1. BASIC CONCEPTS

Where We Are Going and Why
We will develop ways of analyzing ordinary differential equations that take full advantage of the power of calculus and technology. We do this by treating topics from graphical, numerical, and analytical points of view. Because each of these points of view can at times give incomplete information, we always need to compare our results for consistency. The strategies presented in this book will empower you to correctly analyze solutions of a differential equation, even when those solutions are not obtainable as analytical expressions.

In this chapter, we introduce these three points of view by considering differential equations of the form

$$\frac{dy}{dx} = g(x)$$

We start with a brief introduction, definitions, and examples, which make use of prior knowledge of antiderivatives. Then we spend the next two sections illustrating graphical techniques, including slope fields and isoclines. These techniques often let us discover many properties of the solution of a specific differential equation by simply analyzing the differential equation from a graphical point of view. We end this chapter with a discussion of Taylor series, which are especially useful when solutions are given by integrals that do not have simple antiderivatives.

The purpose of this chapter is to illustrate graphical, numerical, and analytical approaches in the familiar setting of antiderivatives. We want to make sure you have a firm foundation in these approaches so you can quickly grasp the new ideas in subsequent chapters.

1.1 Simple Differential Equations and Explicit Solutions

Ever since the ideas of calculus were developed by Newton, Leibniz, and others in the seventeenth century, people have been using differential equations to describe many phenomena that touch our lives. Differential equations are the most common mathematical tool used for the precise formulation of the laws of nature and other phenomena described by a relationship between a function and its derivatives. In this book you will see many examples of such relationships.

The simple definition of a differential equation is an equation that involves a derivative. Thus, many differential equations are solved in beginning calculus courses, perhaps without anyone stating it.

= Two Basic Examples

We start with a familiar example.

Example 1 : Parabolas

Consider the problem of finding the most general antiderivative of the function x. If we call this antiderivative $y(x)$, then the derivative of y is x; that is, y satisfies the differential equation

$$\frac{dy}{dx} = x \tag{1.1}$$

Because any antiderivative of x may be written as

$$y(x) = \frac{1}{2}x^2 + C \tag{1.2}$$

where C is an arbitrary constant, it seems reasonable to call (1.2) solutions of our differential equation (1.1). Because of the arbitrary constant, we have an infinite number of solutions, a different one for each choice of C (collectively called a family of solutions). Some of these solutions are graphed in Figure 1.1 (upward opening parabolas) where we notice that the role of the arbitrary constant is to determine the vertical position. All solutions of this differential equation have the same general shape, and any two solutions will differ from each other by a vertical translation. Thus, two different solutions will not intersect.[1] □

Vertical translation

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Figure 1.1 Some solutions of $dy \quad dx = x$

We now look at another familiar example, one to which we will return frequently in this chapter.

Example 2 : The Natural Logarithm Function

[1] If two functions $y_1(x)$ and $y_2(x)$ have a point x_0 in common, so that $y_1(x_0) = y_2(x_0)$, we say that $y_1(x)$ and $y_2(x)$ intersect, touch, or cross, at $x = x_0$.

We consider the problem of finding the most general antiderivative, $y(x)$, of the function $1\ \ x$ that is, finding solutions of the differential equation

$$\frac{dy}{dx} = \frac{1}{x} \tag{1.3}$$

Because all antiderivatives of $1\ \ x$ may be written as

$$y(x) = \ln\ x\ + C \tag{1.4}$$

or

$$y(x) = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases} \tag{1.5}$$

where C is an arbitrary constant, these are our solutions. Again, because of the arbitrary constant, we have a family of solutions. Some of these solutions are graphed in Figure 1.2, where we notice again that the role of the arbitrary constant is to determine the vertical position. If $x > 0$ all solutions of this differential equation have the same general shape, and any two solutions will differ from each other by a vertical translation. The same is true for $x < 0$. □

Vertical translation

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Some solutions of $dy\ \ dx = 1\ \ x$

Figure 1.2

= Definitions and Comments

These are two of many examples of differential equations covered in calculus. All problems where we found the indefinite integral (or antiderivative) of a function, $g(x)$, could have been formulated as finding $y(x)$ as a solution of

$$\frac{dy}{dx} = g(x) \tag{1.6}$$

The solutions of (1.6) all have the form

$$y(x) = \int g(x)\,dx + C \tag{1.7}$$

where $\int g(x)\,dx$ is any specific antiderivative of $g(x)$.[2] The arbitrary constant C indicates that we have an infinite number of solutions, related to each other by a vertical translation.

Vertical translation

Before developing any methods for finding solutions of differential equations, we give some formal definitions which will be helpful later when we consider more complicated situations.

Differential equations such as (1.1), (1.3), and (1.6) are called first order differential equations, because the first derivative is the highest one that occurs in each equation. Thus we have

Definition 3 : A FIRST ORDER ORDINARY DIFFERENTIAL EQUATION **is an equation that involves at most the first derivative of an unknown function. If y, the unknown function, is a function of x, then we write the first order differential equation as**

$$\frac{dy}{dx} = g(x\ y) \tag{1.8}$$

where $g(x\ y)$ is a given function of the two variables x and y.[3]

Comments about First Order Differential Equations

- The right-hand side of (1.8) may contain x and y explicitly, for example, x^2+y^2. However, in this chapter we will consider the case where $g(x\ y)$ is a function of x alone.
- If y is a function of x then x is called the independent variable and y is called the dependent variable.

We previously noted that (1.2), (1.4), and (1.7) are solutions of (1.1), (1.3), and (1.6), respectively. We know this because if we differentiate these functions and substitute the result into the proper differential equation, we obtain an identity. These solutions are called explicit because they have the dependent variable, y, given solely in terms of the independent variable, x. This prompts our second definition.

Definition 4 : An EXPLICIT SOLUTION **of the first order ordinary differential equation**

$$\frac{dy}{dx} = g(x\ y) \tag{1.9}$$

is any function $y = y(x)$ with a derivative in some interval $a < x < b$, that identically satisfies the differential equation (1.9).

Comments about Explicit Solutions

- Because an explicit solution has a derivative in the interval $a < x < b$, it must be continuous in that interval. (Why?) An explicit solution can never have a vertical tangent. (Why?)

