A similar proof holds for $\lambda < k$ and, hence (4.4) is valid for all $\lambda \neq k$. Note that $\operatorname{sgn} \lambda_{2m,k}$ does not appear in this argument and, hence, (4.4) holds regardless of the direction of the bifurcation at k.

(ii) Assume now that $\operatorname{sgn} \lambda_{2m,k} = -1$ and let $\delta(k) > 0$ be as in Theorem 3.3. By Theorem 3.4 if $\lambda < k$ is such that $|\lambda - k| < \delta(k)$, then

$$d(\Phi(\lambda), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\lambda))}] \cap \mathscr{N}_k^{\nu}) = -1.$$

It then follows from (4.4) and P2 that

$$d(\Phi(\lambda), [U_\lambda^k - \overline{\mathscr{B}_k(\epsilon(k))}] \cap \mathscr{N}_k^{\nu}) = 1. \tag{4.5}$$

We now wish to make a homotopic argument to assert this equality for $\lambda > k$. To do this we note that since no solution to $\Phi(\mu) = 0$ lies on $\partial\mathscr{B}_k^{\nu}(\epsilon(k))$ nor $\partial\mathscr{N}_k^{\nu}$ for all $\mu \in [\lambda, 2k - \lambda]$ and, hence, by P3 Eq. (4.5) is valid for $\bar{\lambda} = 2k - \lambda$. We have then from (4.4) at $\bar{\lambda}$

$$\begin{aligned} 0 &= d(\Phi(\bar{\lambda}), [U_{\bar{\lambda}}^k - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k^{\nu}) \\ &= d(\Phi(\bar{\lambda}), [U_{\bar{\lambda}}^k - \overline{\mathscr{B}_k(\epsilon(k))}] \cap \mathscr{N}_k^{\nu}) \\ &\quad + d(\Phi(\bar{\lambda}), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k^{\nu}), \end{aligned}$$

together with which (4.5) at $\bar{\lambda}$ implies

$$d(\Phi(\bar{\lambda}), [\mathscr{B}_k(\epsilon(k)) - \overline{\mathscr{B}_k(\epsilon'(\bar{\lambda}))}] \cap \mathscr{N}_k^{\nu}) = -1,$$

and inasmuch as $\bar{\lambda} \in (k, k + \delta(k))$ we have, by P1, a contradiction to the last statement of Theorem 3.1, i.e., that the bifurcation at k is to the left when $\operatorname{sgn} \lambda_{2m,k} = -1$. We have thus reached a contradiction which invalidates our original assumption that $\mathscr{S}_k^{\nu} \cap \partial\mathscr{O}_k = \varnothing$. The proof is complete.

We may deduce some corollaries from Theorem 4.1 concerning the topology of the solution branches. The proofs are exactly as in [17] but are sufficiently short for repetition here.

Corollary 4.1. *Under the hypotheses of Theorem* 4.1 *there exists a continuum of solutions* (u, λ) *to problem* N *connecting* $(0, k)$ *to* $\partial\mathscr{O}_k$ *in* $\mathscr{N}_k^{\nu} \times \mathscr{R}$, $\nu = +$ *or* $-$.

Proof. By Lemma 4.1, C_k^{ν} is a compact metric space under the topology induced from B_k. If a continuum of solutions does not exist in $\mathscr{S}_k^{\nu}$ then the sets $A = \{(0, k)\}$, $B = \partial\mathscr{O}_k \cap C_k^{\nu}$ are disjoint-closed subsets of C_k^{ν} and, hence, there exist disjoint-compact subsets K_A, K_B of C_k^{ν} such that $C_k^{\nu} = K_A \cup K_B$,

$A \subset K_A$, $B \subset K_B$ [21]. Since K_A is a compact subset of $\bar{\mathscr{C}}_k$ we can find an open set Ω, $\bar{\Omega} \subset \mathscr{C}_k$ such that $K_A \subset \Omega$ and $K_B \cap \bar{\Omega} = \emptyset$. By Theorem 4.1 there exists a solution $(u, \lambda) \in \mathscr{S}_k^\nu$ on $\partial\Omega$ which then is in $\mathscr{C}_k$ but not $K_A \cup K_B$, a contradiction.

COROLLARY 4.2. *Under the hypotheses of Theorem* 4.1 *there exists for all* $k \geqslant 1$ *a continuum of solutions to problem* N *connecting* $(0, k)$ *to* ∞ *in* $\mathscr{N}_k^\nu \times \mathscr{R}$, $\nu = +$ *or* $-$.

