On Preserving Stability of Volterra Integral Equations under a General Class of Perturbations

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We consider the linear system of integral equations

$$v(t) = \varphi(t) + \int_a^t A(t, s)v(s)\, ds \tag{L}$$

and its perturbation

$$u(t) = \varphi(t) + \int_a^t A(t, s)u(s)\, ds + \int_a^t p(t, s, u(s))\, ds \tag{P}$$

for $t \geq a$, where, following Strauss in [1], we assume that $A(t, s)$ is an $n \times n$ matrix which, for some fixed t_0, is defined for $t \geq s \geq t_0$ and satisfies

$$\text{(H1)} \qquad \begin{cases} \lim\limits_{h \to 0} \displaystyle\int_a^T |A(T+h, s) - A(T, s)|\, ds = 0 \\ \sup\limits_{a \leq t \leq T} \displaystyle\int_a^t |A(t, s)|\, ds < +\infty \\ \lim\limits_{h \to 0} \displaystyle\int_t^{t+h} |A(t+h, s)|\, ds = 0 \text{ uniformly for } a \leq t \leq T \end{cases}$$

for all $T \geq a \geq t_0$. The matrix $A(t, s)$ is assumed to be locally in L^1 in (t, s) for $t \geq s \geq t_0$. Here u, v and φ are continuous (but not necessarily differentiable) n-vector-valued functions. The perturbation term $p(t, s, \xi(s))$ is, for each $t \geq s \geq t_0$, a functional defined for all $\xi \in S(b) = \{\xi \in C^0\,[t_0, +\infty)\colon |\xi|_0 = \max_{t \geq t_0} |\xi(t)| \leq b\}$ for some $b > 0$ which is sufficiently smooth so that solutions of (P) exist locally and are extendable. (For example, p might be continuous for $t \geq s \geq t_0$ and all $\xi \in S(b)$ [2].) Also $p(t, s, 0) \equiv 0$ for all $t \geq s \geq t_0$. Hypothesis (H1) is satisfied for example if $A(t, s)$ is continuous.

If (H1) holds, then for each $\varphi(t)$, continuous for $t \geq t_0$, the linear system (L) has a unique solution existing for all $t \geq a$ (see [2]). Also the existence and

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uniqueness for all $t \geq s \geq t_0$ of the (continuous) *fundamental matrix* $U(t, s)$ as a solution of the matrix system

$$\text{(U)} \qquad U(t, s) = I + \int_s^t A(t, r)U(r, s)\,dr, \qquad t \geq s \geq t_0,$$

is assured. (I is the $n \times n$ identity matrix.) We will also have occasion to assume that

$$\text{(H2)} \begin{cases} \text{(L) has a resolvent } R(t, s) \text{ which is locally in } L^1 \text{ in } (t, s) \text{ for} \\ \quad t \geq s \geq t_0. \end{cases}$$

This resolvent satisfies the matrix system

$$\text{(R)} \qquad R(t, s) = -A(t, s) + \int_s^t A(t, r)R(r, s)\,dr, \qquad t \geq s \geq t_0.$$

The solution of (L) has the representations

$$\text{(1)} \qquad v(t) = U(t, a)\varphi(a) + \int_a^t U(t, s)\varphi'(s)\,ds$$

$$\text{(2)} \qquad v(t) = \varphi(t) - \int_a^t R(t, s)\varphi(s)\,ds$$

for $t \geq a$. Of course the derivative $\varphi'(t)$ must exist for the representation (1). These formulas imply that the solution of the linear nonhomogeneous system

$$\text{(NH)} \qquad u(t) = \varphi(t) + \int_a^t [A(t, s)u(s) + p(t, s)]\,ds$$

is given by

$$\text{(VC)} \qquad u(t) = v(t) + \int_a^t U(t, s)\frac{d}{ds}\int_a^s p(s, r)\,dr\,ds$$

or

$$\text{(RU)} \qquad u(t) = v(t) + \int_a^t p(t, s)\,ds - \int_a^t R(t, s)\int_a^s p(s, r)\,dr\,ds$$

for $t \geq a$. The formula (VC) is a direct generalization of the variation of constants formula for differential equations and may be proved by direct verification. In the case $p(t, s) \equiv A(t, s)q(s)$, it can easily be seen from (2) that (RU) can be rewritten as

$$\text{(RU)}' \qquad u(t) = v(t) - \int_a^t R(t, r)q(r)\,dr.$$

By integration (P) includes integro-differential equations of the form

$$u'(t) = \psi(t) + A(t)u + f(t, u) + \int_a^t [B(t, s)u(s) + g(t, s, u(s))]\,ds.$$

In this case $p(t, s, \xi) = p(s, \xi) \equiv f(s, \xi) + \int_a^s g(s, r, \xi(r))\,dr$.

Our purpose is to prove stability results for (L) and for (P) for certain types of perturbations p. The allowable perturbation terms here are motivated by and are generalizations of certain types which appear often in the theory of ordinary differential equations. The results in Theorems 1 and 2 below are generalizations of certain theorems for systems of differential equations (see

Remark 1) and are proved using (VC). This approach has been taken for integro-differential equations under "higher order" type perturbations (see for example [3]–[5]); here we consider a broader class of perturbations. The assumptions in these theorems are somewhat restrictive for the important class of integral equations of convolution type, although they can be applied in this case as is shown in the Example given below. For this reason we also offer Theorems 3 and 4 below which are proved in a manner very similar to Theorems 1 and 2 except that (RU)′ is used in place of (VC). On the basis of Theorem 1 two conjectures of R. K. Miller [3] can be proved. Theorems 3 and 4 represent generalizations of several results in the literature (see Remark 1). For other related literature see [6]–[19].

In the theory of differential equations (i.e., the case when A and p are independent of t) types of perturbations often considered are of the form $|p(t, s, z)| \equiv |p(s, z)| \leq \omega(s)\,|z|$ for $|z| \leq b$, where $\omega(t)$ is a function which satisfies $\omega(t) \to 0$ as $t \to +\infty$ or $\int_0^{+\infty} \omega\, dt < +\infty$ or $\omega(t) \equiv$ const. Roughly speaking, we wish to extend these results for differential equations to the system (P). That such an extension presents perhaps unexpected difficulties because of the variable t in p is illustrated by several counterexamples in Remark 2 below.

Our approach and arguments lead us to make the following assumption on the perturbation term p. Let $|\cdot|$ denote any n-vector norm.