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[2] In calculus it is customary to have the symbol $\int g(x)\,dx$ include the arbitrary constant C. Here we add the constant explicitly to emphasize geometrical ideas.

[3] Even though all our examples in this chapter are of the form $dy\ dx = g(x)$, we use $dy\ dx = g\,(x\ y)$ in all our definitions so that they also apply to subsequent chapters.

- An explicit solution may contain an arbitrary constant. If it does, we have an infinite number of solutions, called a FAMILY OF EXPLICIT SOLUTIONS. If it does not contain an arbitrary constant, we have a PARTICULAR EXPLICIT SOLUTION. Often particular explicit solutions are just called particular solutions.

- The graph of a particular solution is called a SOLUTION CURVE of the differential equation.

The explicit solutions of the three differential equations mentioned so far all contain an arbitrary constant, so they are families of explicit solutions. This constant may be determined if we know the value of the solution at some specific value of x.

Examples of explicit solutions

Thus, on the one hand, if we specify that the solution (1.4), $y(x) = \ln\ x + C$, of the differential equation (1.3) must pass through the point $(-1\ 0)$, the value of the constant C must satisfy $0 = \ln\ -1 + C$, so $C = 0$. This appears to give the particular solution $\ln\ x$. However, the graph of $\ln\ x$ consists of two disconnected branches (one with $x < 0$, and the other with $x > 0$) and so is not continuous, whereas any particular solution must be continuous. Because our initial point $(-1\ 0)$ is given on the left branch of $\ln\ x$, the particular solution that passes through $(-1\ 0)$ is $\ln\ x$ on the interval $- \quad < x < 0$; that is, $\ln(-x)$. Its graph is shown in Figure 1.3.

On the other hand, if we specify that the solution of (1.3) is to pass through the point $(e\ 6)$, the value of C must satisfy $6 = \ln\ e + C$, so $C = 5$. With similar reasoning to that just used, the particular solution passing through $(e\ 6)$ is $\ln\ x + 5$ on the interval $0 < x <$; that is, $\ln x + 5$. Its graph is also shown in Figure 1.3. If we look at Figure 1.2 through these eyes, we see that it represents 14 particular solutions, not 7, as a cursory glance might indicate.[4]

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Graphs of particular solutions of $dy\ dx = 1\ x$ through $(-1\ 0)$ and $(e\ 6)$

Figure 1.3

[4]A function that merely satisfies a differential equation is sometimes called an **integral** of the differential equation. A solution is an integral that is also continuous. Thus, $\ln\ x + C$ is an integral of $dy\ dx = 1\ x$ but is not a solution. Figure 1.2 shows 7 integral curves and 14 solution curves.

The problem of finding a solution of a differential equation that must pass through a given point such as $(-1\ 0)$ or $(e\ 6)$ in the case of (1.3) is called an INITIAL VALUE PROBLEM, and the point is called an INITIAL VALUE, INITIAL CONDITION, or INITIAL POINT.

Initial condition

Writing Solutions As Integrals

There is another way of expressing the solution of (1.3) when an initial point is specified as $(x_0\ y_0)$. If we use the fact that $\int_{x_0}^{x} g(t)\,dt$ is an antiderivative of $g(x)$, our solution in (1.4) may also be expressed as

$$y(x) = \int_{x_0}^{x} \frac{1}{t}\,dt + C \tag{1.10}$$

if x_0 and x have the same sign. If we substitute $x = x_0$ into (1.10), use the initial condition $y(x_0) = y_0$ and the fact that $\int_{x_0}^{x_0} 1\ \ t\,dt = 0$ (if $x_0 = 0$), we obtain the value of C as $C = y_0$. Thus (1.10) can be written as

$$y(x) = \int_{x_0}^{x} \frac{1}{t}\,dt + y_0$$

Important point

From this example we see that the explicit solution of the initial value problem $\frac{dy}{dx} = g(x)\ \ y(x_0) = y_0$, can be written in the form

$$y(x) = \int_{x_0}^{x} g(t)\,dt + y(x_0) = \int_{x_0}^{x} g(t)\,dt + y_0 \tag{1.11}$$

if $g(t)$ is bounded for t between x_0 and x. This form of the solution is particularly useful when we are unable to evaluate the integral in terms of familiar functions. We demonstrate this in the following example.

Example 5 : The Error Function

An important function, used extensively in applications in probability theory and diffusion processes, is the solution of the differential equation

$$\frac{dy}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{1.12}$$

subject to the initial condition that

$$y(0) = 0 \tag{1.13}$$

This example will recur throughout this chapter.

Explicit solutions of (1.12) can be written as

$$y(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2}\,dx + C \tag{1.14}$$

The usual way to evaluate the constant C so (1.13) is satisfied is to substitute $x = 0$ and $y = 0$ into the solution (1.14) and solve for C. However, the integral in (1.14) cannot be expressed in terms of familiar functions, so this usual way to evaluate C does not work. To bypass this problem we change the form of our solution to the one given in (1.11) and use the fact that $x_0 = 0$ and $y_0 = 0$ to obtain

$$y(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

This explicit solution is called the Error Function, and it is usually denoted by $erf(x)$; that is,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad (1.15)$$

We might ask how we can determine the graph of this function from its form in (1.15).[5] One way would be to construct a table of values of $(x \;\; erf(x))$ by using a numerical method of approximating the integral for specific choices of x (see Exercise 2 on page 7). However, numerical techniques require considerable computation to plot enough points to be confident of the shape of the graph (see Exercises 3 and 4 on page 8). For that reason, in the next sections we develop methods for obtaining qualitative properties of the graph of the solution of our differential equation directly from the differential equation. □

Exercises

1. Solve each of the following first order differential equations.[6] Sketch the explicit solution for three different values of the arbitrary constant C. Then find the specific value of C and the formula for $y(x)$ giving the particular explicit solution that passes through the given point P.

