INFINITE-DIMENSIONAL ANALYSIS AND QUANTUM THEORY

Spring 2024—a topics course proposal

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The course will have a mathematical and a physical content. In mathematics, analysis on Wiener space is a natural and useful generalization of ordinary calculus. In quantum theory, Fock spaces are the way to describe physics of systems with variable particle number. Amazingly, the space of square-integrable Wiener functionals is—up to an isomorphism—the same as a certain bosonic Fock space—a deep fact, first proven by Norbert Wiener. The two spaces are points of departure for many beautiful developments. In physics, Fock space is a proper framework to describe second quantizaton, coherent states, superconductivity and many other topics related to several Nobel prizes (for example, Roy Glauber was awarded the 2005 Nobel prize in physics for his work on coherent states and their role in quantum optics). In mathematics, Wiener space analysis led to the development of Malliavin calculus and its profound applications, in particular to partial differential equations. The isomorphism of the two spaces gave birth to the theory of quantum trajectories with its applications to open quantum systems. Using Poisson space instead of Wiener space, extends the range of these applications. It played a crucial role in the interpretation of Schrödinger Cat experiments which won Serge Haroche the 2012 Nobel prize in physics.

This course will be an introduction to these fascinating topics. It is aimed at graduate students in mathematics, physics, optical sciences—and everybody else who is interested in the subject. I will only assume that the audience is familiar with the basics of Hilbert space theory and probability. The topics I am planning to discuss are:

- Gaussian random variables and Gaussian families
- A short introduction to Itô stochastic calculus
- Wiener chaos representation of square-integrable Wiener and Poisson functionals
- Introduction to Malliavin calculus—calculus on the Wiener and Poisson spaces
- Bosonic Fock spaces, exponential and coherent vectors, basic operators
- Mathematics of second quantization, Weyl group
- Canonical commutation relations
- Fermionic Fock spaces
- Introduction to open quantum systems. Quantum trajectories

Course objective and learning goals:

There are two main objectives:

• To present the basics of calculus of functionals of Wiener and Poisson stochastic processes.

• To discuss many-body quantum systems via the Fock space formalism, known as second quantization.

The overarching goal is to show the deep and intimate relation between these two, seemingly unrelated, topics. The students will become thoroughly familiar with Fock spaces and operators on them, which are among the most important object in analysis and mathematical physics. They will also be introduced to the analysis on infinite-dimensional spaces—the Wiener and Poisson spaces. They will learn important applications of second quantization, and the Malliavin's proof of regularity of solutions to partial differential equations satisfying Hörmander condition. The course will provide an introduction to the dynamically developing theory of open quantum systems and quantum trajectories. Open problems and research opportunities will be discussed. The interdependence of mathematical and physical ideas will be stressed throughout.

Literature: I will not use a single text. The main sources will be:

1. Ph. A. Martin, F. Rothen: Many-Body Problems and Quantum Field Theory. Springer 2004

2. D. Nualart, E. Nualart: Introduction to Malliavin Calculus. Cambridge University Press 2018

3. K. R. Parthasarathy: An Introduction to Quantum Stochastic Calculus. Birkhaüser 1992

4. S. Haroche, J.-M. Raimond: Exploring the quantum: atoms, cavities and photons. Oxford University Press 2006

Please contact me if you have any questions about the course.

Best regards

Janek Wehr