(H3) $\int_a^t p(t, s, \xi(s))\, ds$ is continuously differentiable for $t \geq a$, for all $a \geq t_0$ and all $\xi \in S(b)$, and satisfies

$$\left| \frac{d}{dt} \int_a^t p(t, s, \xi(s))\, ds \right| \leq \omega(t)\, s(\xi; a)\,(t)$$

for all $t \geq a$ and $\xi \in S(b)$ where $s(\xi; a)\,(t) = \max_{a \leq s \leq t} |\xi(s)|$ and $\omega(t) = \rho + \eta(t) + \gamma(t)$. Here $\rho =$ const. ≥ 0 and η, γ are non-negative functions bounded on finite intervals which satisfy

$$\eta(t) \to 0 \text{ as } t \to +\infty \quad \text{and} \quad \gamma^* = \int_{t_0}^{+\infty} \gamma\, dt < +\infty.$$

We will also have occasion to assume

(H3) $p(t, s, \xi) = p_1(t, s, \xi) + p_2(s, \xi)$, where p_1 satisfies (H3) with $\rho = 0$ and p_2 is continuous and satisfies the inequality $|p_2(s, z)| \leq \rho\,|z|$ for all $s \geq t_0$ and for all $z \in R^n$ such that $|z| \leq b$.

Note that if p satisfies (H3)′, then p satisfies (H3).

We make the following definitions for the general system of Volterra equations

$$\text{(V)} \qquad u(t) = \varphi(t) + \int_a^t K(t, s, u(s))\, ds, \qquad t \geq a \geq t_0,$$

where for each $t \geq s \geq t_0$ the kernel $K(t, s, \xi(s))$ is a functional on $S(b)$ satisfying $K(t, s, 0) \equiv 0$.

Definitions. Let N be a normed space of vector-valued functions defined for $t \geq t_0$ with norm $|\cdot|_N$. The zero solution $u(t) \equiv 0$ corresponding to $\varphi(t) \equiv 0$ is called

(1) *stable on N for a given $a \geq t_0$* if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, a) > 0$ such that $|\varphi|_N \leq \delta$, $\varphi \in N$, implies that each solution $u(t)$ of (V) exists and satisfies $|u(t)| \leq \epsilon$ for all $t \geq a$;

(2) *stable on N* if it is stable on N for every $a \geq t_0$;

(3) *uniformly stable on N* if it is stable on N and δ is independent of $a \geq t_0$;

(4) *asymptotically stable on N for a given $a \geq t_0$* if it is stable on N for this $a \geq t_0$ and if there exists a $\delta = \delta(a) > 0$ such that to each $\epsilon > 0$ there corresponds a $T = T(\epsilon, \varphi) \geq a$ for which $|u(t)| \leq \epsilon$ for all $t \geq T$ and $|\varphi|_N \leq \delta$;

(5) *asymptotically stable on N* if it is asymptotically stable on N for every $a \geq t_0$;

(6) *asymptotically stable on N uniformly in $a \geq t_0$* if it is asymptotically stable on N and δ in (4) can be chosen independently of $a \geq t_0$;

(7) *equi-asymptotically stable on N for a given $a \geq t_0$* if T in (4) is independent of $\varphi \in N$, $|\varphi|_N \leq \delta$;

(8) *equi-asymptotically stable on N* if it is equi-asymptotically stable on N for every $a \geq t_0$.

We point out that the space N, or more precisely the norm $|\cdot|_N$, may depend on $a \geq t_0$. Also, for linear systems (L) it is clear that in the definition of asymptotic stability on N the constant $\delta(a)$ can be taken to be $+\infty$ (i.e., asymptotically stable linear systems are globally asymptotically stable) and consequently the asymptotic stability of linear systems is automatically uniform in $a \geq t_0$.

These definitions are equivalent to those made for ordinary differential equations in the case that K and φ are independent of the variable t. Unlike the case of differential equations, however, there is a distinction between stability of (V) for a given $a \geq t_0$ and stability for all $a \geq t_0$ and a distinction between asymptotic stability and equi-asymptotic stability on N. (We would like to thank R. DeFranco for his helpful suggestions relating to the latter distinction.) Examples will be given below (after the proof of Lemma 2).

When we speak simply of the stability of (V) we mean the stability of the zero solution corresponding to $\varphi \equiv 0$.

As is shown by familiar results and examples for differential equations [21], we cannot in general expect a stability property of (L) to hold for (P) under the assumption (H3) without at least the assumption of uniform stability on $R_n = \{\varphi \equiv \text{const.} \in R^n\}$. Further, in the case of differential equations one in fact needs stronger assumptions about (L); for example, uniform asymptotic or exponential stability (unless $\rho = 0$, $\eta = 0$). Referring to Lemma 1 below and Remark 1 we see then that we are justified in assuming that (L) is at least uniformly stable on R_n and also stable on $C_1(a) = \{\varphi(t)\colon \varphi \in C^1[t_0, +\infty), |\varphi|_1 = |\varphi(a)| + |\varphi'|_0 < +\infty\}$ in order to assert the preservation of stability on an arbitrary space N from (L) to (P). Specifically, we will prove

THEOREM 1. *Assume (H1) and (H3). Assume (i) (L) is uniformly stable on R_n and (ii) (L) is stable on $C_1(a)$ for some $a = a_0 \geq t_0$. There exists a constant*

$\rho_0 > 0$ *such that if* $\rho < \rho_0$ (ρ *as in (H3)), then the following conclusions hold.* (*a*) *If (L) is stable on some normed space N for this* $a_0 \geq t_0$, *then (P) is stable on N for this* $a_0 \geq t_0$. (*b*) *If (ii) holds for* $a = t_0$, *then (P) is uniformly stable on a normed space N when (L) is uniformly stable on N.*

Suppose in addition that (L) is asymptotically stable on R_n *for all* $a \geq a_0$ *and that (H3)′ holds.* (*c*) *If (L) is asymptotically stable on a normed space N for* a_0, *then (P) is asymptotically stable on N for this* a_0. (*d*) *Suppose (L) is asymptotically stable on* R_n *for all* $a \geq t_0$ *and that (ii) holds for* $a = t_0$. *If (L) is uniformly stable on a normed space N and asymptotically stable on N uniformly in* $a \geq t_0$, *then so is (P).*

THEOREM 2. (*a*) *Assume that hypothesis (i) is dropped from those of Theorem 1 and that* $\gamma \equiv 0$ *in (H3) and (H3)′. Then the conclusions of Theorem 1 hold.* (*b*) *Assume that hypothesis (ii) is dropped from those of Theorem 1 and that* $\rho = \eta(t) \equiv 0$ *in (H3). Then the conclusions of Theorem 1 hold.*

Theorem 2(b) is stated and proved in [6] and is given here only for completeness and comparison purposes. Well-known examples for differential equations [21] show that Theorem 1 is false if (i) or (ii) is dropped or if (i) is replaced by the assumption of stability on R_n for every $a \geq t_0$.