(a)	$dy \;\; dx = x^3$	$P = (1 \;\; 1)$		(g)	$dy \;\; dx = 1 \;\; x$	$P = (-1 \;\; 1)$
(b)	$dy \;\; dx = x^4$	$P = (1 \;\; 1)$		(h)	$dy \;\; dx = 1 \;\; (1 + x^2)$	$P = (1 \;\; \pi \;\; 4)$
(c)	$dy \;\; dx = \cos x$	$P = (0 \;\; 0)$		(i)	$dy \;\; dx = 1 \;\; [x(1-x)]$	$P = (2 \;\; 1)$
(d)	$dy \;\; dx = \sin x$	$P = (\pi \;\; 2)$		(j)	$dy \;\; dx = \ln x$	$P = (1 \;\; 1)$
(e)	$dy \;\; dx = e^{-x}$	$P = (0 \;\; 1)$		(k)	$dy \;\; dx = x^2 e^{-x}$	$P = (0 \;\; 1)$
(f)	$dy \;\; dx = 1 \;\; x^2$	$P = (1 \;\; 1)$		(l)	$dy \;\; dx = e^{-x} \sin x$	$P = (0 \;\; 1)$

2. **The Error Function.** The purpose of this exercise is to graph the Error Function defined by (1.15), namely,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

by constructing a table of values of $(x \;\; erf(x))$.

   (a) What is the value of $erf(0)$?

   (b) What is the relationship between $erf(x)$ and $erf(-x)$?

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[5]You might also ask how we were able to draw the graph of the function $y = \ln x$ in the previous example. This was done by a computer/calculator. So how did the computer/calculator do it? It constructed a table of numerical values. If the computer/calculator had the ability to construct functions of the sort $\int_0^x g(t)\, dt$, we could use it to graph $erf(x)$. Not many computers/calculators have this facility built in.

[6]The expression solve the differential equation is synonymous with find the explicit solution of the differential equation.

(c) Use a computer/calculator program that performs numerical integration to obtain approximate values (say to 3 decimal places) for $erf(x)$ at $x = 1 \;\; 2$, and 3. Use this information, and the results from parts (a) and (b), to plot $erf(x)$ in the interval $[-3 \;\; 3]$. How confident are you that the graph you have is fairly accurate?

(d) Now repeat part (c) for $x = 0\ 5$, $1\ 5$, and $2\ 5$. Did this change the accuracy of your previous graph for $erf(x)$?

(e) Now repeat part (c) for $x = 0\ 25$, $0\ 75$, $1\ 25$, and $1\ 75$. Did this change the accuracy of your previous graph for $erf(x)$?

(f) What do you think happens to $erf(x)$ as x ?

3. **The Fresnel Sine Integral.** Using (1.11), write down an integral that represents the solution of the initial value problem

$$\frac{dy}{dx} = \sqrt{\frac{2}{\pi}} \sin x^2 \qquad y(0) = 0$$

This solution, known as the Fresnel Sine Integral and denoted by $S(x)$, cannot be expressed in terms of familiar functions.

(a) What is the value of $S(0)$?

(b) What is the relationship between $S(x)$ and $S(-x)$?

(c) Use a computer/calculator program that performs numerical integration to obtain approximate values (say to 3 decimal places) for $S(x)$ at $x = 2$ and 4. Use this information, and the results from parts (a) and (b), to plot $S(x)$ in the interval $[-5 \;\; 5]$. How confident are you that the graph you have is fairly accurate?

(d) Now repeat part (c) for $x = 1$, 3, and 5. Did this change the accuracy of your previous graph for $S(x)$?

(e) Now repeat part (c) for $x = 0\ 5$, $1\ 5$, $2\ 5$, $3\ 5$ and $4\ 5$. Did this change the accuracy of your previous graph for $S(x)$?

(f) What do you think happens to $S(x)$ as x ?

4. **The Sine Integral.** Using (1.11), write down an integral that represents the solution of the initial value problem $\frac{dy}{dx} = g(x) \;\; y(0) = 0$ where

$$g(x) = \begin{cases} \sin x \;\; x & \text{if } x = 0 \\ 1 & \text{if } x = 0 \end{cases}$$

This solution, known as the Sine Integral and denoted by $Si(x)$, cannot be expressed in terms of familiar functions. Use the ideas from Exercise 3 to graph the solution of this differential equation. What do you think happens to $Si(x)$ as x ?

5. Find the family of solutions for each of the differential equations $\frac{dy}{dx} = e^x$ and $\frac{dy}{dx} = -e^{-x}$ Graph these two families of solutions on one plot, using the same scale for the x- and y-axes. What do you notice about the angle of intersection between these two families of solutions?[7] Could you have seen that directly from the differential equations, without solving them?

[7] The angle of intersection between two curves at a point is the angle between the tangent lines to the curves at that point.

6. Write down some odd functions and find their antiderivatives.[8] What property do these antiderivatives share? Make a conjecture that starts: *The antiderivative of an odd function is always*   . Prove your conjecture.

1.2 Graphical Solutions Using Calculus

In the previous section we used antiderivatives to determine the behavior of solutions of $dy \;\; dx = g(x)$ by finding the explicit solution. In this section we discover that there is a wealth of information available about the behavior of such solutions by considering the differential equation itself without finding the explicit solution.

Example 6 : The Natural Logarithm Function

We return to the second example,

$$y \;\; = \frac{dy}{dx} = \frac{1}{x} \tag{1.16}$$

where   indicates differentiation with respect to x. Look at the 14 particular solutions in Figure 1.2. They were sketched directly from the functions $\ln x + C$ for $x > 0$, and $\ln(-x) + C$ for $x < 0$, for different values of C. Based on Figure 1.2, we ask the following questions:

1. *Monotonicity.*[9] Where are the solutions increasing and where are they decreasing?
2. *Concavity.* Where are the solutions concave up and where are they concave down?
3. *Symmetry.* Are there any symmetries?
4. *Singularities.*[10] Is it possible to start on a solution curve where $x < 0$ and proceed along this curve and eventually arrive at positive values of x?
5. *Uniqueness.* Do any solutions intersect?

Based on the graphs in Figure 1.2, the answers to these questions seem to be:

1. *Monotonicity.* Decreasing for $x < 0$ and increasing for $x > 0$.
2. *Concavity.* Concave down for $x = 0$.

