NONTRIVIAL PERIODIC SOLUTIONS OF SOME
VOLTERRA INTEGRAL EQUATIONS

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1. Introductory Remarks. My main purpose in this paper is to prove a bifurcation theorem for nontrivial periodic solutions of a general system of Volterra integral equations. The motivation for considering this problem can be found in models which arise in population dynamics, epidemiology and economics [1,3,6], an example of which appears in §5. The approach taken is that which is usually referred to as the method of Liapunov-Schmidt (or often called the method of alternative problems). This method, which is applicable in a very general setting, is outlined in §2 in a way suitable for the type of problems I have in mind. The fundamental ingredient for this approach in its application to many problems is a Fredholm alternative. A Fredholm alternative for systems of Volterra integral equations is proved in §3. The main bifurcation result (Theorem 4) appears in §4 and an application is given in §5 to a scalar model which has arisen in the mathematical theory of population growth and of epidemics.

2. Some General Remarks. Let X and Y be real normed linear spaces and suppose $L:X\to Y$ is a bounded linear operator with range $R(L)$ and nullspace $N(L)$. The following assumption will in force throughout.

H1: $R(L)$ and $N(L)$ are both closed and admit bounded projections.

Let $P:Y\to R(L)$ be a bounded projection as guaranteed by H1. It follows from H1 that $X = N(L)\oplus M$ where M is closed and that L has a bounded right inverse $A:R(L)\to M$, $LA = I$. Let $B(X,r)$ denote the open ball of radius $r>0$ centered at zero in a normed space X. We consider the operator equation

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$$(1) \qquad Lx = T(x,\lambda) \quad \text{for} \quad (x,\lambda) \in B(X\times\Lambda, r)$$

where Λ is a real Banach space and $T: B(X\times\Lambda, r) \to Y$ is a continuous operator about which more is assumed below. It is easy to see that (x,λ) solves (1) if and only if

$$(2) \qquad z - APT(y+z,\lambda) = 0$$
$$(3) \qquad (I-P)T(y+z,\lambda) = 0$$

where $x = y+z$, $y \in N(L)$, $z \in M$.

The problem of interest here is the existence of nontrivial solutions near $x = 0$ (i.e. small amplitude solutions) under the assumption that $(x,\lambda) = (0,\lambda)$ is a solution for all small $\lambda \in \Lambda$ and thus it is natural to set $x = \varepsilon(y+z)$ for a small real parameter ε in (1) and hence in (2)-(3). Assume

<u>H2</u>: $$N(L) \neq \{0\}.$$

Let E^n denote real Euclidean n-space and $|\cdot|_X$ denote the norm in X. The following will be assumed about the operator T.

<u>H3</u>: $\begin{cases} T(\varepsilon x,\lambda) = \varepsilon\overline{T}(x,\lambda,\varepsilon) \quad \text{for} \quad 0 \le |\varepsilon| < \varepsilon_1 \le +\infty \text{ and } x \in B(X,r) \\ \text{where } \overline{T}: B(X,r)\times B(\Lambda\times E^1, r) \to Y \text{ is } q \ge 1 \text{ times continuously} \\ \text{Fréchet differentiable in } (x,\lambda,\varepsilon). \quad \text{Furthermore, for some} \\ 0 \neq y \in N(L),\ |y|_X < r, \text{ suppose } \overline{T}(y,0,0) = \overline{T}_x(y,0,0) = 0. \end{cases}$

Note that <u>H3</u> implies that $(x,\lambda) = (0,\lambda)$ solves (1) for all small ε. Letting $x = \varepsilon(y+z)$ in (2)-(3) one gets, after cancellation of an ε, the following equations

$$(4) \qquad z - AP\overline{T}(y+z,\lambda,\varepsilon) = 0$$
$$(5) \qquad (I-P)\overline{T}(y+z,\lambda,\varepsilon) = 0$$

which are to be solved for $(z,\lambda) \in M\times\Lambda$ as functions of ε (where y is fixed as in <u>H3</u>.) An existence theorem for (4)-(5) could now easily be stated by means of a straightforward application of the implicit function theorem, provided the Fréchet derivative of the left hand side of (5) with respect to λ at $(z,\lambda,\varepsilon) = (0,0,0)$ is a homeomorphism. This can be seen by applying the implicit function theorem to (4) to obtain a solution $z = z(\lambda,\varepsilon)$, $z(0,0) = 0$, which when substituted into (5) yields an equation solvable for $\lambda = \lambda(\varepsilon)$ again by the implicit function theorem. (Or alternatively one can simply apply the implicit function theorem to the operator defined by the left hand sides of (4) and (5).)

