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Stability of Perturbed Volterra Integral Equations

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We consider the linear system of equations

$$v(t) = f(t) + \int_a^t K(t, s)\, v(s)\, ds, \qquad t \geqslant a \tag{L}$$

and its perturbation

$$u(t) = f(t) + \int_a^t K(t, s)\, u(s)\, ds + \int_a^t p(t, s, u(s))\, ds, \qquad t \geqslant a. \tag{P}$$

Here $f(t)$ is a continuous n-vector valued function of $t \geqslant a \geqslant t_0$ (where t_0 is a fixed point), $p(t, s, z)$ is an n-vector valued function of $(t, s, z) \in \{t \geqslant s \geqslant t_0\} \times \{|\, z \,| \leqslant b, b > 0\}$, and $K(t, s)$ is an $n \times n$ matrix defined for $t \geqslant s \geqslant t_0$ which is locally in L^1 in (t, s) for $t \geqslant s \geqslant t_0$ and satisfies the three conditions:

$$\lim_{h \to 0} \int_{t_0}^T |\, K(T + h, s) - K(T, s)|\, ds = 0,$$

$$\sup_{a \leqslant t \leqslant T} \int_a^t |\, K(t, s)|\, ds < +\infty, \tag{H1}$$

$$\lim_{h \to 0} \int_t^{t+h} |\, K(t + h, s)|\, ds = 0$$

uniformly for $a \leqslant t \leqslant T$ for all $T \geqslant a \geqslant t_0$.

In particular $K(t, s)$ may be assumed continuous in t and s. The functions $v(t)$ and $u(t)$ are of course unknown n-vector valued functions. The symbol $|\cdot|$ will denote any vector norm or any $n \times n$ matrix norm depending on whether it is applied to a vector or matrix, respectively. If N is a normed space of functions f defined for $t \geqslant a$ with norm $|\, f\,|_N$, then (P) (or (L)) is called *stable on N for a fixed $a \geqslant t_0$* if for each $\epsilon > 0$ there corresponds a $\delta = \delta(\epsilon, a) > 0$ such that $|\, f\,|_N \leqslant \delta$, $f \in N$, implies $u(t)$ exists and satisfies

$|u(t)| \leqslant \epsilon$ for all $t \geqslant a$. If N and δ are independent of $a \geqslant t_0$, then (P) (or (L)) is *uniformly stable on* N. If (P) (or (L)) is stable on N for $a \geqslant t_0$ and in addition there exists a $\delta = \delta(a) > 0$ such that to each $\epsilon > 0$ there corresponds a $T = T(\epsilon, f) \geqslant a$ such that $|u(t)| \leqslant \epsilon$ for all $t \geqslant T$ and $|f|_N \leqslant \delta$, then (P) (or (L)) is called *asymptotically stable on* N *for* $a \geqslant t_0$. If T can be chosen independently of $f \in N$, $|f|_N \leqslant \delta$, then (P) (or (L)) is called *equi-asymptotically stable on* N *for* $a \geqslant t_0$.

We are interested in the following question: under what conditions on the perturbation term p is it true that a given stability property of system (L) is also possessed by (P)? In [1–3], Bownds and the author considered this question for various types of perturbations and stability properties. Other closely related results may be found in [4, 5] (for further references see those listed in [1]). In [1–3] it is assumed that p is such that $P(t; \xi) \equiv \int_a^t p(t, s, \xi(s))\, ds \in C^1[a, +\infty)$ for all vector functions $\xi(s) \in C^0(b) = \{\xi \in C^0[a, +\infty): |\xi(t)| \leqslant b$ for all $t \geqslant a\}$ and

$$\left| \frac{d}{dt} P(t; \xi) \right| \leqslant g(t)\, s(\xi; a)(t), \qquad \text{for all } t \geqslant a \quad \text{and} \quad \xi \in C(b), \tag{1}$$

where $s(\xi; a)(t) = \sup_{a \leqslant s \leqslant t} |\xi(s)|$. For example, p may be such that $|p(t, t, z)| \leqslant g_1(t)\,|z|$ and $|p_t(t, s, z)| \leqslant g_2(t, s)\,|z|$ for $t \geqslant s \geqslant a$ and $z \in R^n =$ Euclidean n-space where $g(t) \equiv g_1(t) + \int_a^t g_2(t, s)\, ds$. It is then assumed that $g(t)$ is one of three types (or a linear combination of three types): either (a) $\int_a^{+\infty} g(t)\, dt < +\infty$, (b) $g(t) \to 0$ as $t \to +\infty$, or (c) $g(t) \equiv g_0$ for a sufficiently small constant g_0. Theorem 1 in [1] states, amongst other things, that for these kinds of perturbations (P) preserves the stability of (L) on any space N for a given $a \geqslant t_0$ provided a minimal amount of stability of (L) is present: namely, provided (i) the linear system (L) is uniformly stable on the space of constants $R_n = \{f: f(t) \equiv \text{constant} \in R^n\}$ and (ii) stable on $C_1(a) = \{f \in C^1[a, +\infty): |f|_1 = |f(a)| + |f'|_0 < +\infty$, $|f'|_0 = \sup_{t \geqslant a} |f'(t)|\}$ for $a \geqslant t_0$. In the special case that (L) and (P) are integrated forms of a linear system of ordinary differential equations and its perturbation, respectively (that is, K and p are independent of the variable t), uniform stability of (L) on R_n coincides with the usual notion of uniform stability for differential systems [6] and stability on $C_1(a)$ coincides with uniform asymptotic stability for differential systems (see Remark (1) in [1]). Thus, in this special case, the above cited result reduces to several well, known results for differential systems. For differential systems, however, under the assumption of uniform asymptotic stability one can strengthen the result by allowing $g(t)$ to satisfy a weaker condition than either (a) or (b) above. Stauss and Yorke [7] have shown that asymptotic stability is preserved if $g(t)$ only satisfies the weaker condition $\int_t^{t+1} g\, ds \to 0$ as $t \to +\infty$. More recently, Shanholdt [11] has considered such perturbations for functional differential systems. This motivates our attempt to obtain the conclusions found in [1] under weaker assumptions of this type. (This type of condition on g can also be found in the work of Massera and Schaffer [8] and Coppel [6].) The example in Remark 5 below shows, however, that this is not in general possible. What is necessary is a little more stability from (L) in the sense that assumption (ii) above must be strenghtened slightly. Nonetheless we will still obtain a generalization of the above mentioned result of Strauss and Yorke because the stronger hypothesis we substitute for (ii) (namely (ii) in Theorem 1 below with $p = 1$) is still equivalent to uniform asymptotic stability in the special case of differential systems.