Example. Consider the following problem as posed by Miller in [4]:

$$u'(t) = Au + \int_0^t B(t-s)u(s)\,ds + h(t, u), \qquad t \geq a$$

$$u(t) = f(t), \qquad 0 \leq t \leq a.$$

Here $h(t, u) = o(|u|)$ uniformly in $t \geq 0$ near $u = 0$. Following Miller we assume the linearized system ($h \equiv 0$) is uniformly asymptotically stable as defined by Miller in [4] and that $B \in L^1[0, +\infty)$. This implies that the fundamental matrix $U(t) \to 0$ as $t \to +\infty$ and $U \in L^1[0, +\infty)$ (Theorem 4, [4]). An immediate application of the main perturbation result in [4] is that for $|f|_a = \sup_{0 \leq t \leq a} |f(t)| \leq \delta(\epsilon, a)$ the solution u satisfies $|u|_0 \leq \epsilon$ and $|u| \to 0$ as $t \to +\infty$. Miller conjectures (for $h \equiv u^2$) but does not prove that δ can be chosen independently of $a \geq 0$. We can confirm this conjecture by using Theorem 1. Integrating the integro-differential equation, we find

$$u(t) = \varphi(t) + \int_a^t \left[A + \int_s^t B(r-s)\,dr\right] u(s)\,ds + \int_a^t h(s, u(s))\,ds,$$

where $\varphi(t) = f(a) + \int_a^t \int_0^a B(s-r)f(r)\,dr\,ds$. First we note that the above assumptions on the linearized system imply that it is uniformly stable on both R_n and $C_1(a)$ (since U is bounded and in $L^1[0, +\infty)$ respectively) and asymptotically stable on R_n (since $U \to 0$) for every $a \geq 0$. Further, $p \equiv h(s, u(s)) = o(|u|)$ uniformly in $s \geq 0$ is easily seen to satisfy (H3)′ with $p_1 \equiv 0$ and $\rho < \rho_0$ if $b > 0$ is taken small. Thus, all the conclusions of Theorem 1 hold. Consider the space $N(a) = \{\varphi \in C_1(a)\colon |\varphi'(t)| \to 0\}$. Since $N(a)$ is a subspace of $C_1(a)$, it follows that the linearized and, hence the perturbed system, are uniformly stable on $N(a)$. This means there exists a $\delta_1 = \delta_1(\epsilon) > 0$, *independent of* $a \geq 0$, such that $|\varphi|_1 \leq \delta_1$ implies $|u|_0 \leq \epsilon$. Now $\varphi(a) = f(a)$ and $\varphi'(t) = \int_0^a B(t-r)f(r)\,dr$. Note that $B \in L^1[0, +\infty)$ implies $\int_0^a |B(t-r)|\,dr = \int_{t-a}^t |B(r)|\,dr \to 0$ as $t \to +\infty$ for every $a \geq 0$. Hence, $|\varphi'(t)| \leq \int_0^a |B(t-r)|\,dr\,|f|_a \to 0$ and it follows that $\varphi \in N(a)$ for every $a \geq 0$. Moreover, $|\varphi|_1 \leq (1+B^*)\,|f|_a$, $B^* = \int_0^{+\infty} |B|\,dr$. Thus, if $|f|_a \leq \delta_1^* = \delta_1/(1+B^*)$, then $|u|_0 \leq \epsilon$. Here δ_1^* is independent of $a \geq 0$. To complete the proof of Miller's conjecture we use part (d) of Theorem 1. First we need to note that (L) is asymptotically stable on $N(a)$ *uniformly* in $a \geq 0$ since it is linear. Hence, (P) is asymptotically stable on $N(a)$ uniformly in $a \geq 0$, which is to say that there exists a constant $\delta_2 > 0$, independent of $a \geq 0$, such that $|\varphi|_1 \leq \delta_2$ implies $|u| \to 0$ as $t \to +\infty$. Thus, $|f|_a \leq \delta_2^* = \delta_2/(1+B^*)$ implies $|u| \to 0$. To sum up: if $|f|_a \leq \delta^* = \min(\delta_1^*(\epsilon), \delta_2^*)$, then $|u|_0 \leq \epsilon$ and $|u| \to 0$ as $t \to +\infty$; here δ^* is independent of $a \geq 0$, as conjectured by Miller.

In order to present our second set of results, consider the hypothesis:

$$\text{(H4)}\quad \begin{cases} p(t, s, \xi) = A(t, s)q(s, \xi), \text{ where, for each } s \geq t_0,\ q(s, z) \text{ is a real-} \\ \text{valued function for } s \geq t_0 \text{ and } z \in R^n,\ |z| \leq b, \text{ which satisfies} \\ |q(s, z)| \leq \omega(s)\,|z|, \text{ where } \omega(s) \text{ is as in (H3).} \end{cases}$$

In this case our analysis will utilize (RU)$'$ and hence will require corresponding information about R in place of U; that is, we need different stability assumptions on (L) from (i) and (ii) in Theorem 1. Let $LC = \{\varphi \in C^0[t_0, +\infty) : |\varphi|_L = \int_{t_0}^{+\infty} |\varphi|\,ds < +\infty\}$ and $C_0 = \{\varphi \in C^0[t_0, +\infty) : |\varphi|_0 < +\infty\}$.