If $\dim \Lambda$ and $\operatorname{codim} R(L)$ are finite, then (5) becomes a real algebraic problem of a finite number of real equations for a finite number of real unknowns. Specifically, if

<u>H4</u>: $$m = \operatorname{codim} R(L) < +\infty$$

<u>H5</u>: $$\Lambda = E^m$$

then (5) reduces to m real equations in m real unknowns and states that the components $c = c(z,\lambda,\varepsilon) \in E^m$ of the left hand side of (5) (with respect to any finite set of elements which span a complement of $R(L)$) vanish. The requirement that the Fréchet derivative of the left hand side of (5) with respect to λ at $(z,\lambda,\varepsilon) = (0,0,0)$ be a homeomorphism then becomes the nondegeneracy condition that (note that $c_z(0,0,0) = 0$ by <u>H3</u>)

<u>H6</u>: $$D := \det c_\lambda(0,0,0) \neq 0 \ .$$

The real matrix $c_\lambda(0,0,0)$ is $m \times m$. Under these conditions one gets the following result from the implicit function theorem.

THEOREM 1. *Assume that all of the hypotheses H1 through H6 hold. For some $\varepsilon_0 > 0$, $\varepsilon_0 \leq \varepsilon_1$, there exist operators $z: B(E^1,\varepsilon_0) \to M$, $\lambda: B(E^1,\varepsilon_0) \to E^m$ which are $q \geq 1$ times continuously differentiable and are such that $(x,\lambda) = (\varepsilon y + \varepsilon z(\varepsilon), \lambda(\varepsilon))$ solves (1) for all $|\varepsilon| \leq \varepsilon_0$ where $(z(0),\lambda(o)) = (0,0)$.*

Note that the solutions in Theorem 1 are nontrivial since $x \neq 0$ for $\varepsilon \neq 0$ (since $y \neq 0$ in <u>H3</u>). Theorem 1 is a bifurcation result since the two solution branches $(0,\lambda)$ and $(x(\varepsilon),\lambda(\varepsilon))$ intersect at $(0,0)$. (Moreover, it can be shown that all solutions near $(0,0)$ have the form in Theorem 1 so that the implicit function theorem implies that these nontrivial solutions are unique.)

Before proceding to some applications to Volterra integral equations it may be worthwhile to discuss briefly the hypotheses <u>H1-H6</u>. First of all, <u>H1</u> and <u>H4</u> are motivated by (and are really abstract properties of) so-called Fredholm alternatives for the linear equation $Lx = f$, $f \in Y$. Thus, in applications, the establishment of a Fredholm alternative on suitable spaces would be the preliminary step in making use of Theorem 1. This, of course, is not necessarily easy, but has been done for many types of operators L on certain Banach spaces. For example, for spaces of periodic functions the Fredholm alternative is well known for ordinary differential systems and has been proved for more general Stieltjes-integrodifferential systems [2]. Examples for partial differential equations can be found in [5]. A Fredholm alternative for Volterra integral systems is proved here in §3.

Secondly, H2 is the familiar requirement in bifurcation theory that the linearization at the critical value $\lambda = 0$ be singular.

If Λ is viewed as a finite space of real parameters in equation (1) then the condition H5 requires that the problem have "enough" parameters, namely $m = \operatorname{codim} R(L)$. In many applications (for example, those concerning periodic solutions of autonomous systems) $m = 2$ in which case H5 requires that two parameters be available in the equation. Either parameters which explicitly appear can be used (in most applications there are usually more than enough parameters appearing explicitly) or parameters can be introduced by means of rescalings of independent variables (such as the unknown period in Hopf bifurcation, e.g. see [4,5] and §5 below).

Finally, H6 is a technical, but necessary assumption involving T and a knowledge of $R(L)$. It is necessary in the sense that it is well-known in bifurcation theory that bifurcation does not necessarily occur at the ciritcal value of the linearization and that some further nondegeneracy condition (such as H6) is required to insure that bifurcation take place. In applications, the description of $R(L)$ (and hence c in H6) usually involves an adjoint operator and its null-space. Note that higher order terms in x which appear in T do not contribute to D.

Hypothesis H3 on T is rather routine to check in applications and in fact is usually verified by simple inspection. Besides a minimal amount of smoothness it requires that T be, roughly speaking, higher order in x and λ.

Once a Fredholm alternative has been established for the linear operator L the key hypotheses (the ones requiring the most analysis in applications) in Theorem 1 are H2 and H6.