Before stating our results it is necessary to state some preliminaries and introduce some notation. The assumptions stated above guarantee the existence for all $t \geqslant a$ of a unique solution $v(t)$ of (L) [9] (we consider only continuous solutions); further, the existence of the (unique) fundamental matrix satisfying the matrix equation

$$U(t, s) = I + \int_s^t K(t, r)\, U(r, s)\, dr, \qquad t \geqslant s \geqslant t_0\,,$$

is assured. Let $LBV[a, +\infty)$ denote those functions $f \in C^0[a, +\infty)$ which are of bounded variation on every interval $[a, t]$, $t \geqslant a$. By direct verification it is easy to see that for $f \in LBV[a, +\infty)$ the unique solution of (L) is given by the "variation of constants" formula

$$v(t) = U(t, a) f(a) + \int_a^t U(t, s)\, df(s), \qquad t \geqslant a. \tag{VC}$$

((VC) is identical to the standard variation of constants formula when (L) reduces to a differential system.)

We assume that the perturbation term p in (P) satisfies

(H2) $p(t, s, z)$ is sufficiently smooth for the local existence and continuability of solutions of (P);

(H3) $p^*(t; \xi) \equiv \int_a^t p(t, s, \xi(s))\, ds \in LBV[a, +\infty)$ for every $\xi \in C^0(b)$ and $t \geqslant a$.

Given (H1), hypothesis (H2) is satisfied for example if p is continuous in (t, s, z). For weaker conditions under which (H2) holds see [9]. Hypothesis (H3) is satisfied for example if $dp^*(t; \xi)/dt$ is continuous for all $\xi \in C^0(b)$. Problem (P) can then be seen (using (VC)) to be equivalent to the integral equation

$$u(t) = v(t) + \int_a^t U(t, s)\, dp^*(s; u(s)), \qquad t \geqslant a. \tag{P*}$$

For any integer $p \geqslant 1$ let $\|f\|_{a,p} = |f(a)| + \sup_{t\geqslant a}(\int_t^{t+1} |f'|^p\, ds)^{1/p}$ and let $BM_{a,p}$ denote the normed space of functions $f \in C^1[a, +\infty)$ for which $\|f\|_{a,p} < +\infty$. We also allow $p = +\infty$ and mean by this that $\|f\|_{a,\infty} = |f(a)| + \sup_{t\geqslant a} |f'(t)|$. Without loss in generality we assume $K(t, s) \equiv p(t, s, z) \equiv 0$ for all $t < t_0$ and all s, z.

Our main results are contained in the following two theorems.

THEOREM 1. *Suppose $H1$, $H2$ and $H3$ are satisfied and that in addition the perturbation term p satisfies, for a given $a \geqslant t_0$, the condition*

(H4) $|p^*(t + h; \xi) - p^*(t; \xi)| \leqslant hg(t)\, s(\xi; a)(t)$ *for all* $t \geqslant a$, *sufficiently small* $h \geqslant 0$, *and* $\xi \in C^0(b)$ *where* $g(t) \geqslant 0$ *is a function bounded on finite intervals contained in* $[a, +\infty)$ *such that*

$$0 \leqslant g_p \equiv \limsup_{t\to+\infty} \left(\int_t^{t+1} gp(s)\, ds\right)^{1/p} < +\infty$$

for some integer p, $1 \leqslant p \leqslant +\infty$.

Suppose further that (L) *is* (i) *uniformly stable on R_n and* (ii) *stable on $BM_{a,p}$ for this $a \geqslant t_0$.*

(a) *Then there exists a constant $g_0 > 0$ for which $g_p < g_0$ implies that if* (L) *is stable on a normed space N for this $a \geqslant t_0$, then so is* (P).

(b) *If $H2$, $H3$, $H4$ and* (ii) *hold for all $a \geqslant t_0$ then* (P) *is uniformly stable on any space N on which* (L) *is uniformly stable.*

The next theorem concerns the asymptotic relationship between u and v and has as a corollary the preservation of asymptotic stability.