THEOREM 3. *Assume (H2) and (H4). Assume that (i) (L) is uniformly stable on LC and (ii) (L) is stable on C_0 for some $a = a_0 \geq t_0$. There exists a constant $\rho_0 > 0$ such that if $\rho < p_0$ (ρ as in (H4)) then the following conclusions hold. (a) If (L) is stable on some space N for $a = a_0$, then (P) is stable on N for $a = a_0$. (b) If (ii) holds for all $a = t_0$ and (L) is uniformly stable on some space N, then (P) is uniformly stable on N. (c) Suppose (L) is equi-asymptotically stable on LC_0 for $a = a_0$. If (L) is asymptotically stable on a normed space N for $a = a_0$, then (P) is asymptotically stable on N for $a = a_0$. (d) Suppose (L) is equi-asymptotically stable on LC_0 for every $a \geq t_0$ and (ii) holds for $a = t_0$. If (L) is asymptotically stable on a space N uniformly in $a \geq t_0$ and uniformly stable on N, then so is (P).*

Finally, we have

THEOREM 4. *(a) Suppose that hypothesis (i) is dropped in Theorem 3 and that $\gamma \equiv 0$ in (H4). Then the conclusions of Theorem 3 remain valid. (b) Suppose that hypothesis (ii) is dropped in Theorem 3 and that $\rho = \eta(t) \equiv 0$ in (H4). Then the conclusions of Theorem 3 remain valid.*

Before proving these theorems we need some of the results in the following lemmas concerning the connection between stability of (L) on certain spaces and U or R. Define L to be the space of φ measurable in $s \in [t_0, t]$ for all $t \geq t_0$ for which $|\varphi|_L < +\infty$; LC_+ to be those $\varphi \in C^0[t_0, +\infty)$ for which $|\varphi|_+ = |\varphi|_0 + |\varphi|_L < +\infty$; and LC_0 to be the subspace of LC consisting of those φ satisfying $|\varphi(t)| \to 0$ as $t \to +\infty$.

LEMMA 1. *Assume (H1). (a) (L) is stable on R_n or respectively $C_1(a)$ for a given $a \geq t_0$ if and only if there exists a constant $M(a) > 0$ such that $|U(t, a)| \leq M(a)$ for all $t \geq a$ or respectively $|U(t, a)| \leq M(a)$ for $t \geq a$ and in addition $\sup_{t \geq a} \int_a^t |U(t, s)|\,ds \leq M(a)$; (b) (L) is uniformly stable on R_n or respectively*

$C_1(a)$ if and only if M is independent of $a \geq t_0$ in (a); (c) (L) is equi-asymptotically stable on R_n for a given $a \geq t_0$ if and only if $|U(t, a)| \to 0$ as $t \to +\infty$.

We note that asymptotic stability on R_n of (L) implies its equi-asymptotic stability on R_n (the converse is obvious) since the ball $|\varphi| \leq \delta$, $\varphi \in R_n$, is compact and $T(\varphi, \epsilon)$ is continuous in φ (by continuity with respect to data).

Proof. For the space R_n the solution of (L) is given by $v(t) = U(t, a)\varphi$, $\varphi \in R_n$. It is clear from the definitions that the stability of (L) on R_n for a given $a \geq t_0$ is equivalent to the uniform boundedness in $t \geq a$ of the operator $U(t, a)$ on R_n and that uniform stability on R_n is equivalent to the uniform boundedness of this operator in t and a for $t \geq a \geq t_0$. This proves (a) and (b) for the space R_n. Further, it is clear that if $|U(t, a)| \to 0$ as $t \to +\infty$ then $|v(t)| = |U(t, a)\varphi| \to 0$ as $t \to +\infty$ for all $\varphi \in R_n$. Conversely, suppose (L) is equi-asymptotically stable on R_n for an $a \geq t_0$ so that $|\varphi| \leq \delta = \delta(a)$, $\varphi \in R_n$, implies that for any $\epsilon > 0$ there exists a $T = T(\epsilon) > a$ such that $|U(t, a)\delta\varphi| \leq \epsilon$ for $t \geq T$ and for all $\varphi \in R_n$, $|\varphi| = 1$. Thus $|U(t, a)| = \sup_{|\varphi|=1} |U(t, a)\varphi| \leq \epsilon\delta^{-1}(a)$ for $t \geq T(\epsilon)$; that is, $|U(t, a)| \to 0$ as $t \to +\infty$. This proves the lemma for R_n.

Now consider the space $C_1(a)$. For $\varphi \in C_1(a)$ the solution $v(t)$ of (L) is given by (1). If the conditions of (a) are satisfied, then clearly $|v(t)| \leq M(a)|\varphi|_1$ and (L) is stable on C_1(a); and if M is independent of $a \geq t_0$, then (L) is uniformly stable on $C_1(a)$. Conversely, suppose (L) is stable on $C_1(a)$ for an $a \geq t_0$. Then a priori (L) is stable on R_n and, hence, $|U(t, a)| \leq M_1(a)$ for some constant $M_1(a) > 0$ and all $t \geq a$. Furthermore, we have then that $\left|\int_a^t U(t, s)\varphi'(s)\, ds\right| \leq \epsilon$, $t \geq a$, for all $\varphi \in C_1(a)$, $|\varphi|_1 \leq \delta$. It follows from a result stated in [2, p. 261] that $\sup_{t \geq a} \int_a^t |U(t, s)|\, ds \leq M_2(a) < +\infty$. This proves (a) for the space $C_1(a)$ with $M(a) = \max\{M_1(a), M_2(a)\}$. If (L) is uniformly stable on $C_1(a)$, then M_1 is independent of $a \geq t_0$ and hence M is independent of $a \geq t_0$.

The following results dealing with the resolvent will be useful below.

LEMMA 2. *Assume (H1), (H2), and (H3). (a) (L) is stable on C_0 for a given $a \geq t_0$ if and only if there exists a constant $M(a) > 0$ such that* $\sup_{t \geq a} \int_a^t |R(t, s)|\, ds \leq M(a)$*; (b) (L) is uniformly stable on C_0 if and only if it is stable for all $a \geq t_0$ with M independent of $a \geq t_0$ in part (a); (c) if (L) is stable on LC for some $a \geq t_0$ then there exists a constant $M(a) > 0$ such that* $\operatorname{ess\,sup}_{t \geq s \geq a} |R(t, s)| \leq M(a)$*; (d) if (L) is uniformly stable on LC then M is independent of $a \geq t_0$ in (c); (e) if* $\operatorname{ess\,sup}_{t \geq s \geq a} |R(t, s)| \leq M(a)$ *for some constants $M(a) > 0$ and $a \geq t_0$ then (L) is stable on LC_+ for this $a \geq t_0$; (f) if M is independent of $a \geq t_0$ in (e) then (L) is uniformly stable on LC_+; (g) (L) is equi-asymptotically stable on LC_0 for a given $a \geq t_0$ if and only if* $\lim_{t \to +\infty} \operatorname{ess\,sup}_{s \in [a, t]} |R(t, s)| = 0$.

Proof. (a) The solution of (L) for $\varphi \in C_0$ is given by (2). Clearly the assumed bound on R implies $|v(t)| \leq (1 + M(a))\, |\varphi|_0$ and consequently the stability of (L) for this $a \geq t_0$. Conversely, if (L) is stable on C_0 for $a \geq t_0$, then $\left|\int_a^t R(t, s)\varphi(s)\, ds\right| \leq \epsilon$ for $|\varphi|_0 \leq \delta(a, \epsilon)$ for each $\epsilon > 0$ and some $\delta(a, \epsilon) > 0$. As before, a theorem stated in Miller [2, p. 61] implies the existence of the desired constant $M(a) > 0$.