3. Linear Volterra Operators. Assume that

A1: $\begin{cases} k(t) \text{ is a real } n\times n \text{ matrix valued function defined} \\ \text{and piecewise continuous on an interval } 0 \le t \le a,\ 0 < a < +\infty . \end{cases}$

Then the operator

$$(6) \qquad Ly := y(t) - \int_{t-a}^{t} k(t-s)y(s)\,ds \quad , \quad 0 < a < +\infty$$

is a bounded linear operator from $X(p)$ into itself where $X(p)$ is the Banach space of real continuous, n-vector valued p-periodic functions under the supremum norm $|y|_p = \sup_{0\le t\le p}|y(t)|$. The integral appearing in L is continuous in t for any p-periodic function $y \in L^1[0,a]$.

Any function $y \in X(p)$ is square integrable on $0 \le t \le p$ so that it is associated with a unique Fourier series $\sum_{-\infty}^{+\infty} a_j \exp(ijwt)$, $w = 2\pi/p$ where $a_{-j} = a_j'$, $j \ge 0$ ($'$ denotes complex conjugation) and $\sum_{-\infty}^{+\infty} |a_j|^2 < +\infty$. The Fourier coefficients of Ly are

$$(7) \qquad (I_n - k_j(p))a_j \quad \text{where} \quad k_j(p) := \int_0^a k(s)\exp(-ijws)\,ds \quad .$$

Here I_n is the $n\times n$ identity matrix. If $f \in R(L)$ then $Ly = f$ for some $y \in X(p)$ which implies

$$(A_j) \qquad\qquad (I_n - k_j(p))a_j = f_j \quad , \quad j \ge 0$$

(where the f_j are the Fourier coefficients of f) are solvable for a_j for all $j \ge 0$. The following lemma establishes the converse.

LEMMA 1. $f \in R(L)$ *if and only if* (A_j) *is solvable for* a_j *for all* $j \ge 0$, *in which case the* a_j (*with* $a_{-j} = a_j'$ *for* $j > 0$) *are the Fourier coefficients of a* $y \in X(p)$ *such that* $Ly = f$ *for all* t .

Proof. It is only necessary to prove the converse. Let a_j solve (A_j) and define $a_{-j} = a_j'$, $j > 0$. A (Stieltjes) integration of $k_j(p)$ by parts shows that

$$|k_j(p)| \le C/j \ , \ j > 0$$

for some constant $C > 0$ independent of j (but depending on p). Here $|k_j|$ means any matrix norm, say the largest absolute value of the entries in k_j. This means

(8) $$(I_n - k_j(p))^{-1} \text{ exists for } j \ge j_o \ge 0$$

for some $j_o \ge 0$ and is bounded uniformly in j so that

$$|a_j| \le \tilde{C}|f_j| \quad , \quad j \ge j_o \ .$$

From this and the fact that $f \in X(p)$ implies the square summability of the f_j follows the square summability of the a_j. Thus these a_j determine a p-periodic function $y \in L^2[0,p]$ (by the Riesz-Fischer theorem). The Fourier coefficients of Ly are easily seen to be given by (7) and consequently, by the definition of a_j as a solution of (A_j), the functions Ly and f have identical Fourier coefficients and must be equal almost everywhere. However, the integral in L is continuous (as is f) so we conclude that y is an L^2 function which is equal almost everywhere to a continuous function. The function y can then be redefined on a set of measure zero so as to be continuous and hence lie in $X(p)$. Since this redefinition does not change the integral in L one obtains a function $y \in X(p)$ such that $Ly = f$ everywhere. §§

The operator

$$L_A y := y(t) - \int_t^{t+a} k^T(s-t)y(s)\, ds$$

will be called the <u>adjoint</u> of L. Lemma 1 applies with L replaced by L_A and (A_j) replaced by

(A_j^T) $$(I_n - k_j^T(p))a_j = f_j \quad , \quad j \ge 0$$

where k_j^T is the conjugate transpose of k_j. Thus

$$k_j^T(p) := \int_0^a k^T(s)\exp(ijws)\, ds \quad , \quad w = 2\pi/p \; .$$

Since the coefficient matrix in (A_j^T) is the conjugate transpose of that in (A_j) one obtains the following result from (8) and Lemma 1 with $f_j = 0$.

THEOREM 2. _The nullspaces of both_ L _and_ L_A _in_ $X(p)$ _are finite dimensional and_ $m = \dim N(L) = \dim N(L_A) < +\infty$.

The number m can be computed by finding the nullity ν_j of each singular matrix $I_n - k_j(p)$ (by (8) there are at most a finite set J of j's for which this matrix is singular). Each independent complex vector a_j in the nullspace of this matrix gives rise to two independent real solutions of $Ly = 0$ if $j > 0$ (namely the real and imaginary parts of $a_j\exp(ijwt)$) and one if $j = 0$ (namely a_o itself). Thus, $m = \nu_o + 2\sum_{0 \neq j \in J} \nu_j$.

Define

$$(x,y) := p^{-1}\int_0^p x(t)\cdot y(t)\, dt$$

for $x,y \in X(p)$. The next theorem describes the range of L and the complements of the range and nullspace of L.