[8] Odd functions have the property that $f(-x) = -f(x)$ for all values of x in the domain of f.

[9] A function is monotonic on an interval if it is either increasing on the entire interval or decreasing on the entire interval.

[10] A function $f(x)$ is singular at $x = a$ if $\lim_{x \quad a} f(x)$ does not exist.

3. *Symmetry.* Yes across the y-axis. However, no particular solution has this symmetry. **It is the family of solutions that has this symmetry**.

4. *Singularities.* Not for any particular solution if the y-axis is a vertical asymptote, as it appears to be.

5. *Uniqueness.* On this graph, the answer appears to be yes near the y-axis.

Now imagine that we are unable to integrate (1.16) explicitly, and so we are unable to draw the particular solutions in Figure 1.2. How much of this information (monotonicity, concavity, symmetry, singularities, and uniqueness) can we obtain directly from the differential equation (1.16) using our knowledge of calculus?

From calculus we know that $y > 0$ on an interval requires that y be increasing on that interval, and $y < 0$ means that y is decreasing. We also know that $y > 0$ on an interval requires the function to be concave up on that interval, whereas $y < 0$ means the function is concave down. In Figure 1.4 we show the general shapes of solution curves for the four cases: $y < 0$, $y > 0$; $y > 0$, $y > 0$; $y > 0$, $y < 0$; and $y < 0$, $y < 0$.

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Possible shapes of solution curves determined by the first and second derivatives

Figure 1.4

With this information, let s return to our original questions and try to justify these answers.

1. *Monotonicity.* From (1.16) we see that the derivative of $y(x)$ is positive for $x > 0$, and so y increases there. Similar reasoning shows that y decreases when $x < 0$.

2. *Concavity.* If we differentiate (1.16) with respect to x, we have

$$y = \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

which is negative for $x = 0$. Thus, the solutions must be concave down for all $x = 0$.

3. *Symmetry.* To be symmetric across the y-axis means that we have no change in the family of solutions if x is replaced by $-x$ on both sides of (1.16). If in (1.16) we replace x by $-x$ we have

$$\frac{dy}{-dx} = \frac{1}{-x}$$

or $y = 1 \; x$, which is exactly (1.16). Thus, the family of solutions that satisfies (1.16) is unchanged by the interchange of x by $-x$, and so must be symmetric across the y-axis.

4. *Singularities.* Because (1.16) is undefined at $x = 0$, we anticipate problems at $x = 0$.

5. *Uniqueness.* The statement that two solutions intersect means that there is a common point $(x_0 \; y_0)$ through which two distinct particular solutions of (1.16), say $y_1(x)$ and $y_2(x)$, pass. Because both y_1 and y_2 are solutions of (1.16), we must have $y_1 = 1 \; x$ and $y_2 = 1 \; x$, so that $y_1 = y_2$, or $(y_1 - y_2) = 0$ From this we have $y_1(x) - y_2(x) = C$ The fact that $y_0 = y_1(x_0)$ and $y_0 = y_2(x_0)$ requires that $C = 0$, so that $y_1(x) = y_2(x)$. In other words, the two curves $y_1(x)$ and $y_2(x)$ are one and the same. This means that only one solution of (1.16) can pass through any point $(x_0 \; y_0)$. Another way of saying this is that a solution of the differential equation (1.16) that passes through any given point is unique. Consequently, contrary to our conjecture based on Figure 1.2, solutions do not intersect. In fact, this argument can be used on any differential equation of the form $y = g(x)$ to show that their solutions cannot intersect (see Exercise 7 on page 13). □

Caution! **Exercise care when drawing conclusions from graphical analysis about whether curves intersect.**

From the preceding analysis we see that just by using calculus we can obtain much qualitative information about solution curves without knowing the explicit solution. Let s see how we can use this to sketch the Error Function, *erf*(x), by going through the checklist we have developed.

Example 7 : The Error Function *Using the techniques of calculus, sketch the family of solutions of the differential equation*

$$y = \frac{2}{\overline{\pi}} e^{-x^2} \tag{1.17}$$

1. *Monotonicity.* The derivative of y is always positive, so all solutions are increasing.

2. *Concavity.* If we differentiate (1.17) with respect to x we find $y = -\frac{4}{\overline{\pi}} x e^{-x^2}$ From this we see that $y > 0$ when $x < 0$, and $y < 0$ when $x > 0$. Thus, all solutions are concave up when $x < 0$ and concave down when $x > 0$.

3. *Symmetry.* If we replace x with $-x$ on both sides of (1.17), the right-hand side is unchanged but the left-hand side changes sign. So the family of solutions is not symmetric across the y-axis. However, if we simultaneously replace x with $-x$ and replace y with $-y$, then we obtain (1.17) back again. So the family of solutions of (1.17) is unchanged under simultaneous interchange of x with $-x$ and y with $-y$. This means that the family of solutions is symmetric about the origin.[11]

4. *Singularities.* There are no obvious points where the derivative fails to exist.

5. *Uniqueness.* From arguments similar to those at the end of Example 6 on page 9, we see that solutions cannot intersect.

Based on this qualitative information, we can sketch by hand the family of solutions of (1.17), which is shown in Figure 1.5.[12] This family of solutions contains the particular solution curve that passes through the point P with coordinates (0 0) namely, *erf* (x). □

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Some hand-drawn solution curves of $y\ = 2e^{-x^2}\ \pi^{1\ 2}$

Figure 1.5

Exercises

1. Use monotonicity, concavity, symmetry, singularities, and uniqueness to sketch various solution curves for each of the following first order differential equations. Then draw the particular solution curve that passes through the point P. When you have finished, compare your

[11] Recall that a graph is symmetric about the origin if it is unchanged when rotated 180^o about the origin.

[12] Throughout the text we make reference to hand-drawn solutions. Of course, they were drawn by machine.

answers with those you found for Exercise 1, Section 1.1.