(b) The proof is as in (a) where M is independent of $a \geq t_0$ (since δ is independent of $a \geq t_0$).

(c) The unique solution of (L) is again given by (2). We first consider the scalar case $n = 1$. If (L) is stable on LC then for each $t \geq a$, $v(t)$ is a continuous linear functional on LC for each $t \geq a$. Moreover, since $|v|_0 < +\infty$ (for $|\varphi|_L$ small) this family of linear functionals indexed by $t \geq a$ is uniformly continuous. Also the linear functional defined on LC by $\varphi(t)$ for fixed $t \geq a$ is continuous for almost all $t \geq a$ and again, by the uniform boundedness principle and the fact that $|\varphi|_0 < +\infty$, this family of functionals is uniformly continuous. Thus, from (2) we see that the family of linear functionals on LC defined by $F_t\varphi \equiv \int_a^t R(t, s)\varphi(s)\, ds$ is uniformly continuous for almost all $t \geq a$. By the Hahn-Banach Theorem, each linear functional F_t can be extended (with the same norm) to the space L and, hence, for $t \geq a$ we have a family of uniformly continuous linear functionals F_t^*, $t \geq a$, defined on L. For each $t \geq a$ the Riesz representation theorem tells us that $F_t^*\varphi = \int_a^t R^*(t, s)\varphi(s)\, ds$, where, because LC is dense in L, we have that $R^*(t, s) \equiv R(t, s)$ almost everywhere for $t \geq s \geq a$. Theorem 5 ([22], p. 289) implies that $|F_t^*| = \text{ess sup}_{t\geq s\geq a} |R(t, s)|$ for each $t \geq a$. But the family F_t^* is uniformly continuous and hence $|F_t^*| \leq M(a)$ for some constant $M(a) > 0$. This proves (c) in the scalar case $n = 1$.

For the nonscalar case we proceed as follows: take $|\cdot|$ to be the vector norm $|x| = \max |x_i|$ and the matrix norm $|M|$ to be $|M| = \max_{ij} |m_{ij}|$. This is no loss in generality by the equivalence of all vector norms on R_n and matrix norms on $n \times n$ matrices. Then $v(s) \in L$ if and only if $v_i(s) \in L$ for each i. Suppose ess $\sup_{t\geq s\geq a} |R(t, s)|$ is not finite. Then $R(t, s)$ must have a component $r_{km}(t, s)$ for which ess $\sup_{t\geq s\geq a} |r_{km}(t, s)|$ is not finite. If we define the extended functional on L by $\int_a^t R_{km}(t, s)\varphi(s)\, ds$, where $R_{km}(t, s) \equiv (r_{ij}(t, s))$ with $r_{ij}(t, s) \equiv 0$ for $(i, j) \neq (k, m)$ and $t \geq s \geq a$, then by the above scalar result this family of linear functionals defined on L is not uniformly bounded in $t \geq a$. Consequently there exist a sequence of unit elements $\varphi_i \in L$ and a sequence $t_i \to +\infty$ such that $\left|\int_a^{t_i} R_{km}(t_i, s)\varphi_i(s)\, ds\right| \to +\infty$ as $i \to +\infty$. Thus, $\int_a^{t_i} R(t_i, s)\varphi_i(s)\, ds$, which has $\int_a^{t_i} R_{km}(t_i, s)\varphi_i(s)\, ds$ as a component, must be unbounded as $i \to +\infty$. This proves (c) in the general case.

(d) If (L) is uniformly stable on LC, then the above argument may be repeated, and hence $M(a)$ is independent of $a \geq t_0$.

(e)–(f) For $\varphi \in LC_+$ the solution $v(t)$ of (L) is given again by (2). If $R(t, s)$ is essentially bounded by $M(a)$ for $t \geq s \geq a$, then we have the obvious estimate $|v|_0 \leq (1+M(a))\, |\varphi|_+$ from which follows the stability (and uniform stability if M is independent of $a \geq t_0$) of (L) on LC_+.

(g) Let F_t be the linear functional considered in part (a) restricted to LC_0 and F_t^* be the extension of F_t^* from LC_0 to L. If (L) is asymptotically stable on LC_0 then arguing as in (c) we can show that $|F_t^*| = \text{ess sup}_{t\geq s\geq a} |R(t, s)| \to 0$ as $t \to +\infty$. (Here we use the fact that LC_0 is dense in L.) The converse is obviously true from (2).

At this point we can give an example which illustrates the distinction between the stability of (L) for a given $a \geq t_0$ and stability for *all* $a \geq t_0$. We take $t_0 = 0$ and the kernel $A(t, s)$ to be the solution of (R) with $R(t, s) \equiv t-2s$. For this system we have $U(t, s) \equiv ts-s^2+1$ which is bounded in t for $s = 0$ but unbounded in t for $s \neq 0$. Thus, (L) in this case is stable on R_1 for $a = 0$ but unstable for $a \neq 0$.

We also can give an example of a system (L) which is asymptotically stable at $a = 0$ but not equi-asymptotically stable at $a = 0$ on the space $N = \{\varphi \in C_0: |\varphi(t)| \to 0 \text{ as } t \to +\infty \text{ with norm } |\varphi|_+ < +\infty\}$. With $n = 1$ and $a = 0$ we set $A(t, s) \equiv -1$ in (L), in which case $R(t, s) = \exp(s-t)$. By Lemma 2(e) this system is stable on LC_+ for $a = 0$ and hence on the subspace N. From (2) it is easy to show that $|v(t)| \to 0$ as $t \to +\infty$ for all $\varphi \in N$, and consequently (L) is asymptotically stable on N for $a = 0$. However, (L) is *not* equi-asymptotically stable on N for $a = 0$. This can be shown by consideration of the sequence $\varphi_i \in N$ defined by

$$\varphi_i(t) = \begin{cases} 0; & 0 \le t \le i-1 \\ t-i+1; & i-1 \le t \le i \\ 1; & i \le t \le i+1 \\ -t+i+1; & t \ge i+1 \end{cases}$$

for which $|\varphi_i|_+ = 3$ for all i. That (L) is not equi-asymptotically stable on N follows from the fact that $v_i(i+1) = e^{-1} - e^{-2} = \text{const.}$ for all i, where $v_i(t)$ is the solution of (L) corresponding to φ_i given by (2).