THEOREM 2. *Assume in addition to the hypotheses of Theorem* 1 *with $g_p = 0$ that* (L) *has the property*

$$\lim_{t\to+\infty} \int_a^T |U(t, s)|\, ds = 0, \tag{2}$$

for all $T \geqslant a$. There exists a constant $\delta = \delta(a) > 0$ such that if $|f|_N \leqslant \delta$ then $|u(t) - v(t)| \to 0$ as $t \to +\infty$.

COROLLARY. *Suppose in addition to the assumptions of Theorem* 1 *with $g_p = 0$ that the linear system* (L) *is asymptotically stable on R_n. Then* (P) *is (equi-)asymptotically stable for $a \geqslant t_0$ on any space N on which* (L) *is (equi-)asymptotically stable for $a \geqslant t_0$.*

This corollary follows immediately from Theorem 2 and the Lemma, part (b), below since the assumption of asymptotic stability of (L) on R_n implies (2).

Remark 1. Hypothesis (H4) is satisfied by any perturbation term p satisfying (1) for suitable $g(t)$. Moreover the condition on $g(t)$ in (H4) is fulfilled if $(\int_a^{+\infty} g^p\, ds)^{1/p} < +\infty$ or if $g(t) \to 0$ as $t \to +\infty$ for in either case $(\int_t^{t+1} g^p\, ds)^{1/p} \to 0$ as $t \to +\infty$ and hence $g_p = 0$. (H4) is also fulfilled if $g(t) \equiv \text{constant} = g_p$; it is this case which can be used to deal with perturbations $p(t, s, z)$ which are higher order in z (by taking $b > 0$ small if necessary) as would naturally arise under the usual procedure of linearization. Consequently all perturbations considered in [1] fulfill (H4) for $p = 1$. Theorems 1 and 2 for $p = 1$ require, however, more stability of (L) than the theorems in [1]; specifically assumption (ii) is stronger than the assumption of stability of (L) on $C_1(a)$ since $C_1(a)$ is a proper subspace of $BM_{a,1}$ (also see Remark 4).

Remark 2. Intervals of unit length were used above purely for convenience in the definition of the spaces $BM_{a,p}$ and in (H4). Intervals of any fixed, finite length $c > 0$ could be used (in which case the integral $\int_t^{t+1} g^p\, ds$ would be replaced by $c^{-1} \int_t^{t+c} g^p\, ds$).

Remark 3. If K and p are continuous and independent of t, then the above Corollary with $p = 1$ reduces to a theorem of Strauss and Yorke [7, Theorem 3.2]. This is because it is true that stability on $BM_{a,1}$ in this case can be seen to be equivalent to uniform asymptotic stability of (L). (See the Lemma, part (c), below and [1, Remark 1].)

Remark 4. Unlike the case of differential equations (as pointed out in Remark 3) stability of (L) on $C_1(a)$ is not equivalent to stability on $BM_{a,1}$; although, clearly stability on $BM_{a,1}$ implies stability on the subspace $C_1(a)$. An illustration of this is furnished by the following example. Take $a = t_0 = 0$ and $n = 1$. We will construct an example for which $| U(t, s)| \leqslant 1$ for $0 \leqslant s \leqslant t$, $\int_0^t | U(t, s)|\, ds \leqslant \pi^2/6$, $t \geqslant 0$, and $\int_0^t u(t, s) f'(s)\, ds$ is unbounded in t for a specific $f \in BM_{0,1}$. Thus, (L) will be uniformly stable on R_n and stable on $C_1(0)$ (see the Lemma, parts (a) and (c), below) but unstable on $BM_{0,1}$. For simplicity we will use functions with step discontinuities although it will be clear conceptionally how the example could be "smoothed" to make K and f' continuous. Define a function $U(t, s)$ for $t \geqslant s \geqslant 0$ such that for each positive integer $n \geqslant 1$

$$U(2n + n^{-2}, s) = \begin{cases} 1 & \text{for } s \in \bigcup_{i=1}^{n} [2i, 2i + i^{-2}], \\ 0 & \text{elsewhere.} \end{cases}$$

Then for all $n \geqslant 1$

$$\int_0^{2n+n^{-2}} | U(2n + n^{-2}, s)| \, ds = \sum_{i=1}^{n} i^{-2} \leqslant \pi^2/6.$$

Define $U(t, s)$ for $t \neq 2n + n^{-2}$ and $0 \leqslant s \leqslant t$ such that $0 \leqslant U(t, s) \leqslant 1$, $U(t, t) = 1$ and $\int_0^t U(t, s) \, ds \leqslant \pi^2/6$ and such that U is (at least piecewise) continuous. Let $f(t)$ be defined for $t \geqslant 0$ such that $f(0) = 0$ and

$$f'(t) = \begin{cases} i & \text{for } t \in [2i, 2i + i^{-2}], \\ 0 & \text{elsewhere.} \end{cases}$$

Then for all $t \geqslant 0$ we have $\int_t^{t+1} |f'| \, ds \leqslant (j/2)^{-1}$ where j is the closest even integer to t. Thus, $\int_t^{t+1} |f'| \, ds \to 0$ as $t \to +\infty$ so that $f \in BM_{0,1}$. However, from (VC) the solution of (L) for this f (the kernel K can be constructed from U by making U smooth enough so that $\partial U(t, s)/\partial s$ is continuous and noting that this derivative is the resolvent R of (L) the equation for which can be solved for continuous K when R is known) is given by $v(t) = \int_0^t U(t, s) f'(s) \, ds$ and hence (L) is unstable since this v is unbounded:

$$v(2n + n^{-2}) = \int_0^{2n+n^{-2}} U(2n + n^{-2}, s) f'(s) \, ds = \sum_{i=1}^{n} i^{-1}.$$

Note that $f \notin C_1(0)$ since $\int_0^{+\infty} |f'| \, ds = \sum_1^{+\infty} i^{-1}$.