THEOREM 3. $X(p) = N(L) \oplus M$ _and_ $X(p) = R(L) \oplus N(L_A)$ _where_

$$M = \{x \in X(p):\ (x,y) = 0 \ \text{for all} \ y \in N(L)\}$$

$$R(L) = \{x \in X(p):\ (x,y) = 0 \ \text{for all} \ y \in N(L_A)\}$$

are closed.

Proof. That M and $R(L)$ are closed is obvious, as is the decomposition $X(p) = N(L) \oplus M$. It remains only to establish that $R(L)$ is given as described in the theorem. By Lemma 1 and well-known algebraic facts, $f \in R(L)$ if and only if the Fourier coefficients f_j are all orthogonal to the nullspaces of $I_n - k_j^T(p)$ which is the same as to say if and only if f is orthogonal to solutions of $L_A y = 0$ (by Lemma 1 for L_A and by (A_j^T) with all $f_j = 0$). §§

It follows from Theorem 3 that $R(L)$ and $N(L)$ admit continuous projections. Thus, with regard to the hypotheses of the previous section §2, this theorem yields the following result.

COROLLARY 1. The linear operator $L: X(p) \to X(p)$ defined by (6) satisfies H1 and H4 when $k(t)$ satisfies A1 .

4. A Bifurcation Theorem for Volterra Integral Equations. Consider the systems

$$x(t) - \int_{t-a}^{t} k(t-s)x(s)ds = T(x,\lambda) \quad , \lambda \in E^m , \; 0 < a < +\infty \tag{9}$$

$$y(t) - \int_{t-a}^{t} k(t-s)y(s)ds = 0 \tag{10}$$

where $k(t)$ satisfies A1 and T satisfies H3 with $X = Y = X(p)$ for some $p > 0$ and $\Lambda = E^m$. In order to apply Theorem 1 it follows from Corollary 1 that one needs only H2, H5 and H6. Hypotheses H2 and H5 are equivalent to the following assumption.

A2: For some period $p > 0$ the linear homogeneous system (10) has exactly $m > 0$ independent, nontrivial p-periodic solutions $y_j \in X(p)$, $1 \leq j \leq m$.

By Theorem 2 the adjoint equation

$$y(t) - \int_{t}^{t+a} k^T(s-t)y(s)ds = 0$$

also has exactly m independent solutions $y_j^A \in X(p)$ which span $N(L_A)$. Assume without loss in generality that

$$(y_j^A, y_k^A) = \delta_{jk} .$$

(Such adjoint solutions can be found by choosing independent solution vectors of (A_j^T) with $f_j = 0$ to be orthonormal.) The components of the projection $I-P$ of any function $x \in X(p)$ onto $N(L_A)$ are then (x, y_j^A) so that c in H6 is given by

$$c(z,\lambda,\varepsilon) = \mathrm{col}((\bar{T}(y+z,\lambda,\varepsilon),y_j^A)) .$$

Thus <u>H6</u> becomes

$$\underline{A3}: \begin{cases} D \neq 0 \quad \text{for some solution } y \in X(p),\ y \neq 0,\ |y|_p < r \text{ of } (10) \\ \text{where} \\ \qquad D := \det((\bar{T}_i(y,0,0),y_j^A)) \ ,\ \bar{T}_i = (\partial/\partial\lambda_i)\bar{T} \ ,\quad \lambda = \mathrm{col}(\lambda_i) \\ \text{and} \quad 1 \leq i,j \leq m. \end{cases}$$

Theorem 1 of §1 now gives the following result.

THEOREM 4. <u>Assume</u> <u>A1</u>, <u>A2</u> <u>and</u> <u>A3</u> <u>hold and that</u> T <u>satisfies</u> <u>H3</u> <u>with</u> $X = Y = X(p)$ <u>and</u> $\Lambda = E^m$. <u>Then there is an</u> $\varepsilon_0 > 0$ <u>such that</u> (9) <u>has nontrivial p-periodic solutions of the form</u> $x = \varepsilon(y+z)$, $\lambda = \lambda(\varepsilon)$ <u>for</u> $|\varepsilon| \leq \varepsilon_0$ <u>where</u> $(z,y_j) = 0,\ 1 \leq j \leq m,\ z(0) = 0,\ \lambda(0) = 0$ <u>and</u> $z(\varepsilon)$, $\lambda(\varepsilon)$ <u>are</u> $q \geq 1$ <u>times Fréchet differentiable in</u> ε.