Finally, before proving our theorems we make some observations.

Remark 1. In the case that (L) reduces to a differential system (i.e., A is independent of t), uniform stability (and not asymptotic stability) on $C_1(a)$ is equivalent to uniform asymptotic (or exponential) stability as defined for differential equations. This can be seen from Lemma 1 in the case $U(t, s) = Y(t)Y^{-1}(s)$, where $Y(t)$ is a fundamental solution matrix of the homogeneous system and from theorems in [23, p. 85] and [24, p. 290]. Thus, in this case, Theorem 1 (or Theorem 2(a)) with $\eta \equiv \gamma \equiv 0$ reduces to a well-known perturbation result for differential systems (see [21, p. 68] and [25]). Also Theorem 1 with $\rho = 0, \gamma \not\equiv 0$ yields a result of Strauss and Yorke [26]. Theorem 2(b) yields another well-known perturbation result (see [21], [27]). An important fact from the theory of differential equations is that for perturbations of higher order the hypothesis of uniform asymptotic stability cannot be weakened to that of either asymptotic stability or uniform stability (for examples, see [21], [27]). This points out the importance of stability on $C_1(a)$ in Theorem 1. The assumption of stability on $C_1(a)$ for a fixed $a \ge t_0$ also appears in perturbation theory for differential equations [24].

Also, if $\gamma \equiv \eta \equiv 0$ in Theorem 4(a) we obtain a result due to Strauss [1], which is itself a generalization of a result of Miller, Nohel and Wong [11].

Remark 2. Here we offer some counterexamples which perhaps lend some necessity to the strength of assumption (H3) and/or (H3)$'$ on the perturbation term p. If $n = 1$ and $A = -1$, then (L) is uniformly asymptotically stable as a differential equation or equivalently stable on $C_1(a)$ for all a. If we take $p = \epsilon u \sin(t-s)$, then the solution u of (P) with $a = 0$ and $\varphi \equiv c = \text{const.}$ has Laplace transform $Lu = c(s^2+1)/(s^3+s^2+(1-\epsilon)s+1)$. It is easy to show (using the Routh-Hurwitz criterion) that for all small $\epsilon > 0$ the denominator has two complex conjugate roots in the right half-plane, and hence that u is unbounded for any initial function c. Consequently, in marked contrast to the case when p is independent of t, perturbations p satisfying $|p(t, s, u)| \le \epsilon|u|$ for arbitrarily small ϵ uniformly in $t \ge s \ge t_0$ do not in general preserve stability even when the unperturbed system (L) is exponentially stable. The same example, only with $p = u\exp(t-s)$ or $p = u\exp(s-t)$, shows that the integrability on $[0, +\infty)$ or the tending to zero in either variable t or s (holding the other variable fixed) is not sufficient for preservation of stability.

Finally we give a higher-order example to illustrate that even for exponentially stable linear systems stability is not preserved under perturbations satisfying $p(t, s, u) = o(|u|)$ near $u = 0$ uniformly in $t \geq s \geq t_0$. This is again in marked contrast to the case of differential equations. Again let $a = 0$ and $A = -1$ and take $p(t, s, u) = p^*(t, s)u^2$, where $p^*(t, s) = c(t) + \int_s^t c(r)\,dr$. Here $c(t)$ is any differentiable function defined for $t \geq 0$ satisfying $0 < c(t) < M$, $c \in L^1[0, +\infty)$, and $\limsup_{t\to+\infty} c(t) > 0$. Under these conditions $p^*(t, s)$ is bounded uniformly in $t \geq s \geq 0$ and hence $p = o(|u|)$ uniformly in $t \geq s \geq 0$. The perturbed equation

$$u(t) = \varphi - \int_0^t u(s)\,ds + \int_0^t p^*(t, s)u^2(s)\,ds, \qquad \varphi \in R_1, \tag{3}$$

as pointed out, has a linearization which is exponentially stable (hence, is uniformly and asymptotically stable on R_1 and stable on $C_1(a)$ for every $a \geq 0$) and satisfies all hypotheses of Theorem 1. We now show that this perturbed equation (3) is unstable on R_1. The equation (3), from (VC), is equivalent to the equation

$$u(t) = \varphi e^{-t} + e^{-t}\int_0^t e^{-s}\frac{d}{ds}\int_0^s p^*(s, r)u^2(r)\,dr\,ds$$

or, by the way p^* was chosen, to the equation

$$u(t) = \varphi e^{-t} + c(t)\int_0^t u^2(s)\,ds. \tag{4}$$

Let $w(t) = u(t)/c(t)$. Then $w(t) = \varphi e^{-t}/c(t) + \int_0^t c^2(s)w^2(s)\,ds$, and clearly $w(t) > 0$ for all $t \geq 0$ and $\varphi > 0$. Moreover, clearly $\liminf_{t\to+\infty} w(t) > 0$ and hence $w(t) \geq \delta > 0$ for all $t \geq 0$ and some constant δ; i.e., $u(t) \geq \delta c(t) > 0$ for $t \geq 0$. From (4), $u(t) \geq \delta^2 c(t)\int_0^t c^2(s)\,ds > 0$, $t \geq 0$. Now note that

$$+\infty \geq \limsup_{t\to+\infty} c(t)\int_0^t c^2(s)\,ds = b > 0,$$

for otherwise $b = 0$ would imply $c(t) \to 0$, contrary to the way in which $c(t)$ was chosen. Thus, for all $\varphi > 0$ we have $\limsup_{t\to+\infty} u(t) \geq \delta^2 b > 0$. The constant $\delta^2 b$ being independent of φ implies that (3) is unstable. (In fact, if $c(t)$ is chosen so that $c^2 \notin L^1[0, +\infty)$, then $b = +\infty$ and solutions $u(t)$ are unbounded, for $\varphi > 0$.)

Proof of Theorem 1. (a) We first prove that (P) is stable on N for the given $a = a_0 \geq t_0$ if (L) is; that is, for any $\epsilon > 0$ satisfying $\epsilon < b$ we wish to show that for this a_0 the solution $u(t)$ of (P) exists and satisfies $|u(t)| \leq \epsilon$ for all $t \geq a_0$ provided that $|\varphi|_N \leq \delta$ for some $\delta = \delta(\epsilon, a_0) > 0$.