Proof. The component of $\mathscr{S}_k^\nu$ containing $(0, k)$, since it is locally compact, becomes a compact topological space under the one point compactification of adding ∞. By Corollary 4.2 every neighborhood of ∞ intersects this component.

5. THE SPECTRUM

The interesting question still remains: how does $(u, \lambda) \in \mathscr{S}_k^\nu$ go to ∞? Obviously at least one of the components $\| u \|$ or λ must tend to ∞, and we should expect that some knowledge of $f(u, \theta)$ will yield theorems along these lines. We will not study this question in depth, but will offer only some results which follow easily from the nature of the problem and/or our results above.

Clearly if $(u_n, \lambda_n) \to (u, \lambda)$ in B_k then $u_n \to u$ under $\| \cdot \|_{0,\bar{D}}$. Conversely, if $(u_n, \lambda_n) \to (u, \lambda)$ under $\| \cdot \|_{0,\bar{D}}$, then $\| u_n \|_{0,\bar{D}} \leqslant K_1$, $| \lambda_n | \leqslant K_1$, for some $K_1 = \text{const}$, independent of n and on such a bounded set the hypotheses of f guarantee that f is bounded and satisfies a Lipschitz condition in u uniformly in θ. Thus, from (2.3) we have [if u, u_n also satisfy $\Phi(u, \lambda) = 0$]

$$\begin{aligned} \| u_n - u \| &\leqslant K \| \lambda_n f(u_n, \theta) - \lambda f(u, \theta) \|_{0,\partial D} \\ &\leqslant K K_1 L \| u_n - u \|_{0,\bar{D}} + K K_2 | \lambda_n - \lambda |, \end{aligned}$$

where L is the Lipschitz constant of f depending only on K_1 and

$$\| f(u_n, \theta) \|_{0,\partial D} \leqslant K_2$$

and, hence, $(u_n, \lambda_n) \to (u, \lambda)$ in B_k.

LEMMA 5.1. *The results of Corollary* 4.1, 4.2 *are valid when* B_k *is endowed with the norm* $\| \cdot \|_{0,\bar{D}}$ *instead of* $\| \cdot \|$.

This lemma follows directly from the definition of continuum [21] and the preceding remarks.

If u_0 is a constant for which $f(u_0, \theta) \equiv 0$, $\theta \in [-\pi, \pi]$, then we call u_0 a *u-zero of f*; by (H2) $u_0 = 0$ is a u-zero of f. Note that if u_0 is a u-zero of f then so is $-u_0$ by (H3). Set $\Lambda_k^\nu = \{\lambda : (u, \lambda) \in K_k^\nu\}$ where K_k^ν is a continuum of solutions joining $(0, k)$ to ∞ in $\mathscr{N}_k^\nu \times \mathscr{R}$; Λ_k^ν is thus the set of eigenvalues for $\Phi(u, \lambda) = 0$ corresponding to solutions lying on branches from $(0, k)$. If $\lambda = 0$ then the only solution to problem N is $u \equiv$ const.; thus, if $u \in B_k$, $u \equiv 0$ and, hence, $\lambda = 0 \notin \Lambda_k^\nu$ ($0 \notin \mathscr{N}_k^\nu$ by definition). As a result $\Lambda_k^\nu \subseteq \mathscr{R}^+$ for all k and ν since Λ_k^ν is a continuum.

THEOREM 5.1. *If f satisfies* (H1)–(H4)$_k$ *and f has a non-zero u-zero, then Λ_k^ν is an unbounded interval in $\mathscr{R}^+$. Moreover, the solutions u on K_k are uniformly bounded in the $\| \cdot \|_{0,\bar{D}}$ norm by the smallest positive u-zero of f.*

Proof. Suppose Λ_k^ν were bounded. Then we know that $\{\| u \| : (u, \lambda) \in K_k^\nu$ for some $\lambda \in \Lambda_k^\nu\}$ is unbounded and, hence, by Lemma 5.1 there exists a u for which $\| u \|_{0,\bar{D}} = u^0$, where u_0 is a positive u-zero of f. Let $\theta_0 \in [\pi, \pi]$ be such that $u(1, \theta_0) = u_0$; then by (2.1) we have $\partial u/\partial r = 0$ at θ_0 where u attains its maximum. This contradicts a well-known theorem [16] which asserts $\partial u/\partial r > 0$ at θ_0; Λ_k^ν is accordingly unbounded. This last remark, of course, remains valid whenever $\| u \|_{0,\bar{D}} = u_0$ for a solution on K_k^ν and since K_k^ν is a continuum the proof is complete.

The following lemma is a direct consequence of Corollary 4.2.