The key hypotheses in any application of Theorem 4 (see §5) are <u>A2</u> and <u>A3</u>. Note that in order for <u>A2</u> to hold it is necessary (see Lemma 1 and (A_j)) that the "characteristic function" of (10)

$$\det(I_n - \int_0^a k(s)\exp(-zs)ds)$$

have purely imaginary roots. Also note that any higher order (i.e. $O(|x|_p^2)$) terms in T yield order ε terms in $\bar{T}$ and hence make no contribution to D in <u>A3</u>.

With the Fredhom alternative for Stieltjes-integrodifferential systems given in [2] , a theorem similar in content to Theorem 4 can easily be stated for such systems by means of Theorem 1. This theorem would be a considerable generalization of that given in [2] (and would include as a special case the existence part of the well-known Hopf bifurcation theorem for ordinary differential systems).

Note that T need not be independent of t so that the period p can be prescribed by the equation. In autonomous problems p can be used as one of the parameters (as in Hopf type bifurcation) although the analysis is frequently greatly simplified if explicit parameters are used.

5. An Application. The scalar integral equation

$$(11) \qquad N(t) = r\int_{t-1}^{t-d} N(s)(1-N(s))ds \quad , \quad 0<r \quad , \quad 0\leq d<1$$

has arisen in the mathematical theory of epidemics and population dynamics [1,3,6]. For $d = 0$ there are no nonconstant periodic solutions [1,6], but since d represents a delay due to a certain indubation period one might expect nonconstant periodic solutions for at least some $d>0$. Numerical evidence of such periodic solutions was reported in [1]; we will confirm their existence here.

In order to apply Theorem 4 several changes of variable must be made. If

$$(12) \qquad r > 1/(1-d)$$

then (11) has a positive equilibrium $e = 1 - r^{-1}(1-d)^{-1}$. Let $x = N - e$. First (11) will be studied with d held fixed and r treated as a bifurcation parameter. Since the null space of the linearization turns out to have dimension $m = 2$ ((11) is autonomous in the sense that time t translates of solutions are solutions) a second parameter is needed which will be taken (in the classical Hopf bifurcation manner) to be an unknown period p. After a change of independent variable from t to tp^{-1} is made, solutions are then sought in $X(1)$. If these changes of dependent and independent variables are made in (11) and if, for notational simplicity, a variable

$$\beta := \frac{2-(1-d)r}{1-d}$$

is defined, then (11) reduces to

$$(13) \qquad x(t) = p\int_{t-1/p}^{t-d/p} (\beta x(s) - f(\beta)x^2(s))\, ds$$

$$f(\beta) := \frac{2-(1-d)\beta}{1-d} \quad .$$

Bifurcation from the trivial solution $x = 0$ will occur at some critical values of β (i.e. r) and the period p. Let β_o and p_o

denote these (yet to be determined) critical values, set $\lambda_1 = p-p_o$ $\lambda_2 = \beta-\beta_o$ and rewrite (13) in the form of (9) with $n = 1$, $a = 1/p_o$ and

$$k(s) = \begin{cases} 0 , & 0 \leq s < d/p_o \\ \beta_o p_o , & d/p_o \leq s \leq 1/p_o \end{cases}$$

$$T(x,\lambda) = (\lambda_1+p_o)\int_{t-1/(\lambda_1+p_o)}^{t-d/(\lambda_1+p_o)} ((\lambda_2+\beta_o)x(s) - f(\lambda_2+\beta_o)x^2(s))\,ds$$
$$- \int_{t-a}^{t} k(s)x(s)\,ds \quad .$$

This kernel $k(s)$ satisfies <u>A1</u>. It is also easy to see that this T satisfies <u>H3</u> with $X = Y = X(1)$, $\Lambda = E^2$ and $q = 1$. (The operator $\overline{T}$ is just T above with an ε placed in front of $x^2(s)$.) Thus, to apply Theorem 4 one need only justify <u>A2</u> and <u>A3</u> with $m = 2$.

<u>(1)</u> To find solutions $y \in X(1)$ of (10) one can turn to the scalar equations (A_j) with $f_j = 0$. There exist exactly two independent 1-periodic solutions of (10) if and only if $1-k_j(1) \neq 0$ for $j \neq 1$ and $1-k_1(1) = 0$. A straightforward calculation shows that $1-k_o(1) = 1-\beta_o(1-d)$ and for $j \geq 1$ that $1-k_j(1)$ equals

$$(1 - (\beta_o p_o/\pi j)\cos\pi j(1+d)/p_o \sin\pi j(1-d)/p_o)$$
$$+ i\,(\beta_o p_o/\pi j)\sin\pi j(1+d)/p_o \sin\pi j(1-d)/p_o \ .$$

From this it is seen that $m = 2$ if and only if p_o is chosen so that

$$\sin\pi(1+d)/p_o = 0 \quad , \quad \sin\pi(1-d)/p_o \neq 0$$
(14)
$$(-1)^{\alpha}\sin\pi j(1-d)/p_o \neq j(-1)^{j\alpha}\sin\pi(1-d)/p_o \quad \text{for all } j \geq 2,$$