Let M_1 and $M_2(a_0)$ be as in Lemma 1(b) and 1(a) respectively (cf. (i) and (ii)). Suppose $\rho < \rho_0 = 1/M_2(a_0)$. Then there exists a small constant $\theta > 0$ such that $\rho M_2(a_0) + \theta < 1$. Referring to (H3) we choose $T = T(\epsilon, a_0) \geq t_0$ so large that

$$0 \leq M_1\int_T^\infty \gamma(s)\,ds + M_2(a_0)\max_{s\geq T}|\eta(s)| \leq \theta/4.$$

First we show that $u(t)$ exists and can be made small on the finite interval $[a_0, T]$, provided that $|\varphi|_N$ is chosen small. (If $T \leq a_0$, then this step of the proof is not needed.) From (VC), (H3), and (P) we have that $u(t)$ satisfies

$$\text{(P)}' \qquad u(t) = v(t) + \int_{a_0}^{t} U(t, s)P(s)\, ds,$$

where $P(s) = d(\int_{a_0}^{s} p(s, r, u(r))\, dr)/ds$ for as long as $u(t)$ exists for $t \in [a_0, T]$. That $u(t)$ exists for some $t \geq a_0$ was assumed at the outset. Note that $v(t)$ exists for all $t \geq a_0$ by (VC). From (H3) we have the estimate

$$(5) \qquad |P(s)| \leq M_3 s(a_0; u)(s),$$

where M_3 is a positive constant such that $|\omega(s)| \leq M_3$ for $s \in [a_0, T]$ ($\omega(s)$ is bounded on finite intervals by (H3)). This estimate (5) holds for those s for which $u(s)$ exists. From (P)$'$ we have that

$$|u(t)| \leq |v(t)| + MM_3 \int_{a_0}^{t} s(u; a_0)(s)\, ds,$$

and, by maximizing both sides over the interval $[a_0, t]$, that

$$0 \leq s(u; a_0)(t) \leq s(v; a_0)(T) + MM_3 \int_{a_0}^{t} s(u; a_0)(s)\, ds,$$

which implies by Gronwall's lemma that

$$(6) \qquad |u(t)| \leq s(u; a_0)(t) \leq s(v; a_0)(T) \exp(MM_3(T - t_0))$$

for those $t \in [a_0, T]$ for which $u(t)$ exists. By the assumed extendability property of solutions of (P) it is true that so long as $|u(t)| < b$ the solution $u(t)$ can be continued as a solution of (P). From (6) and the fact that the assumed stability of (L) on N implies $s(v; a_0)(T)$ can be made small for $|\varphi|_N$ small, it clearly follows that for $|\varphi|_N$ small $u(t)$ exists on $[a_0, T]$ and is small. Specifically, choose $\delta_1 = \delta_1(\epsilon, a_0) > 0$ so small that

$$(7) \qquad s(v; a_0)(T) < \epsilon \min(1, (4M_3M_2(a_0))^{-1}\theta) \exp(M_1M_3(t_0 - T)).$$

Then from (6) we have $|u(t)| < \epsilon < b$ for as long as $u(t)$ exists on $[a_0, T]$ which, as pointed out, implies $u(t)$ exists on the whole interval $[a_0, T]$. In addition, this choice of δ_1 implies from (6) that

$$(8) \qquad |u(t)| < \epsilon \min(1, (4M_3M_2(a_0))^{-1}\theta), \qquad t \in [a_0, T].$$

Since $|u(T)| < \epsilon$, $u(t)$ actually exists beyond T. We wish now to show in fact that $|u(t)| \leq \epsilon$ for all $t \geq a_0$ if $|\varphi|_N$ is small enough. Specifically, choose $\delta_2 = \delta_2(\epsilon, a_0) > 0$ such that $|v(t)| \leq \epsilon\theta/4$ for all $t \geq a_0$ and for $|\varphi|_N \leq \delta_2$. This is possible by the assumed stability of (L) on N at a_0. Set $\delta = \min(\delta_1, \delta_2)$ and assume that $|\varphi|_N \leq \delta$. For purposes of contradiction suppose that there exists a first point T' such that $T < T' < +\infty$ for which $|u(T')| = \epsilon$. From (H3) and

$$(9) \qquad u(t) = v(t) + \int_{a_0}^{T} U(t, s)P(s)\, ds + \int_{T}^{t} U(t, s)P(s)\, ds$$

for $t \in [T, T']$ we obtain the estimate

$$|u(t)| \leq \epsilon\theta/4 + \int_{a_0}^{T} |U(t, s)|\, \omega(s)\, ds\, s(u; a_0)(T) + \epsilon \int_T^t |U(t, s)|\, (\gamma(s) + \eta(s) + \rho)\, ds,$$

or, from (7) and (8),

$$\begin{aligned} |u(t)| &\leq \epsilon\theta/4 + M_3 M_2(a_0)\epsilon(4M_3M_2(a_0))^{-1} \\ &\quad + \Big(M_1 \int_T^t \gamma(s)\, ds + M_2(a_0) \max_{s \geq T} |\eta(s)| + \rho M_2(a_0)\Big)\epsilon \\ &\leq \epsilon(3\theta/4 + \rho M_2(a_0)). \end{aligned}$$

Thus, for all $t \in [T, T']$, $|u(t)| < \epsilon(\theta + M_2(a_0)) < \epsilon$, which in particular implies the contradiction $|u(T')| < \epsilon$.

Thus, if $|\varphi|_N \leq \delta = \delta(\epsilon, a_0)$ we have that $|u(t)| < \epsilon$ for all $t \geq a_0$; i.e., (P) is stable on N for this a_0.

(b) If hypothesis (ii) holds for $a = t_0$, then $M_2(a_0)$ in the above argument can be replaced throughout by $M_2(t_0)$. It is easily seen then that δ in the above argument is independent of $a_0 \geq t_0$.

(c) Suppose now that (L) is asymptotically stable on N for $a = a_0$ and that (H3)$'$ holds. We wish to show that (P) is also asymptotically stable on N for $a = a_0$. But by definition, asymptotic stability on N implies stability on N and we have already shown (P) preserves stability on N if $\rho < 1/M_2(a_0)$. (Note: (H3)$'$ implies (H3).) Consequently, we need only show that, for $a = a_0$, $|u(t)| \to 0$ as $t \to +\infty$ provided only that $|\varphi|_N$ is small enough.