(a)	y	$=$	x^3	$P = (1\ \ 1)$	(g)	y	$=$	$1\ \ x$	$P = (-1\ \ 1)$
(b)	y	$=$	x^4	$P = (1\ \ 1)$	(h)	y	$=$	$1\ \ (1 + x^2)$	$P = (1\ \ \pi\ \ 4)$
(c)	y	$=$	$\cos x$	$P = (0\ \ 0)$	(i)	y	$=$	$1\ \ [x(1 - x)]$	$P = (2\ \ 1)$
(d)	y	$=$	$\sin x$	$P = (\pi\ \ 2)$	(j)	y	$=$	$\ln x$	$P = (1\ \ 1)$
(e)	y	$=$	e^{-x}	$P = (0\ \ 1)$	(k)	y	$=$	$x^2 e^{-x}$	$P = (0\ \ 1)$
(f)	y	$=$	$1\ \ x^2$	$P = (1\ \ 1)$	(l)	y	$=$	$e^{-x} \sin x$	$P = (0\ \ 1)$

2. Intuition suggests that if $g(x)$ 0 as x then the solutions of the differential equation $y = g(x)$ will have horizontal asymptotes. Explain why this suggestion might be plausible. Explain why this suggestion is wrong.

3. For the differential equation

$$y = \frac{4}{x\,(x-4)}$$

find the explicit solution satisfying the initial condition (a) $y(-1) = 0$, (b) $y(1) = 0$, (c) $y(5) = 0$.

4. For what values of a and x_0 is the solution of the initial value problem

$$y = \frac{1}{x\,(x-a)} \qquad y(x_0) = 0$$

valid for all $x > 0$?

5. **The Fresnel Sine Integral.** Use monotonicity, concavity, symmetry, singularities, and uniqueness to sketch various solution curves for the differential equation $y = \sqrt{\frac{2}{\pi}} \sin x^2$ Then draw the graph of the particular solution that satisfies $y(0) = 0$ What do you think happens to $y(x)$ as x ? Compare your answers with the one you found for Exercise 3, Section 1.1.

6. **The Sine Integral.** Use monotonicity, concavity, symmetry, singularities, and uniqueness to sketch various solution curves for the differential equation $y = g(x)$ where

$$g(x) = \begin{cases} \sin x\ \ x & \text{if } x = 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then draw the graph of the particular solution that satisfies $y(0) = 0$ What do you think happens to $y(x)$ as x ? Compare your answers with the one you found for Exercise 4, Section 1.1.

7. **The Uniqueness Theorem**. Show that if $y_1(x)$ and $y_2(x)$ are solutions of the initial value problem $y = g(x)$, $y(x_0) = y_0$, where $g(x)$ is continuous, then $y_1(x) = y_2(x)$. How does this guarantee that different solutions of the differential equation $y = g(x)$ cannot intersect?

8. Show that the family of antiderivatives of an even function is symmetric about the origin.[13] Under what conditions will an antiderivative of an even function be an odd function? Give some examples.

[13]Even functions have the property that $f(-x) = f(x)$ for all values of x in the domain of f.

1.3 Slope Fields and Isoclines

In the previous section we saw that using the techniques of calculus, we can determine much qualitative information about solutions of $y = g(x)$ from the signs of the first and second derivatives. However, there is still more information contained in the differential equation, because, in addition to the signs, it also gives us the magnitude of the slope at each point on a solution curve. In this section we exploit this information.

What Are Slope Fields?

Example 8 : Lines

We start with a very simple differential equation that describes the function whose rate of change is always 1, namely,

$$y = 1 \tag{1.18}$$

Let s use our knowledge of calculus to sketch some solution curves of (1.18). Because the right-hand side of (1.18) is positive (namely, 1), we know that the solutions of (1.18) are increasing everywhere. From (1.18) we also know that for all values of x and y, the solution of this differential equation has a tangent line whose slope is 1. To transfer this information to a graph we can select various coordinates $(x\ y)$ and draw short line segments with slope 1, as shown in Figure 1.6.

From calculus we know that a differentiable function may be approximated near a point on the curve by its tangent line at that point. Another way of stating this is that each tangent segment gives the slope of the solution of a differential equation at that point. Such a collection of short line segments is known as a SLOPE FIELD of the differential equation, as it gives a short segment of the tangent line to the solution curve at each selected point.[14]

Slope field

We now construct a solution curve such that the tangent lines to this curve are consistent with the slope field. If we try to draw a curve whose tangent line has the slope 1 everywhere, we will end up drawing a straight line with slope 1. In fact the solution curves of (1.18) are the family of straight lines $y = x + C$. □

How to Sketch the Slope Field for $y = g(x\ y)$
Purpose: To sketch the slope field for $y = g(x\ y)$

Process

1. Select a rectangular window in the xy-plane in which to view the slope field.
2. Subdivide this rectangular region into a grid of equally spaced points $(x\ y)$. The number of points in the x and y directions may be different.

[14]Slope fields are sometimes called direction fields.

empty

Slope field for $y \ = 1$

Figure 1.6

3. At each of these points $(x\ y)$, find the numerical value of $g(x\ y)$ and draw a short line segment at $(x\ y)$ with slope $g(x\ y)$.

Comments about Slope Fields

- All the slope fields in this book were computer generated.
- **From now on we assume you either have access to a computer/calculator program that displays slope fields or are willing to construct slope fields by hand.**

Example 9 : Parabolas

Now consider the differential equation that models the situation where the rate of change of the unknown function is equal to the value of the independent variable,

$$y \ = x \tag{1.19}$$

This is the first example we considered in Section 1.1.

Monotonicity, concavity, symmetry

Again using our knowledge of calculus, we see that (1.19) tells us that solutions $y = y(x)$ increase if $x > 0$ and decrease if $x < 0$. Moreover, because $y \ \ = 1$, the second derivative of y is always positive, so y is concave up everywhere. Finally, if we replace x by $-x$ in (1.19), the differential equation remains unchanged, so the family of solutions is symmetric across the y-axis.

Singularities and uniqueness

Because the right-hand side of (1.19) is defined for all values of x there are no singularities. As shown in Exercise 7 on page 13, all solutions of differential equations of the form $y \ = g(x)$, $y(x_0) = y_0$ are unique if $g(x)$ is continuous, so distinct solutions of (1.19) do not intersect.