LEMMA 5.2. *If (λ, u) is a solution to problem* N *satisfying an a priori estimate of the form $\| u \| \leqslant M(\lambda)$ where $M(\lambda)$ is a nonnegative, real-valued function defined (and finite) for $\lambda \in \mathscr{R}^+$ which is bounded on finite intervals then $\Lambda_k^\nu \subseteq \mathscr{R}^+$ is an unbounded interval.*

THEOREM 5.2. *Let f satisfy* (H1)–(H4)$_k$. *If there exist a constant $M > 0$ such that*

$$|f(\xi, \theta)| \leqslant M |\xi|^p \quad \textit{for } (\xi, \theta) \in (-\infty, +\infty) \times [-\pi, \pi] \textit{ and some } p \in [0, 1), \tag{5.1}$$

then $\Lambda_k^\nu \subseteq \mathscr{R}^+$ is an unbounded interval for all k, ν.

Using (2.3) and (5.2) we have

$$\| u \| \leqslant K |\lambda| \, \|f(u, \theta)\|_{0,\partial D} \leqslant KM\lambda \| u \|^p$$

and, consequently, we may set $M(\lambda) = K_2 \lambda^{1/(1-p)}$, $K_2 = (KM)^{1/(1-p)}$, in Lemma 5.2.

Whenever Λ_k^ν is an unbounded interval for all $k \geqslant 1$ then for $\lambda \in (k, k+1]$ there exist solutions in each set $\mathscr{N}_l^\nu$, $1 \leqslant l \leqslant k$, $\nu = +$ or $-$. Thus, from Theorems 5.1, 5.2 we have

COROLLARY 5.1. *If the hypotheses of Theorems 5.1 or 5.2 are valid, then for each $\lambda \in (k, k+1]$, where k is an arbitrary positive integer, there exists at least $2k$ nontrivial solutions to problem N in B_1, one in each of $\mathcal{N}_l^{\nu}$, $1 \leq l \leq k$, $\nu = +$ and $-$. The solution in $\mathcal{N}_l^{-}$ is the negative of the one in $\mathcal{N}_k^{+}$.*

Note that $B_k \subset B_1$, $k > 1$. This corollary is valid whenever an *a priori* estimate of the type appearing in Lemma 5.2 holds for problem N.

We finish this section with a theorem which gives conditions which insure that the *entire* spectrum of $\Phi(u, \lambda) = 0$ is positive. Applying Green's first identity to any nonconstant solution (u, λ) we have

$$\lambda \int_{-\pi}^{\pi} uf(u, \theta)\, d\theta = \int_{-\pi}^{\pi} u \frac{\partial u}{\partial r}\, d\theta = \iint_D \sum_{i=1,2} (D_i u)^2\, dx > 0. \quad (5.2)$$

THEOREM 5.3. *If f is continuous in its arguments and $\xi f(\xi, \theta) \geq 0$ for $(\xi, \theta) \in (-\infty, +\infty) \times [-\pi, \pi]$ then no nonconstant solution to problem N exists in $C^1(\bar{D})$ for $\lambda \leq 0$.*

6. AN EXAMPLE

The exact mathematical theory of steady, progressing surface waves on an incompressible fluid of infinite depth was reduced by Levi-Civita [12] to the problem of finding a harmonic function on D subject to the condition $\partial u/\partial r = \lambda e^{-3v} \sin u$ for $r = 1$ where v is the harmonic conjugate of u vanishing at the origin and $\lambda = gl/2\pi c^2$; here l is the wavelength and c the velocity of the waveform, g is the gravitational constant, and u is the angle of the velocity vector as measured from the horizontal. As an approximation to this problem one could consider $f \equiv \sin u$ in (2.1). The function $\sin u$ satisfies the hypotheses of all of our theorems [$(\text{H4})_k$ is satisfied for all k] and, consequently, we conclude that for each $k \in Z^+$ there exist two contina of solutions K_k^{ν} differing only in sign, connecting $(0, k)$ to ∞ with the nodal structure of $r^k \sin k\theta$; that $\Lambda_k^{\nu} \subset \mathscr{R}^+$ is an unbounded interval; and that for any solution (u, λ) on any one of these branches $\| u \|_{0,\bar{D}} < \pi$, since $u_0 = \pi$ is the smallest positive u-zero of $\sin u$. By Theorem 3.1 the bifurcation of K_k^{ν} at $(0, k)$ is to the right, since $\lambda_{2,k} < 0$ (cf. [6]). More, however, can be shown for this special problem. If $(u(r, \theta), \lambda) \in K_1^{\nu}$ then it is easy to show that $(u_n, n\lambda) \in K_n^{\nu}$ where $u_n \equiv u(r^n, n\theta)$. (This is in fact true in general for problem N if f is independent of θ.) Furthermore, by the uniqueness result in [4] together with $\| u \|_{0,\bar{D}} < \pi$ it follows that for a given λ at most one solution exists with a given nodal structure; in particular, then, the branches from $(0, k)$ are single-valued functions of λ. Furthermore, from the same paper it follows that $(u_1, \lambda) \in K_1^{\nu}$, $(u_k, \lambda) \in K_k^{\nu}$ implies