By Lemma 1(c) we have that $|U(t, s)| \to 0$ as $t \to +\infty$ for each $s \geq a_0$. Let $\delta_1(a_0)$ be the constant in the definition of asymptotic stability of (L) on N and $\delta_2(\epsilon)$ the constant in the definition of stability of (P) on N. Set $\delta(a_0) = \min(\delta_1(a_0), \delta_2(b/2))$. Then $|\varphi|_N \leq \delta(a_0)$ implies that $|v(t)| \to 0$ and that $u(t)$ exists and is bounded by $b/2$ for all $t \geq a_0$. For purposes of contradiction, suppose that $\limsup_{t\to+\infty} |u(t)| = u^* > 0$. Assume $\rho < 1/4M_2(a_0)$. Using (H3)$'$ we choose $T \geq a_0$ so large that

$$(b/2)\,(M_2(a_0) \max_{s \geq T} |\eta(s)| + M_1 \int_t^\infty \gamma(s)\, ds) \leq u^*/3 \tag{10}$$

and so large that in addition

$$|u(t)| \leq \theta^{-1} u^*, \qquad t \geq T, \tag{11}$$

where θ is some fixed constant, $3/4 < \theta < 1$. Note that $\rho M_2(a_0)\theta^{-1} < 1/3$. From (H3)$'$ and from (P)$'$ for $t \geq T$ we have

$$|u(t)| \leq |v(t)| + \int_{a_0}^{T} |U(t, s)|\, |P(s)|\, ds + \int_T^t |U(t, s)|\, |u(s)|\, ds + (b/2) \int_T^t |U(t, s)|\, (\eta(s) + \gamma(s))\, ds.$$

Thus (10) and (11) imply that

$$|u(t)| \leq |v(t)| + \int_{a_0}^{T} |U(t, s)|\, |P(s)|\, ds + u^*/3 + M_2(a_0)\theta^{-1} u^*.$$

Let $t \to +\infty$. Then $|v(t)| \to 0$ and by the Lebesgue dominated convergence theorem $\int_{a_0}^{T} |U(t, s)|\, |P(s)|\, ds \to 0$. ($P(s)$ is bounded on $[a_0, T]$ by (H3)′.) Consequently we find that $0 < u^* < u^*/3 + M_2(a_0)\theta^{-1}u^* < 2u^*/3$, a contradiction. Hence, $u^* = 0$ and (P) is asymptotically stable on N for this a_0.

(d) If (L) is asymptotically stable uniformly in $a \geq t_0$, then δ in part (c) above is independent of $a = a_0$, and hence the proof of (c) as given yields the fact that (P) is asymptotically stable uniformly in $a \geq t_0$.

Proof of Theorem 2. The proof of part (a) is exactly as the above proof of Theorem 1 except that in (9) we have $\gamma^* = 0$, which follows from (8) with $\gamma(s) \equiv 0$. The reason hypothesis (i) of Theorem 1 can be dropped in this case is that the uniform bound on $|U(t, s)|$ is no longer needed. This same remark in the proof of the preservation of asymptotic stability of (P) above holds. For the proof of part (b) see [6].

Proof of Theorem 3. The idea of the proof is exactly that of the proof of Theorem 1 except that we begin with the following integral equation for the solution $u(t)$ of (P) which arises from (RU)′:

$$u(t) = v(t) - \int_{a_0}^{t} R(t, s)\, q(s, u(s))\, ds, \qquad t \geq a_0. \tag{12}$$

(a) Given $\epsilon > 0$ sufficiently small we choose $T = T(\epsilon) \geq t_0$ so large that $|\eta(t)| \leq \epsilon$ for $t \geq T$ (see (H4)). The proof breaks into two parts as does the proof of Theorem 1. First we show that $u(t)$ exists and satisfies $|u(t)| < \epsilon$ for $t \in [a_0, T]$ provided that $|\varphi|_N \leq \delta_1$ for some $\delta_1 = \delta_1(\epsilon, a_0) > 0$. Since this step is very much like the corresponding step in the proofs of Theorem 1 (and is essentially a continuity-with-respect-to-data argument on finite intervals), we omit its details.

By the continuation property of solutions to (P) we know that for $|\varphi|_N \leq \delta_1$ we have that $|u(t)| < \epsilon$ for $t \in [a_0, T')$ for some $T' > T$. We wish to show $T' = +\infty$ and proceed to do this as above by a contradiction argument. Suppose $T' > T$ is the first point for which $|u(T')| = \epsilon$. We estimate $u(t)$ by using (12) on $t \in [T, T']$ where $|u(t)| \leq \epsilon$:

$$\begin{aligned} |u(t)| &\leq |v(t)| + \int_{a_0}^{T} |R(t, s)|\, |q(s, u(s))|\, ds + \int_{T}^{t} |R(t, s)|\, |q(s, u(s))|\, ds \\ &\leq s(v; a_0)(t) + \int_{a_0}^{T} |R(t, s)|\, |q(s, u(s))|\, ds \\ &\qquad + M(\rho+\epsilon)s(u; T)(t) + M \int_{T}^{t} \gamma(s) s(u;T)(s)\, ds, \end{aligned}$$

where M is the larger of the two constants appearing in Lemma 2(a) and 2(d). In a manner similar to the proof of Theorem 1 it can be shown that for $|\varphi|_N \leq \delta_2(\epsilon, a_0)$ we have for all $t \geq t_0$

$$s(v; a_0)(t) + \int_{a_0}^{T} |R(t, s)|\, |q(s, u(s))|\, ds \leq \tfrac{1}{2}\epsilon m^{-1} \exp(-mM\gamma^*),$$

where $m = (1 - M(\rho+\epsilon))^{-1}$. Here ϵ and ρ_0 (with $\rho < \rho_0$) have been chosen so small that $0 < m < +\infty$. Consequently, maximizing over $[T, t]$ both sides of the inequality, we arrive, after some manipulations, at

$$s(u; T)(t) \leq m\epsilon(2m)^{-1} \exp(-mM\gamma^*) + mM \int_T^t \gamma(s)s(u; T)(s)\, ds$$

for $t \in [T, T']$. An application of Gronwall's lemma leads immediately to $s(u; T)(t) \leq \epsilon/2$ for $t \in [T, T']$, which obviously implies the contradiction $|u(T')| < \epsilon$ for $|\varphi|_N \leq \delta_2$. Thus, for $|\varphi|_N \leq \delta = \min(\delta_1, \delta_2)$ we have $|u(t)| < \epsilon$ for all $t \geq a_0$ if ϵ and ρ_0 are small enough. This proves that (P) is stable on N at $a = a_0$.

(b) As usual, part (b) follows from the proof of (a) with M, and hence δ, independent of $a \geq t_0$.

(c)–(d) Starting from (13) and using Lemma 2 we can prove parts (c) and (d) exactly as parts (c) and (d) of Theorem 1 were proved with U replaced by R.

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