Slope field

If we construct the slope field as shown in Figure 1.7, we can obtain more information: we have a zero slope when $x = 0$, and the slopes of the short line segments of the slope field increase as x increases. Notice that the slope field appears to be symmetric across the y-axis.

Because the slope field for a differential equation gives the inclination of the tangent line to solutions at many points, the graph of any particular solution of this differential equation must be consistent with these tangent lines. Notice that the solution curves of (1.19) have horizontal tangents for $x = 0$, positive slopes for $x > 0$, and negative slopes for $x < 0$. Also note that these slopes become larger as x increases.

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Slope field for $y \ = x$

Figure 1.7

Drawing solution curves

To manually draw a solution curve on the graph of the slope field, we start at some point; for instance, where we have already drawn a short tangent line. Because the solution will be a differentiable function, its graph is well approximated by its tangent line near every point. Thus, we may proceed in the direction given by this tangent line for a short distance to the right and then see what the slope field looks like there. We then adjust the direction of our curve so it changes in a manner consistent with this slope field.

To show this for a specific case in Figure 1.7, consider the solution curve that passes through the point $(0\ 1)$. As we move to the right from this point, the curve changes from being horizontal in such a manner that the slope is continually increasing. This gives the curve labeled A shown in Figure 1.8. (Note that this results in a curve that is concave up.) Figure 1.8 also shows some other hand-drawn solution curves (all of which are parabolas), each having a different y-intercept. □

How to Manually Sketch Solution Curves from the Slope Field for $y \ = g(x\ y)$

Purpose: To sketch, by hand, solution curves from the slope field for

$$y \ = g(x\ y) \tag{1.20}$$

Process

1. Sketch the slope field for (1.20).
2. Start at some initial point and put a dot there. If this dot lies on a short line segment, you have the slope of your solution curve at that point. If

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Hand-drawn family of solution curves for $y = x$

Figure 1.8

not, estimate the value of the slope of the tangent line at that point by looking at nearby slopes. This gives the direction of the slope field at that point.

3. Proceed in this direction for a short distance to the right. Place a dot at the point where you finish.

4. Adjust your direction so it is consistent with the direction of the slope field in the vicinity of the point where you finished.

5. Repeat steps 3 and 4, as often as needed, joining the dots with a curve.

6. Start with a new initial point, and return to step 2.

Example 10 : The Natural Logarithm Function

Once more we return to the differential equation

$$y = \frac{1}{x} \tag{1.21}$$

Based on our previous analysis we know we have a concave down, decreasing shape for $x < 0$, and a concave down, increasing shape for $x > 0$. We also know the family of solutions will be symmetric across the y-axis.

Slope field

We now draw the slope field for (1.21) (see Figure 1.9) and from it confirm the major properties of its solution. We see from Figure 1.9 that the solution curves consistent with this slope field are increasing for $x > 0$ and decreasing for $x < 0$ and that all the slopes above a specific x location are equal. Figure 1.10 shows a few solution curves drawn on this slope field. Note that the solution curves are concave down everywhere, and that the slope field appears to be symmetric across the y-axis. Also note that the solution curves appear to be vertical translations of each other. You should measure the vertical distances between the curves to verify this conjecture. □

Vertical translation

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Slope field for $y \; = 1 \; x$

Figure 1.9

empty

Hand-drawn solution curves and slope field for $y \; = 1 \; x$

Figure 1.10

Now that we have some solution curves on the slope field in Figure 1.10, it is apparent that **we can think of a slope field as what remains after plotting many solution curves and then erasing parts of them, leaving only some short segments here and there that look like straight lines**. In this sense, the challenge in finding solution curves from a slope field is to fill in the gaps between the short line segments of the slope field. This is a major use of slope fields   namely, to determine the graph of a solution of a differential equation whether or not an explicit solution is readily obtainable in terms of familiar functions. Slope fields are also used to check consistency with your findings concerning monotonicity and concavity.

Example 11 : The Error Function

We already have an example where we needed to draw a solution curve of an initial value problem without having an explicit solution in terms of familiar functions   namely, the initial value problem that generates the Error Function,

$$y = \frac{2}{\overline{\pi}} e^{-x^2} \qquad y(0) = 0 \tag{1.22}$$

Slope field

Because we want the solution of this equation that starts at the point (0 0), we construct the slope field that includes this point, as shown in Figure 1.11.

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Slope field for $y = 2e^{-x^2} \ \pi^{1 \ 2}$

Figure 1.11

Monotonicity, concavity, symmetry

As expected, the slope field indicates that the solution curve that passes through (0 0) is increasing everywhere, concave up when $x < 0$ and concave down when $x > 0$. Also notice that the slope field appears to be symmetric about the origin. Figure 1.12 shows a hand-drawn solution curve for (1.22) that passes through the origin, the graph of $y = erf(x)$.

We could also use a numerical integration technique to obtain values for $erf(x)$ at different values of x from its definition, namely,

$$erf(x) = \frac{2}{\overline{\pi}} \int_0^x e^{-t^2}\, dt$$

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Hand-drawn graph of $y = erf(x)$

Figure 1.12

For example, we used Simpson s rule on this integral to create Table 1.1.[15] (Here we set the number of subintervals to 16 and rounded the answers to three decimal places.) Figure 1.13 shows the slope field, these numerical values, and a hand-drawn solution curve. Notice the agreement between this solution curve and a plot of these numerical values. □

Table 1.1 Simpson s rule for $erf(x)$

x	$y(x)$
0.0	0.000
0.5	0.520
1.0	0.843
1.5	0.966
2.0	0.995

Caution!

Slope fields can sometimes be misleading **see Exercise 8 on page 27.** **We must make sure that any conclusions drawn from slope fields are confirmed by other means.** One way is to make use of the analytical and graphical techniques learned from the previous two sections, where the first and second derivatives of a function give us information about the function itself.

= What Are Isoclines?

We now exploit the slope field concept in a different way, to obtain additional properties of solution curves directly from the differential equation.

Example 12 : The Natural Logarithm Function

We start by reconsidering the differential equation

$$y = \frac{1}{x} \tag{1.23}$$

[15]See a calculus text to remind yourself of Simpson s rule.