β_o is given by

$$\beta_o = (\pi/p_o)(-1)^{\alpha}\csc\pi(1-d)/p_o \quad , \quad \alpha = (1+d)/p_o$$

and $\beta_o \neq 1/(1-d)$. Note that (14) implies that α is a positive integer. Moreover since it is required that (12) hold, β_o must satisfy $\beta_o < 1/(1-d)$. The problem then is: given d, choose p_o such that all of these conditions hold. Note that (14) fails to hold for any p_o if $d = 0$. This means no bifurcation occurs when no "incubation" or "delay" is present.

An in depth study of this problem for p_o will not be taken up here. One can however easily observe that (14) and all of the conditions on β_o hold if $p_o = 1+d$ and if d satisfies

$$(15) \qquad \sin\pi(1-d)/(1+d) > 1/2 \quad .$$

In this case

$$(16) \qquad \beta_o = -\pi(1+d)^{-1}\csc\pi(1-d)/(1+d) < 0 \quad .$$

Then $m = 2$ and $y_1(t) = \sin 2\pi t$, $y_2(t) = \cos 2\pi t$. Since (A_j) and (A_j^T) are scalar equations the adjoint solutions are the same, that is $y_j(t) = y_j^T(t)$.

Note that (15) holds if and only if $1/11 < d < 5/7$. Inequality (15) is sufficient, but not necessary for $m = 2$. I conjecture that $m = 2$ for suitably chosen p_o and β_o for in fact any d satisfying $0 < d < 1$. §§

<u>(2)</u> Equation (11) is autonomous in the sense that time translates of solutions are solutions and as a result one loses no generality in assuming that $y = \sin 2\pi t$ in <u>A3</u> (as opposed to any other nontrivial linear combination of y_1 and y_2). A straightforward, but rather lengthy calculation shows that D in <u>A3</u> is a nonzero constant multiple of $1 - \cos 2\pi(1-d)/(1+d)$ and consequently $D \neq 0$ for d satisfying (15). §§

In terms of the original variables Theorem 4 now yields the following result for equation (11).

THEOREM 5. <u>If</u> $1/11 < d < 5/7$ <u>then</u> (11) <u>has solutions of the following form for small</u> $|\varepsilon|$:

$$(17) \qquad N(t) = e + \varepsilon \sin 2\pi t/p + \varepsilon z(t/p, \varepsilon)$$

<u>for</u> $p = 1 + d + \lambda_1(\varepsilon)$, $r = 2(1-d)^{-1} - \beta_o - \lambda_2(\varepsilon)$ <u>and</u>

$e = 1-r^{-1}(1-d)^{-1} > 0$ where $\beta_o < 0$ is given by (16); where $z(\cdot,\varepsilon)$ is 1-periodic, is Fréchet continuously differentiable in ε with $z(t,0) \equiv 0$ and is orthogonal to $\sin 2\pi t$ and $\cos 2\pi t$; and where each $\lambda_j(\varepsilon)$ is a continuously differentiable real valued function of ε satisfying $\lambda_j(0) = 0$.

Note that the solution (17) of (11) is p-periodic in t and that (12) holds (i.e. $e > 0$) for small $|\varepsilon|$. If $d = 1/2$ Theorem 5 proves the existence of those nonconstant periodic solutions found numerically in [1] for p near 3/2 and r near $4 + 4\pi/3\sqrt{3}$.

In Theorem 5 the constant d is held fixed in (11) and the two parameters r and p are used in the application of Theorem 4. It is also possible to use the explicitly appearing parameters d and r as the bifurcation parameters in Theorem 4, in which case p is held fixed. If this is done the details are similar to (and in fact simpler than) those given above for the proof of Theorem 5 and are consequently omitted here. The following theorem results from this alternative approach.

THEOREM 6. Given any p satisfying $12/11 < p < 12/7$ there exist p-period solutions of (11) for small $|\varepsilon|$ of the form

$$N(t) = e + \varepsilon \sin 2\pi t/p + \varepsilon z(t,\varepsilon), \quad e = 1 - r^{-1}(1-d)^{-1}$$

for $d = p - 1 + \lambda_1(\varepsilon)$, $r = 2(2-p)^{-1} - \beta_o - \lambda_2(\varepsilon)$ where

$$\beta_o = -\pi p^{-1}\csc \pi(2-p)/p < 0 ;$$

where $z(\cdot,\varepsilon)$ is p-periodic and orthogonal to both $\sin 2\pi t/p$, $\cos 2\pi t/p$ and is Fréchet continuously differentiable in ε with $z(t,0) \equiv 0$; and where each $\lambda_j(\varepsilon)$ is a continuously differentiable real valued function of ε satisfying $\lambda_j(0) = 0$.