Remark 5. We can also give an example to show that hypothesis (ii) in Theorems 1 and 2 above cannot be replaced by the assumption of stability on $C_1(a)$ as is done in [1]. Let U, K and f all be defined as above in Remark 4. Note that $U(t, s)$ can be constructed such that $U(t, 0) \equiv 0$ for $t \geqslant 2$. Consider the scalar equation (L) and its perturbation (P) with $p(t, s, z) \equiv f'(s) z$. Then p satisfies (H4) with $g = f'$ and $p = 1$ (hence, $g_1 = 0$). By the way $U(t, s)$ was constructed (L) is uniformly stable on R_n and stable on $C_1(0)$ (see Lemma, parts (a) and (c), below) and, consequently, all the hypotheses of Theorem 1 are fulfilled with the space $C_1(0)$ replacing $BM_{0,1}$ in (ii). We will to show, however, that the perturbed system (P) has an unbounded solution for all $f = c \in R^1$. By (VC) the solution of (P) in this example is given by

$$u(t) = U(t, 0) \, c + \int_0^t U(t, s) f'(s) \, u(s) \, ds.$$

Let $W(t, s)$ be the fundamental matrix for this linear equation:

$$W(t, s) = 1 + \int_s^t U(t, r) f'(r) \, W(r, s) \, dr. \tag{3}$$

Then

$$u(t) = W(t, 0)\, c + \int_0^t W(t, s)\, U_s(s, 0)\, ds.$$

Thus, for $t > 2$, $u(t) \equiv W(t, 0)c$. Since U and f' are nonnegative it is clear from (3) that $W(t, 0) \geqslant 0$ for all $t \geqslant 0$. Hence, from (3) we have $W(t, 0) \geqslant 1$ and in turn

$$W(t, 0) \geqslant 1 + \int_0^t U(t, s) f'(s)\, ds.$$

But the latter integral is unbounded in t as constructed in Remark 4. Thus, $u(t) = W(t, 0)c$ for $t \geqslant 2$ is unbounded.

This example demonstrates that for integral equations the results of [1] are not valid for the larger class of perturbations described in (H4) without some strenghtening of the assumptions on (L) (as provided by (ii)).

To prove the above theorems we need the following lemma which describes the connection between U and stability on various spaces.

LEMMA. *Assume* (H1).

(a) (L) *is uniformly stable on* R_n *if and only if there exists a constant* $m \geqslant 0$ *such that* $|U(t, s)| \leqslant m$ *for all* $t \geqslant s \geqslant t_0$.

(b) (L) *is asymptotically (or equi-asymptotically) stable on* R_n *for a given* $a \geqslant t_0$ *if and only if* $|U(t, a)| \to 0$ *as* $t \to +\infty$.

(c) (L) *is stable on* $C_1(a)$ *for a given* $a \geqslant t_0$ *if and only if there exists a constant* $m = m(a) > 0$ *such that for all* $t \geqslant a$

$$\int_a^t |U(t, s)|\, ds \leqslant m, \qquad |U(t, a)| \leqslant m.$$

(d) *Suppose* (L) *is uniformly stable on* R_n. *Then* (L) *is stable on* $BM_{a,p}$ *for a given* $a \geqslant t_0$ *if and only if there exists a constant* $m = m(a) > 0$ *such that for all* $t \geqslant a$ *and* $1/p + 1/q = 1$

$$\int_{a-1}^t \left(\int_s^{s+1} |U(t, r)|^q\, dr\right)^{1/q} ds \leqslant m, \qquad \text{if} \qquad p \neq 1 \tag{4}$$

or

$$\int_{a-1}^t \max_{s \leqslant r \leqslant s+1} |U(t, r)|\, ds \leqslant m, \qquad \text{if} \qquad p \neq 1.$$

Proof. Parts (a), (b) and (c) are proved in [1]. We need only prove (d).

We take $p \neq 1$, the case $p = 1$ being similar. First, suppose (4) holds. From (VC) the unique solution of (L) is given by

$$v(t) = U(t, a)f(a) + \int_a^t U(t, r)f'(r)\,dr, \qquad t \geqslant a.$$

Let $m(a)$ be the larger of the constants in (a) (by assumption (L) is uniformly stable on R_n) and (4). Then for all $t \geqslant a$

$$\begin{aligned}
|v(t)| &\leqslant m(a)\,|f(a)| + \int_a^t |U(t,r)|\,|f'(r)|\,dr \\
&= m(a)\,|f(a)| + \int_a^t\int_{r-1}^r |U(t,r)|\,|f'(r)|\,ds\,dr \\
&= m(a)\,|f(a)| + \int_{a-1}^t\int_s^{s+1} |U(t,r)|\,|f'(r)|\,dr\,ds \\
&\quad - \int_{a-1}^a\int_{a-1}^r |U(t,r)|\,|f'(r)|\,ds\,dr - \int_t^{t+1}\int_{r-1}^t |U(t,r)|\,|f'(r)|\,ds\,dr \\
&\leqslant m(a)\,\|f\|_{a,p} + \int_{a-1}^t\int_s^{s+1} |U(t,r)|\,|f'(r)|\,dr\,ds \\
&\leqslant m(a)\,\|f\|_{a,p} + \int_{a-1}^t \left(\int_s^{s+1} |U(t,r)|^q\,dr\right)^{1/q} \left(\int_s^{s+1} |f'(r)|^p\,dr\right)^{1/p} ds.
\end{aligned}$$

Thus, $|v(t)| \leqslant m(a)\,\|f\|_{a,p} + m(a)\,\|f\|_{a,p} = 2m(a)\,\|f\|_{a,p}$ which implies the stability of (L) on $BM_{a,p}$.