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Numerical values and hand-drawn graph of $y = erf(x)$

Figure 1.13

Slope field

which has the slope field shown in Figure 1.9. Our objective is to show that this slope field is reasonable.

In calculus, a first step in plotting the graph of a function was to compute its derivative, but (1.23) supplies the derivative of all the solution curves without any further work. In calculus we then set the derivative equal to 0 to find horizontal tangents. Because the derivative in this case, 1 x is never equal to 0, no solution curve has a horizontal tangent.

Even though there are no points on any of our solution curves that have a horizontal tangent, we can set the derivative equal to another constant and see for what value (or values) of x the solution curves would have that constant slope. For example, because 1 $x = 1$ for $x = 1$, the short line segments will have a slope of 1 at all points on the slope field where x equals 1. In general we can say that solution curves will have a slope equal to m whenever

$$\frac{1}{x} = m$$

that is,

$$x = \frac{1}{m}$$

which is a vertical line through the point (1 m 0).

Isoclines

This vertical line is called an ISOCLINE (equal inclination) of the differential equation (1.23). All solution curves of (1.23) will have the same slope, m, as they cross the isocline at this value of x. For example, the solution curves of (1.23) will have a slope of 1 when $x = 1$, a slope of 1 2 when $x = 2$, a slope of 2 when $x = 1$ 2 a slope of -1 when $x = -1$, a slope of -1 2 when $x = -2$, and a slope of -2 when $x = -1$ 2. We can see that Figure 1.14 is consistent with this information, which shows isoclines for $m = \pm 1$ 2 and $m = \pm 1$. To make sure you understand isoclines, add the isoclines for $m = 2$ and $m = -2$ to this figure. Is there an isocline for $m = 0$?

We hope that with this additional information you feel very confident in drawing solution curves consistent with the slopes, the isoclines, the monotonicity, and the concavity we have determined (Figure 1.15). □

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Isoclines ($m = \pm 1\ \ 2, \pm 1$) and slope field for $y\ = 1\ \ x$

Figure 1.14

empty

Hand-drawn solution curves and slope field for $y\ = 1\ \ x$

Figure 1.15

Definition 13 : An ISOCLINE CORRESPONDING TO SLOPE m **of the differential equation** $y \ = g(x \ y)$ **is the curve characterized by the equation** $g(x \ y) = m$.

Comments about Isoclines

- For any particular m, the isocline corresponding to slope m may consist of more than one curve.
- An isocline corresponding to slope m is also called an isocline for slope m.
- If $g(x \ y)$ does not include y, isoclines are vertical lines.
- In the general case where $g(x \ y)$ depends on both x and y, isoclines may not be lines. For example, if $g(x \ y) = x^2 + y^2$ then the isoclines $x^2 + y^2 = m$ are circles centered at the origin with radius $\overline{m}$.

How to Sketch Isoclines for $y \ = g(x \ y)$
Purpose: To sketch isoclines for $y \ = g(x \ y)$

Process

1. Set
$$g(x \ y) = m \tag{1.24}$$
where m is constant.

2. Pick several different values for m. For each m try to solve (1.24) for y in terms of x and m, or for x in terms of y and m. If this cannot be done, try to identify the curves defined implicitly by (1.24). This gives the isocline corresponding to slope m.

3. For each value of m, plot the isocline corresponding to slope m.

4. If you are constructing slope fields by hand, draw short line segments with slope m crossing the appropriate isocline.

Example 14 : The Error Function

Now we return to the differential equation giving rise to the Error Function,

$$y \ = \frac{2}{\overline{\pi}} e^{-x^2}$$

Isocline

The isocline corresponding to slope m is given by

$$\frac{2}{\overline{\pi}} e^{-x^2} = m \tag{1.25}$$

Notice that this guarantees that there are no isoclines for slope $m \leq 0$ or for slope $m > 2 \quad \overline{\pi} \approx 1\ 128$. (Why?) If we solve (1.25) for x we obtain $x = \pm \ \ln[2 \ (m \ \overline{\pi})]^{\ 1 \ 2}$ as the equation of the isocline corresponding to

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Isoclines ($m = 0\ 1$, $0\ 3$, $0\ 7$) and slope field for $y\ = 2e^{-x^2}\ \pi^{1\ 2}$

Figure 1.16

slope m. The isoclines for slope 0 1, 0 3, and 0 7 are shown in Figure 1.16. Notice that in this case each isocline consists of two vertical lines. □

To summarize results found to this point, we gather together some **general observations about differential equations of the special form** $y\ = g(x)$

Summary

- All solutions are explicit.
- Once we have found one member of the family of solutions, other members of the family can be generated from this member by vertical translations.
- If $y\ = g(x)$ remains unchanged after the interchange of x by $-x$, then the family of solutions is symmetric across the y-axis.
- If $y\ = g(x)$ remains unchanged after the simultaneous interchange of y by $-y$ and x by $-x$, then the family of solutions is symmetric about the origin.
- For the case $y\ = g(x)$ all isoclines are vertical lines; that is, parallel to the y-axis.

Exercises

1. Sketch the slope field for each of the following first order differential equations. In each case draw some isoclines to confirm your sketch. Use your sketch to draw various solution curves. Then draw the solution curve that passes through the point P. When you have finished, compare your answers with those you found for Exercise 1, Section 1.1, and

Exercise 1, Section 1.2.

(a)	y	$=$	x^3	$P = (1\ 1)$	(g)	y	$=$	$1\ x$	$P = (-1\ 1)$
(b)	y	$=$	x^4	$P = (1\ 1)$	(h)	y	$=$	$1\ (1 + x^2)$	$P = (1\ \pi\ 4)$
(c)	y	$=$	$\cos x$	$P = (0\ 0)$	(i)	y	$=$	$1\ [x(1-x)]$	$P = (2\ 1)$
(d)	y	$=$	$\sin x$	$P = (\pi\ 2)$	(j)	y	$=$	$\ln x$	$P = (1\ 1)$
(e)	y	$=$	e^{-x}	$P = (0\ 1)$	(k)	y	$=$	$x^2 e^{-x}$	$P = (0\ 1)$
(f)	y	$=$	$1\ x^2$	$P = (1\ 1)$	(l)	y	$=$	$e^{-x} \sin x$	$P = (0\ 1)$

2. Explain why it is useful to plot isoclines corresponding to an infinite slope, even though no point on a solution curve can have a vertical tangent.