Theorem 5 and 6 are easily seen to yield the same bifrucation phenomenon and to be identical to lowest order in ε.

It is of interest to know the signs of λ_j as functions of ε in the above results, for they determine the "direction of bifurcation". I hope to deal with this problem as well as to apply Theorem 4 to other more general models (in particular, to systems of Volterra integral equations such as appear in [1,3]) in future work.

6. Some Further Applications to Scalar Equations. Consider the scalar (n = 1) equation

$$(18)\quad x(t) = \int_{t-a}^{t-b} (k(t-s,\beta)x(s) + g(t-s,\beta,x(s)))ds\ , \quad \beta \in E^1$$

$0 \leq b < a < +\infty$ where g satisfies

A4: $g: [a,b] \times E^1 \times B(E^1,r) \to E^1$ is continuously differentiable and $|g(t,\beta,x)| = o(|x|)$ uniformly for $a \leq t \leq b$ and on compact β sets.

Suppose that the linearized equation

$$(19)\qquad y(t) = \int_{t-a}^{t-b} k(t-s,\beta_0)y(s)\, ds$$

has exactly two independent p_0-periodic solutions for some period $p_0 > 0$ and some $\beta_0 \in E^1$. If without loss in generality p_0 is the minimal period then these solutions are $\sin 2\pi t/p_0$, $\cos 2\pi t/p_0$, which because (19) is scalar are also solutions of the adjoint equation. If the changes of variables $\bar{t} = t/p$ and $\bar{x}(\bar{t}) = x(\bar{t}p)$ are made in (18), then (18) can be written in the form (9) with $n = 1$, with $k(t)$ replaced by $p_0 k(p_0 t,\beta_0)u(t;b/p_0)$ where $u(t;c)$ is the unit step function at c, with a replaced by a/p_0, with $\lambda_1 = \beta-\beta_0$ and $\lambda_2 = p-p_0$, and finally with

$$T(x,\lambda) := (\lambda_2+p_0)\int_{t-a/(\lambda_2+p_0)}^{t-b/(\lambda_2+p_0)} (k((\lambda_2+p_0)(t-s),\lambda_1+\beta_0)x(s) +$$
$$g((\lambda_2+p_0)(t-s),\lambda_1+\beta_0,x(s)))ds - p_0\int_{t-a/p_0}^{t-b/p_0} k(p_0(t-s),\beta_0)x(s)ds$$

(the bars on x and t having been dropped for convenience). By A4 this T satisfies H3 for $q = p = 1$ provided k is continuously differentiable in its arguments. Theorem 4 can now be applied with $p = q = 1$ and $m = 2$ provided $D \neq 0$ in A3 with $y(t) = \kappa_1 \sin 2\pi t + \kappa_2 \cos 2\pi t$, $\kappa_1^2 + \kappa_2^2 \neq 0$. In the usual Hopf-type bifurcation theorems this nondegeneracy condition is related to the transversal crossing of the imaginary axis by a conjugate pair of roots of the characteristic equation. This can also be done here as follows. In order that (19) have exactly $m = 2$ independent

p_o-periodic solutions it is sufficient (but not necessary) that the characteristic equation

$$(20) \qquad h(z,\beta) := 1 - \int_b^a k(s,\beta)\exp(-zs)ds = 0$$

have two and only two (conjugate) purely imaginary roots $z = \pm 2\pi i/p_o$ when $\beta = \beta_o$. If the root $z = 2\pi i/p_o$ is simple (i.e. $h_z(2\pi i/p_o, \beta_o) \neq 0$) then the implicit function theorem implies that (20) has a unique continuously differentiable branch of solutions $z = z(\beta)$, $z(\beta_o) = 2\pi i/p_o$ near $\beta = \beta_o$. By implicit differentiation one can compute $z'(\beta_o)$. A lengthy, but straightforward calculation shows that D, as calculated in A3, is a non-zero constant multiple of $\mathrm{Re}\, z'(\beta_o)$. The result is the following Hopf-type bifurcation theorem for the scalar equation (18).