Conversely, suppose (L) is stable on $BM_{a,p}$. We prove (4) holds in the scalar case $n = 1$. From this, the general case $n > 1$ can be proved following the techniques used in [9] or [1]. Define the linear functional

$$L_t^{(1)}\gamma \equiv \int_a^t U(t, r) \int_{r-1}^r \gamma(s, r)\,ds\,dr,$$

for arbitrary, but fixed $t \geqslant a$. We first establish that $L_t^{(1)}$ is a bounded linear functional (uniformly in $t \geqslant a$) on the normed space S of function $\gamma = \gamma(s, r)$ for which $\gamma = 0$, $(s, r) \notin [a, t] \times [s, s + 1]$, γ is continuous in $s \in [a, t]$ and in $r \in [s, s + 1]$, and $\|\gamma\|_{S,p} = \sup_{s\geqslant a}(\int_s^{s+1} |\gamma(s, r)|^p\,dr)^{1/p} < +\infty$. Now

$$\begin{aligned}
\int_t^{t+1} \left| \int_{r-1}^r \gamma(s,r)\,ds \right|^p dr &\leqslant \int_t^{t+1}\int_{r-1}^r |\gamma(s,r)|^p\,ds\,dr \\
&\leqslant \int_{t-1}^{t+1}\int_s^{s+1} |\gamma(s,r)|^p\,ds\,dr \qquad (5) \\
&\leqslant 2 \cdot \sup_{s\geqslant a} \int_s^{s+1} |\gamma(s,r)|^p\,dr
\end{aligned}$$

and, hence, for all $t \geqslant a$

$$\left(\int_t^{t+1} \left| \int_{r-1}^{r} \gamma(s, r)\, ds \right|^p dr\right)^{1/p} \leqslant 2^{1/p} \| \gamma \|_{S,p} . \tag{6}$$

Now by the assumed stability of (L) on $BM_{a,p}$ we see that the linear functionals $\int_a^t U(t, r) f'(r)\, dr$ are bounded uniformly in $t \geqslant a$ with respect to the norm $\| f \|_{a,p}$. It follows that the linear functionals defined by

$$A_t\psi \equiv \int_a^t U(t, r)\, \psi(r)\, dr,$$

are bounded (uniformly in $t \geqslant a$) with respect to the norm $\| \psi \|_{a,p}^0 = \sup_{t \geqslant a}(\int_t^{t+1} | \psi(r)|^p\, dr)^{1/p}$. Now (6) says that

$$\left\| \int_{r-1}^{r} \gamma(s, r)\, ds \right\|_{a,p}^0 \leqslant 2^{1/p} \| \gamma \|_{S,p}$$

and inasmuch as $L_t^{(1)}\gamma = A_t(\int_{r-1}^{r} \gamma(s, r)\, ds)$ we see that $L_t^{(1)}$ is bounded (uniformly in $t \geqslant a$) on S with respect to the norm $\| \cdot \|_{S,p}$.

Next we write $L_t = L_t^{(1)} + L_t^{(2)} + L_t^{(3)}$ where

$$L_t\gamma = \int_{a-1}^{t} \int_s^{s+1} U(t, r)\, \gamma(s, r)\, dr\, ds,$$

$$L_t^{(2)}\gamma = \int_{a-1}^{a} \int_s^{a} U(t, r)\, \gamma(s, r)\, dr\, ds,$$

$$L_t^{(3)}\gamma = \int_t^{t+1} \int_{r-1}^{r} U(t, r)\, \gamma(s, r)\, ds\, dr.$$

We wish to show that L_t, like $L_t^{(1)}$, is bounded (uniformly in $t \geqslant a$) as a functional on S. This will be done by showing $L_t^{(2)}$ and $L_t^{(3)}$ have this property. Using the assumption of uniform stability on R_n and the constant m from part (a) of the Lemma, we obtain

$$\begin{aligned} | L_t^{(3)}\gamma | &\leqslant \int_t^{t+1} \int_{r-1}^{r} | U(t, r)|\, | \gamma(s, r)|\, ds\, dr \\ &\leqslant \int_{t-1}^{t+1} \int_s^{s+1} | U(t, r)|\, | \gamma(s, r)|\, dr\, ds \\ &\leqslant \int_{t-1}^{t+1} \left(\int_s^{s+1} | U(t, r)|^q\, dr\right)^{1/q} dr\, \| \gamma \|_{S\ p} \\ &\leqslant 2m \| \gamma \|_{S,p} . \end{aligned}$$

Also, another use of Hölder's inequality implies

$$| L_t^{(2)}\gamma | \leqslant \int_{a-1}^{a}\int_{s}^{a} | U(t,r)| \, | \gamma(s,r)| \, dr\, ds$$

$$\leqslant \int_{a-1}^{a}\int_{s}^{s+1} | U(t,r)| \, | \gamma(s,r)| \, dr\, ds$$

$$\leqslant m \| \gamma \|_{S,p} .$$

Thus, L_t is bounded (uniformly in $t \geqslant a$) on S.