3. **The Fresnel Sine Integral**. Use slope fields and isoclines for the differential equation $y\ = \sqrt{\frac{2}{\pi}} \sin x^2$ to draw various solution curves. Then draw the solution curve that satisfies $y(0) = 0$ What do you think happens to $y(x)$ as x ? Compare your answers with those you found for Exercise 3, Section 1.1, and Exercise 5, Section 1.2.

4. **The Sine Integral.** Consider the differential equation $y\ = g(x)$ where

$$g(x) = \begin{cases} \sin x\ \ x & \text{if } x = 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Use slope fields and isoclines to draw various solution curves. Then draw the solution curve that satisfies $y(0) = 0$ What do you think happens to $y(x)$ as x ? Compare your answers with those you found for Exercise 4, Section 1.1, and Exercise 6, Section 1.2.

5. Figure 1.17 shows one member of a family of solutions of the differential equation $y\ = g(x)$ where $g(x)$ is a given function.

   (a) Use this information to plot other members of the family of solutions. Do not attempt to find $y(x)$ or $g(x)$.

   (b) Can every solution of the differential equation be obtained by the technique used in part (a)?

6. Figures 1.18 and 1.19 are mystery slope fields, believed to be the slope fields for two of the following differential equations:

$$y\ = \frac{x^2+1}{x^2-1} \qquad y\ = \frac{x^2-1}{x^2+1} \qquad y\ = -\frac{x^2+1}{x^2-1} \qquad y\ = -\frac{x^2-1}{x^2+1}$$

   (a) Identify to which differential equation each of the mystery slope fields belongs. Confirm all the information using calculus and isoclines. Do not plot any slope fields to answer this question.

   (b) Now superimpose the two mystery slope fields, perhaps by placing one on top of the other and holding both up to the light. [Another way to do this is to plot the slope fields for

$$y\ = a\frac{x^2+1}{x^2-1} + b\frac{x^2-1}{x^2+1}$$

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Figure 1.17 Graph of $y(x)$

empty

Figure 1.18 Mystery slope field 1

empty

Figure 1.19 Mystery slope field 2

with $a = \pm 1$ and $b = 0$, and then with $a = 0$ and $b = \pm 1$.] What do you notice? Would this have made your previous analysis in part (a) easier?

7. Consider the following four differential equations:

$$y = x + 1 \qquad y = x - 1 \qquad y = \ln x \qquad y = x^2 - 1$$

   (a) The slope field of one of the preceding equations is given in Figure 1.20. Match the correct equation with the figure, carefully stating your reasons. Do not plot any slope fields to answer this question.
   (b) Briefly outline a general strategy for this matching process.

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Figure 1.20 Identify the slope field

8. Use a computer/calculator program to sketch the slope field for $y = 1 + \cos(1000x)$ in the window $-10 < x < 10$, $-10 < y < 10$. Use your sketch to draw the solution curve that passes through the point (0 0). Now solve the differential equation and find the formula for the particular explicit solution that passes through the point (0 0). Plot this solution on top of your previous sketches, and comment on what you see. What lesson can be learned from this exercise?

9. If an object falls out of an airplane, its downward velocity after x seconds is often crudely approximated by

$$y = \frac{g}{k}\left(1 - e^{-kx}\right)$$

where $g = 9\ 8$ m/sec^2 and $k = 0\ 2$ sec^{-1}. Here $y(x)$ is the distance fallen at time x so $y(0) = 0$. If this object falls from 5000 meters above the ground, estimate how many seconds it falls before it hits the ground, by

   (a) using slope fields, monotonicity, isoclines, and concavity, and
   (b) finding the explicit solution.

What Have We Learned?

Main Ideas

- A FIRST ORDER ORDINARY DIFFERENTIAL EQUATION is an equation that involves at most the first derivative of an unknown function. If y, the unknown function, is a function of x, then we write the first order equation as
$$\frac{dy}{dx} = g(x \; y)$$
where $g(x \; y)$ is a given function of x and y.

- An EXPLICIT SOLUTION of
$$\frac{dy}{dx} = g(x \; y) \tag{1.26}$$
is any function $y = y(x)$ (differentiable in some interval $a < x < b$) that identically satisfies the differential equation (1.26).

- If an explicit solution contains an arbitrary constant, the infinite number of solutions it generates is called a FAMILY OF EXPLICIT SOLUTIONS. If an explicit solution contains no arbitrary constant, it is called a PARTICULAR EXPLICIT SOLUTION.

- The graph of a particular solution is called a SOLUTION CURVE of the differential equation.

- Looking for a solution of a differential equation that must pass through a given point is called an INITIAL VALUE PROBLEM, and the point is called an INITIAL VALUE, INITIAL CONDITION, or INITIAL POINT.

- There is one and only one solution of any differential equation of the form $y \ = g(x)$ that passes through a given point $(x_0 \; y_0)$ if $g(x)$ is continuous. See the Uniqueness Theorem on page 13.

- To sketch slope fields, see *How to Sketch the Slope Field for* $y \ = g(x \; y)$ on page 14.

- To hand-draw solution curves from slope fields, see *How to Manually Sketch Solution Curves from the Slope Field for* $y \ = g(x \; y)$ on page 16.

- An ISOCLINE CORRESPONDING TO SLOPE m of the differential equation $y \ = g(x \; y)$ is the curve characterized by the equation $g(x \; y) = m$.

- To sketch isoclines, see *How to Sketch Isoclines for* $y \ = g(x \; y)$ on page 23.

- When the explicit solution is left in the form of a definite integral, it is frequently possible to put it in an alternative form. This is done by expanding the integrand using Taylor series, and then integrating the result term by term. See page **??**.

Words of Caution

- Exercise care when drawing conclusions from graphical analysis about whether curves intersect.

- Make sure that any conclusions drawn from one technique are confirmed by other techniques.