THEOREM 7. Assume g satisfies A4 and that $k(t,\beta)$ is continuously differentiable in t and β on $0 \leq b \leq t \leq a < +\infty$ and for β near β_o. Assume that (20) has two and only two purely imaginary roots $\pm 2\pi i/p_o$, $p_o > 0$ for $\beta = \beta_o$, that these roots are simple and that $\mathrm{Re}\, z'(\beta_o) \neq 0$ where $z(\beta)$ is the unique branch of roots of (20) satisfying $z(\beta_o) = 2\pi i/p_o$. Then for small real ε equation (18) has p-periodic solutions of the form

$$x(t) = \varepsilon y(t/p) + \varepsilon z(t/p,\varepsilon) \quad , \quad \beta = \beta_o + \lambda_1(\varepsilon) \ , \quad p = p_o + \lambda_2(\varepsilon)$$

where $y(t) = \kappa_1 \sin 2\pi t + \kappa_2 \cos 2\pi t$, $\kappa_1^2 + \kappa_2^2 \neq 0$, where $z(t,\varepsilon)$ is a 1-periodic function of t and where z and λ_i have the properties in Theorem 4 with $q = 1$.

As a final application consider the scalar equation with two parameters given by

$$(21) \qquad x(t) = \int_{t-a}^{t} (\beta_1 k_1(t-s) + \beta_2 k_2(t-s))x(s)\, ds \ + \ R(x,\beta)$$

$$\beta = \mathrm{col}(\beta_1,\beta_2) \in E^2 \quad , \quad \int_0^a k_i(s)ds = 1 \quad \text{for} \quad i = 1,2 \ .$$

Suppose that the linear equation

$$(22) \qquad y(t) = \int_{t-a}^{t} (\beta_1 k_1(t-s) + \beta_2 k_2(t-s))y(s)\, ds$$

has, for an isolated pair $\beta_o = \mathrm{col}(\beta_1^o, \beta_2^o) \in E^2$, exactly two independent p-periodic solutions (namely $\sin 2\pi t/p$ and $\cos 2\pi t/p$ which because (22) is scalar are also the adjoint solutions) for some period $p > 0$. A simple Fourier analysis shows that this is true if and only if

$$W := C_1(1)S_2(1) - S_1(1)C_2(1) \neq 0 \ , \ (C_1(1)-1)S_2(1) \neq (C_2(1)-1)S_1(1)$$

$$(23) \qquad \beta_1^0 = S_2(1)/W \quad \text{and} \quad \beta_2^0 = -S_1(1)/W$$

and for every integer $n \geq 2$ either $S_2(1)S_1(n) - S_1(1)S_2(n) \neq 0$ or $S_2(1)C_1(n) - S_1(1)C_2(n) \neq W$. Here

$$S_i(n) := \int_0^a k_i(s)\sin 2\pi ns/p \, ds \ , \quad C_i(n) := \int_0^a k_i(s)\cos 2\pi ns/p \, ds.$$

The primary reason for mentioning these details here is to show that the assumption that (22) has, for an isolated $\beta_0 \in E^2$, exactly two independent p-periodic solutions implies $W \neq 0$. As will be seen below this in turn will imply the nondegeneracy condition $D \neq 0$.

Equation (21) has the form (9) with $k(t) = \beta_1^0 k_1(s) + \beta_2^0 k_2(s)$ and

$$T(x,\lambda) := \int_{t-a}^{t} (\lambda_1 k_1(t-s) + \lambda_2 k_2(t-s))x(s) \, ds \quad + \quad S(x,\lambda)$$

where $\lambda = \beta - \beta_0 \in E^2$ and $S(x,\lambda) := R(x, \lambda + \beta_0)$. Assume

<u>A5</u>: $R\colon B(X(p),r) \times E^2 \to X(p)$ is $q \geq 1$ times continuously Fréchet differentiable and $|R(x,\beta)|_p = o(|x|_p)$ near $x = 0$ uniformly on compact β sets.

Then T satisfies <u>H3</u>. Inasmuch as an easy calculation shows that D given in <u>A3</u> is equal to $D = -(\kappa_1^2 + \kappa_2^2)W/4 \neq 0$ for $y(t) = \kappa_1 \sin 2\pi t/p + \kappa_2 \cos 2\pi t/p$, $\kappa_1^2 + \kappa_2^2 \neq 0$, Theorem 4 now yields the following result for the scalar equation (21).

THEOREM 8. <u>Assume that the kernels</u> k_i <u>satisfy</u> <u>A1</u> <u>and that</u> R <u>satisfies</u> <u>A5</u>. <u>If the linear equation</u> (22) <u>has, for an isolated</u> $\beta = \mathrm{col}(\beta_1^0, \beta_2^0) \in E^2$, <u>exactly two independent p-periodic solutions for some period</u> $p > 0$, <u>then</u> (21) <u>has a branch of nontrivial p-periodic</u>

solutions for $\beta_i = \beta_i^o + \lambda_i(\varepsilon)$, β_i^o *given by* (23), *as described in Theorem 4 with* $m = 2$.

All of the bifurcation theorems in this paper are purely existence results. It would also, of course, be of interest to study the stability of the nontrivial periodic solutions found in the theorems above, a problem not addressed here.

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