For each $s \in [a, t]$ there exists a sequence of functions $\gamma_n(s, r) \in L^p[s, s+1]$ with respect to r such that

$$\left(\int_{s}^{s+1} \gamma_n(s,r)^p \, dr\right)^{1/p} = 1,$$

$$\int_{s}^{s+1} U(t,r)\,\gamma_n(s,r)\, dr \to \left(\int_{s}^{s+1} | U(t,r)|^q \, dr\right)^{1/q},$$

as $n \to +\infty$ [10, p. 285]. (That γ_n is continuous in $s \in [a, t]$ is clear from their construction in [10].) Since $C^0[s, s+1]$ is dense in $L^p[s, s+1]$ we may assume γ_n is continuous in $r \in [s, s+1]$. A straightforward application of Lebesgue's dominated convergence theorem yields (since the sequence $\int_s^{s+1} U(t,r)\,\gamma_n(s,r)\,dr$ is by Hölder's inequality bounded uniformly by $m(\int_s^{s+1} | \gamma_n(s,r)|^p \, dr)^{1/p} = m$) the limit

$$L_t\gamma_n \to \int_{a-1}^{t} \left(\int_{s}^{s+1} | U(t,r)|^q \, dr\right)^{1/q} ds,$$

as $n \to +\infty$ for each fixed $t \geqslant a$. Thus, the norm of L_t has the lower bound

$$| L_t | \geqslant \int_{a-1}^{t} \left(\int_{s}^{s+1} | U(t,r)|^q \, dr\right)^{1/q} ds.$$

Inasmuch as the opposite inequality is obvious (from Hölder's inequality) we find that

$$| L_t | = \int_{a-1}^{t} \left(\int_{s}^{s+1} | U(t,r)|^q \, dr\right)^{1/q} ds.$$

But we have shown above that L_t is bounded uniformly in $t \geqslant a$; i.e., $| L_t | \leqslant m(a)$ for some constant $m(a) > 0$ and all $t \geqslant a$. This completes the proof of the lemma.

Proof of Theorem 1. (a) Assume without loss in generality that $t_0 \geqslant 0$.

By assumption the solution of (P) can be extended as a solution so long as it remains bounded by $b > 0$. Let $m = m(a)$ be the larger of the two constants in parts (a) and (d) of the Lemma. Set $g_0 = 1/4m$ and assume $g_p < g_0$. Then $1 - 2g_0 m > 0$. Let $\epsilon > 0$ be given ($\epsilon < b$). By assumption (L) is stable on a space N for the given $a \geqslant t_0$; thus, there exists a constant $\delta_1 = \delta_1(\epsilon, a) > 0$ such that $|f|_N \leqslant \delta_1$ implies

$$| v |_0 \leqslant \epsilon(1 - 2g_p m)/3. \tag{7}$$

Referring to (H4) we may pick $a^* = a^*(\epsilon) \geqslant a + 1$ so large that for $t \geqslant a^* - 1$

$$\left(\int_t^{t+1} g^p(s)\, ds\right)^{1/p} \leqslant 2g_p .$$

The first step in our argument is to prove that for $|f|_N$ sufficiently small the solution $u(t)$ of (P) exists on the interval $[a, a^*]$. Specifically, suppose $|f|_N \leqslant \delta_2 = \delta_2(\epsilon, a)$ where δ_2 is so small that

$$| v |_0 \leqslant \tfrac{1}{2}\epsilon \exp(-mka^*)$$

where k is a constant such that $|g(t)| \leqslant k$ for $t \in [a, a^*]$ (cf. (H4)). Then from (P*) and (H4) we have for $t \geqslant a$

$$\begin{aligned} | u(t)| &\leqslant | v |_0 + \int_a^t | U(t, s)|\, g(s)\, s(u; a)(s)\, ds \\ &\leqslant | v |_0 + mk \int_a^t s(u; a)(s)\, ds. \end{aligned}$$

Thus, replacing t with s and taking the supremum of both sides from a to t we obtain

$$s(u; a)(t) \leqslant | v |_0 + mk \int_a^t s(u; a)(s)\, ds,$$

from which it follows by the well known Gronwall lemma that

$$s(u; a)(t) \leqslant | v |_0 \exp(mk(t - a))$$

and consequently (since $a \geqslant t_0 \geqslant 0$)

$$| u(t)| \leqslant s(u; a)(t) \leqslant | v |_0 \exp(mkt) \leqslant \tfrac{1}{2}\epsilon < \epsilon < b, \tag{8}$$

for as long as $u(t)$ exists on $[a, a^*]$. The continuability property which follows from the assumptions (H1) and (H2) implies that $u(t)$ exists on $[a, a^*]$. We have in fact shown $| u(t)| \leqslant \frac{1}{2}\epsilon$ on $[a, a^*]$ for $|f|_N \leqslant \delta_2$. It also follows that $u(t)$ exists as a solution of (P) locally beyond a^*; and, by continuity, $|u(t)| < \epsilon$ locally beyond a^*.