$$\| u_1 \|_{0,\bar{D}} > \| u_k \|_{0,\bar{D}}, \qquad k > 1.$$

These results have some interesting features. First of all, solutions on K_k^ν, being constructed as above, give rise physically to identically the same waves as u_1 (see [5] and [12]), with, for fixed propagation velocity c, the wavelength taken as kl, l being the *smallest* wavelength. Thus, for this *approximate* nonlinear theory we have proved *globally* the long standing conjecture of Levi-Civita that all possible waves generated by solutions on K_k^ν are generated by solutions lying on the first branch K_1^+ alone; to this date this had been shown only locally [5, 11]. It must be pointed out, however, that we have not ruled out the existence of solutions not on any branch K_k^ν and/or solutions with nodal structure different from that of all $r^k \sin k\theta$ (cf. [5, 11] where such waves are also ruled out).

Secondly, this nonlinear problem apparently does not adequately approximate locally the full nonlinear problem of Levi-Civita since the bifurcation at (0, 1) is to the right while for the full nonlinear problem solutions exist for $\lambda < 1$ [12].

Finally, it might appear that since $\Lambda_1^+ = (1, +\infty)$ we have a contradiction to the result in [11] which gives $\lambda \in [a, b]$, a, b finite. The theory in [11], however, deals with solutions $\| u \|_{0,\bar{D}} < \pi/6$ and this then is not necessarily incompatible with our results.

Appendix

For a complete treatment of Leray-Schauder degree theory see [10].

Let B be a Banach space, Ω a bounded open subset of B, and $\Phi(u)$ a completely continuous map of $\bar{\Omega} \to B$. Let $b \in B$, $b \notin \Phi(\partial\Omega)$. The degree of the map is an integer $d(\Phi, \Omega, b)$ which has the following properties:

P1: $d(\Phi, \Omega, b) = 0$ *if* $b \notin \Phi(\Omega)$.

P2: *If* $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \varnothing$, *where* Ω_i *are bounded open subsets of* B *and if* $b \notin \Phi(\partial\Omega_i)$, $i = 1, 2$, *then* $d(\Phi, \Omega, b) = d(\Phi, \Omega_1, b) + d(\Phi, \Omega_2, b)$.

P3: (Homotopy Invariance) *Let* $[a, b]$ *be a finite interval in* $\mathscr{R}$ *and* U *be a bounded open set in* $B \times [a, b]$ *under the product topology. Set* $U_\lambda = \{u \in B : (u, \lambda) \in U\}$. *If* $\Phi(u, \lambda)$ *is a completely continuous mapping of* $U \to B$ *and* $b \in B$, $b \notin \Phi(\partial U_\lambda, \lambda)$ *for all* $\lambda \in [a, b]$, *then* $d(\Phi(u, \lambda), U_\lambda, b)$ *is constant for* $\lambda \in [a, b]$.

By definition, $d(\Phi, \varnothing, b) = 0$.

If u is an isolated solution of $\Phi(u) = b$ and $\mathscr{B}(\epsilon)$ is a ball centered at u of radius ϵ, then $\lim_{\epsilon\to 0} d(\Phi, \mathscr{B}(\epsilon), b) = i(\Phi, u, b)$ is well defined and called the index of u.

P4: *Let* $\Phi(u) = u - N(u)$ *be a completely continuous mapping of* $B \to B$ *and* $\Phi(u_0) = 0$. *Suppose one is not a characteristic value of the linear operator* L *which is the Fréchet derivative of* N *at* u_0. *Then* u_0 *is an isolated solution of* $\Phi(u) = 0$ *and* $i(\Phi, u_0, 0) = (-1)^m$ *where* m *is the sum of the orders of the characteristic values of* L *lying in* $(0, 1)$.

P5: *If* Ω_0 *is an open subset of* B, $\Phi^{-1}(b) \cap \Omega_0 \subset \Omega$, *then* $d(\Phi, \Omega, b) = d(\Phi, \Omega_0, b)$, *provided* $b \notin \Phi(\partial\Omega), \Phi(\partial\Omega_0)$.

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