To finish the argument we will show, by contradiction, that for small $|f|_N$ the solution $u(t)$ actually exists and satisfies $|u(t)| < \epsilon$ for all $t \geqslant a$. Suppose this is not the case and let $T \geqslant a^*$ be the *first* point at which $|u(T)| = \epsilon$. Then $|u(t)|$ and, hence, $s(u; a)(t)$ are both $\leqslant \epsilon$ on $[a, T]$. For $t \in [a^*, T]$ we have

$$u(t) = v(t) + \int_a^{a^*} U(t, s)\, dp^*(s, u(s))\, ds + \int_{a^*}^t U(t, s)\, dp^*(s, u(s))\, ds. \tag{9}$$

Let δ_1 be as above. We can, without loss of generality, assume that the constant $\delta_2 = \delta_2(\epsilon, a)$ above is chosen (smaller, if necessary) so that $|f|_N \leqslant \delta_2$ implies, in addition to (8),

$$|u(t)| \leqslant \epsilon(1 - 2g_p m)(3mka^*)^{-1}, \tag{10}$$

for $t \in [a, a^*]$. Now let $\delta = \delta(\epsilon, a) = \min(\delta_1, \delta_2) > 0$ and suppose $|f|_N \leqslant \delta$. From (9) we obtain the estimate, using (7), (8) and (9),

$$\begin{aligned}
|u(t)| &\leqslant \epsilon(-2g_p m)/3 + m\int_a^{a^*} g(t)\, s(u; a)(s)\, ds \\
&\quad + \int_{a^*}^t \int_{s-1}^s |U(t, s)|\, g(s)\, s(u; a^*)(s)\, dr\, ds \\
&\leqslant \epsilon(1 - 2g_p m)/3 + mk\epsilon(1 - 2g_p m)(3mka^*)^{-1}(a^* - a) \\
&\quad + \int_{a^*-1}^t \int_r^{r+1} |U(t, s)|\, g(s)\, s(u; a)(s)\, ds\, dr \\
&\leqslant 2\epsilon(1 - 2g_p m)/3 + \epsilon \int_{a^*-1}^t \left(\int_r^{r+1} |U(t, s)|^q\, ds\right)^{1/q} \cdot \left(\int_r^{r+1} g(s)^p\, ds\right)^{1/p} dr,
\end{aligned}$$

for $t \in [a^*, T]$. By the manner in which a^* was chosen we have, continuing with the inequalities,

$$|u(t)| \leqslant 2\epsilon(1 - 2g_p m)/3 + 2\epsilon g_0 m = 2\epsilon(1 + g_p m)/3$$

and, hence since $g_p < g_0 = 1/4m$, we find that

$$|u(t)| \leqslant 2\epsilon(1 + 1/4)/3 = 5\epsilon/6,$$

for $t \in [a^*, T]$. Thus, we have arrived at the contradiction $\epsilon = |u(T)| \leqslant 5\epsilon/6$. This proves part (a) of Theorem 1.

(b) If hypothesis (ii) holds for all $a \geqslant t_0$ then, since (i) is true, we see that m in part (d) of the Lemma can be chosen independently of $a \geqslant t_0$. This is because

$$\int_{a-1}^{t} \left(\int_{s}^{s+1} | U(t, r)|^q \, dr\right)^{1/q} ds \leqslant \int_{t_0-1}^{t} \left(\int_{s}^{s+1} | U(t, r)|^q \, dr\right)^{1/q} ds \leqslant m(t_0),$$

for all $t \geqslant a \geqslant t_0$. Consequently in the event that (L) is uniformly stable on a space N the constants δ_1 and δ_2 and hence δ in the preceding proof of part (a) are independent of $a \geqslant t_0$; i.e., (P) is uniformly stable on N.

Proof of Theorem 2. From Theorem 1 we know that both $v(t)$ and $u(t)$ are bounded for $|f|_N \leqslant \delta$. Let $\epsilon > 0$ be arbitrary but fixed. Choose $T = T(\epsilon) > a$ so large that for $t \geqslant T(\epsilon)$

$$\left(\int_{t}^{t+1} g(s)^p \, ds\right)^{1/p} \leqslant \epsilon/k_1 m,$$

where $| u(t)| \leqslant k_1$ for $t \geqslant a$. This is possible since $g_p = 0$. Now for $t \geqslant T(\epsilon)$

$$\begin{aligned} | u(t) - v(t)| &= \left| \int_a^t U(t, s) \, dp^*(s; u(s)) \right| \\ &\leqslant k_1 \int_a^T | U(t, s)| \, g(s) \, ds + k_1 \int_T^t | U(t, s)| \, g(s) \, ds \\ &\leqslant k_1 k \int_a^T | U(t, s)| \, ds + k_1 \int_T^t \int_{s-1}^{s} | U(t, s)| \, g(s) \, dr \, ds, \end{aligned}$$

where $k \geqslant | g(t)|$ for $t \in [a, T]$ (cf. (H4)). Thus

$$\begin{aligned} | u(t) - v(t)| &\leqslant k_1 k \int_a^T | U(t, s)| \, ds + k_1 \int_{T-1}^{t} \int_{r}^{r+1} | U(t, s)| \, g(s) \, ds \, dr \\ &\leqslant k_1 k \int_a^T | U(t, s)| \, ds + k_1 \int_{a-1}^{t} \left(\int_{r}^{r+1} | U(t, s)|^q \, ds\right)^{1/q} (\epsilon/k_1 m) \, dr \\ &\leqslant k_1 k \int_a^T | U(t, s)| \, ds + \epsilon, \end{aligned}$$

for $t \geqslant T(\epsilon)$. Letting $t \to +\infty$ we obtain

$$\limsup_{t \to +\infty} | u(t) - v(t)| \leqslant \epsilon.$$

Whereas $\epsilon > 0$ was arbitrary we conclude that $\lim_{t \to +\infty} | u(t) - v(t)| = 0$.

In conclusion we briefly point out that in the special case that

$p(t, s, z) \equiv K(t, s)\, q(s, z)$, a case treated frequently in the literature, it is possible to obtain other results (independent from those obtainable through Theorems 1 and 2) in exactly the same manner as above except that the starting point is the representation formula

$$u(t) = v(t) - \int_a^t R(t, s)\, q(s, u(s))\, ds,$$

obtained using the resolvent of (L) instead of (P*) obtained from the variation of constants formula (VC) for (L). In this approach R replaces U in the above arguments and we must make the hypothesis

(H5) (L) has a resolvent $R(t, s)$ which is locally in L^0 in (t, s) for $t \geqslant s \geqslant t_0$.

With the obvious changes in the necessary spaces and the obvious modifications of the Lemma (with R in place of U) we can prove, exactly as above, the following theorem.

THEOREM 3. *Suppose* (H1), (H2) *and* (H5) *with* $p(t, s, z) \equiv K(t, s)\, q(s, z)$. *Assume q satisfies*

$$| q(s, z)| \leqslant g(s)\, | z |,$$

for all $s \geqslant a$ *and all* $| z | \leqslant b$, $z \in R^n$, *where* $g(s)$ *is as in* (H4) *for some* p. *Assume R satisfies the two conditions*

(i) $\operatorname{ess\,sup}_{t \geqslant s \geqslant a} | R(t, s)| \leqslant m_1,\ t \geqslant s \geqslant a,$

(ii) $\int_{a-1}^t (\int_s^{s+1} | R(t, r)|^q\, dr)^{1/q}\, ds \leqslant m_2(a),\ t \geqslant a,$

for $1/p + 1/q = 1$. *Then the conclusions* (a) *and* (b) *of Theorem* 1 *hold. If further* $g_p = 0$ *and*

$$\lim_{t \to +\infty} \int_a^T | R(t, s)|\, ds = 0$$

for all $T > a$ *then the conclusion of Theorem* 2 *holds.*

With a slight modification of the statement and proof of this theorem the (rather strong) assumption (i) can be dropped. Toward this end suppose

$$g_p{}^* = \sup_{t \geqslant a} \left(\int_t^{t+1} g(s)^p\, ds \right)^{1/p} < +\infty \tag{11}$$

and that (L) is stable on a space N for $a \geqslant t_0$. Then for a given $\epsilon > 0$ we know $| f |_N \leqslant \delta = \delta(\epsilon, a)$ implies $| v |_0 \leqslant \min(\epsilon/2m_2(a), \epsilon/2)$ and hence $| u(a)| = | v(a)| < \epsilon$. We know then that $u(t)$ exists and satisfies $| u(t)| < \epsilon$

locally beyond a. Let $T > a$ be the first point at which $|u(T)| = \epsilon$. For $t \in [a, T]$ and $|f|_N \leqslant \delta$ we have

$$|u(t)| \leqslant |v|_0 + \int_a^t |R(t,s)|\, g(s)\, |u(s)|\, ds$$

$$\leqslant \epsilon/2 + \int_{a-1}^t \left(\int_r^{r+1} |R(t,s)|^q\, ds\right)^{1/q} \left(\int_r^{r+1} g(s)^p\, ds\right)^{1/p} dr$$

$$\leqslant \epsilon/2 + \epsilon m g_p\, .$$

Thus, if $g_p < 1/2m_2$ we have that $|u(t)| < \epsilon$, $t \in [a, T]$, and the contradiction $\epsilon = |u(T)| < \epsilon$. Thus under this assumption, T cannot exist and $|u(t)| < \epsilon$ for all $t \geqslant a$ if $|f|_N \leqslant \delta$.

THEOREM 4. *Suppose* (H1), (H2) *and* (H5) *with* $p(t, s, z) \equiv K(t, s)\, q(s, z)$ *where* $|q(s, z)| \leqslant g(s)\, |z|$ *for all* $s \geqslant a$ *and* $|z| \leqslant b$, $z \in R^n$ *with* g *as in* (H4). *Suppose* R *satisfies* (ii) *in Theorem* 3. *Then for* $g_p{}^*$ (*as defined in* (11)) *sufficiently small the conclusions* (a) *and* (b) *of Theorem* 1 *hold.*

Suppose further that $g_p = 0$ *and* $\int_a^T |R(t, s)|\, ds \to 0$ *as* $t \to +\infty$ *for all* $T \geqslant a$. *Then the conclusion of Theorem* 2 *holds*

The proof of the last assertion in this theorem is exactly as that of Theorem 2 except that (R) is used instead of (P*).

A theorem midway between Theorems 3 and 4 is possible in which (i) in Theorem 3 is replaced by the assumption that $u \equiv 0$ is the ***unique*** solution of (P) corresponding to $f \equiv 0$ and that $\int_a^T |R(t, s)|\, ds$ is bounded in t on finite intervals. The proof is almost exactly as that of Theorem 2 so the details will not be given. The uniqueness assumption is used to guarantee the continuity of u on finite intervals (namely, on $[a, a^*]$) with respect to f ([9, Chapter II.4]) and consequently allow us to perform the first step in the proof. Simple estimates show that the bound on R (which replaces U in the proof of Theorem 2) can be dispenced with in favor of the added, weaker assumption that $\int_a^T |R(t, s)|\, ds$ be bounded in t on finite intervals.

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