Nonuniformly Expanding 1D Maps

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The simplest paradigms of chaotic behavior in dynamical systems are found in uniformly expanding and uniformly hyperbolic (or Anosov) maps. Allowing expanding and contracting behaviors to mix leads to a multitude of new possibilities. In spite of much progress, the analysis of most nonuniformly hyperbolic systems has remained hopelessly difficult. One-dimensional maps are an exception. The situation in 1 dimension is made tractable by the fact that the worst enemy of expansion is the critical set, *i.e.*, the set on which f' vanishes, and for typical 1D maps, this set is finite. It has been shown that by controlling the orbits starting from this finite set, the dynamics on the rest of the phase space can be tamed.

A nonuniform theory for 1D maps was developed in a series of papers in the late 1970s and 1980s ([J], [M], [CE], [BC1], [R], [BC2], [NS], and others). These ideas are later exploited in the study of attractors with a single direction of instability, beginning with the Hénon maps ([BC2], [BY] etc.) and culminating recently in a general theory of rank-one attractors that can live in phase spaces of arbitrary dimensions [WY2]. In the course of these developments, some of the original 1D arguments have been extended and improved. This paper is written in response to numerous requests from the dynamical systems community to make more accessible a certain body of ideas in 1 dimension, both for its independent interest and as an introduction to the study of rank-one maps in higher dimensions.

The content of this paper can be summarized as follows: Let I be a closed interval or a circle, and let $C^2(I,I)$ denote the set of C^2 maps from I to itself. We seek to identify a reasonably large class of maps $\mathcal{G} \subset C^2(I,I)$ with controlled nonuniform expansion, and to give a description of its dynamical properties. This is carried out in the following 3 steps:

(1) First we identify a set $\mathcal{M} \subset C^2(I,I)$ defined by strong expanding conditions.

(2) Our class of "good maps" $\mathcal{G} \subset C^2(I,I)$ is obtained by relaxing these conditions. Maps in $\mathcal{G}$ are shown to have absolutely continuous invariant measures.

(3) We show that $\mathcal{G}$ is "large" in the sense that for every typical 1-parameter family $\{f_a\}$ passing through $\mathcal{M}$, the set $\{a : f_a \in \mathcal{G}\}$ has positive Lebesgue measure.

We first cite the main references directly related to (1)–(3): The class $\mathcal{M}$ is a slight generalization of the maps studied in [M]. In the special case of the quadratic family $f_a(x) = 1 - ax^2$, the existence of absolutely continuous invariant measures for a positive measure set of parameters is the well known theorem of Jakobson [J]; for other proofs of Jakobson's theorem, see [BC1], [BC2] and [R]. A key idea used in [BC1] and [BC2], namely the exponential growth of derivatives along critical orbits, is first introduced in [CE]. An

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analysis along the lines of (1)–(3) above for unimodal maps was carried out in [TTY]. A version of Jakobson's theorem for multimodal maps is given in [T].

While the results of this paper as stated have not been published before, we do not claim that the ideas of the proofs are new. In outline, our proofs follow those in [BC1] and Section 2 of [BC2]. The generalization from $f_a(x) = 1 - ax^2$ to more general maps is along the lines of [TTY]. We have also borrowed heavily from [WY1] and especially [WY2], both in terms of setting and the way in which the arguments are carried out. More detailed references are given at the end of each section.

Organization of paper The class $\mathcal{M}$ in (1) above is discussed in Section 1; the class $\mathcal{G}$ is introduced in Section 2. The result on invariant measures (Theorem 1) is stated and proved in Section 3. The result on positive measure sets of parameters (Theorem 2) is stated in Sect. 4.1 and proved in Sections 4–7.

Part I DYNAMICAL PROPERTIES

1 The Class $\mathcal{M}$

1.1 Definition and expanding property

For $f \in C^2(I, I)$, let $C = C(f) = \{f' = 0\}$ denote the critical set of f, and let C_δ denote the δ-neighborhood of C in I. For $x \in I$, let $d(x, C) := \min_{\hat{x} \in C} |x - \hat{x}|$.

Definition 1.1 *We say $f \in C^2(I, I)$ is in the class $\mathcal{M}$ if the following hold for some $\delta_0 > 0$:*
(a) Outside of C_{δ_0}: *there exist $\lambda_0 > 0, M_0 \in \mathbb{Z}^+$ and $0 < c_0 \leq 1$ such that*
(i) for all $n \geq M_0$, if $x, f(x), \cdots, f^{n-1}(x) \notin C_{\delta_0}$, then $|(f^n)'(x)| \geq e^{\lambda_0 n}$;
(ii) if $x, f(x), \cdots, f^{n-1}(x) \notin C_{\delta_0}$ and $f^n(x) \in C_{\delta_0}$, any n, then $|(f^n)'(x)| \geq c_0 e^{\lambda_0 n}$.
(b) Inside C_{δ_0}: *(i) $f''(x) \neq 0$ for all $x \in C_{\delta_0}$;*
(ii) for all $\hat{x} \in C$ and $n > 0$, $d(f^n(\hat{x}), C) \geq \delta_0$;
(iii) for all $x \in C_{\delta_0} \setminus C$, there exists $p_0(x) > 0$ such that $f^j(x) \notin C_{\delta_0}$ for all $j < p_0(x)$ and $|(f^{p_0(x)})'(x)| \geq c_0^{-1} e^{\frac{1}{3}\lambda_0 p_0(x)}$.

Remark 1. The maps in $\mathcal{M}$ are among the simplest with nonuniform expansion: The phase space is divided into two regions, C_{δ_0} and $I \setminus C_{\delta_0}$. Condition (a) in Definition 1.1 says that on $I \setminus C_{\delta_0}$, f is essentially uniformly expanding. (b)(iii) says that for $x \in C_{\delta_0} \setminus C$, even though $|f'(x)|$ is small, the orbit of x does not return to C_{δ_0} again until its derivative has regained a definite amount of exponential growth; *i.e.*, if n is the first return time of $x \in C_{\delta_0}$ to C_{δ_0}, then $|(f^n)'(x)| \geq e^{\frac{1}{3}\lambda_0 n}$. (To see this, use (b)(iii) followed by (a)(ii)).

Remark 2. We identify two properties of the critical orbits of $f \in \mathcal{M}$ that will serve as the basis of the generalization in Section 2. Let $\hat{x} \in C$.

(1) $d(f^n(\hat{x}), C) \geq \delta_0$ for all $n > 0$, *i.e.*, (b)(ii) in Definition 1.1. (This condition is redundant and is included solely for emphasis; it follows from (b)(iii) together with the observation that $p_0(x) \to \infty$ as $d(x, C) \to 0$.)

(2) $|(f^n)'(f(\hat{x}))| \geq c_0' e^{\lambda_0 n}$ for all $n > 0$ where $c_0' = (\max|f'|)^{-M_0}$. This follows from (b)(ii) and (a)(i).

We record for future use the following important fact about the behavior of $f \in \mathcal{M}$ outside of C_δ for arbitrary $\delta < \delta_0$:

Lemma 1.1 *There exists $c_0'' > 0$ depending only on f such that for all $\delta < \delta_0$ and $n > 0$:*
(a) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$, then $|(f^n)'(x)| \geq c_0'' \delta e^{\frac{1}{3}\lambda_0 n}$;
(b) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$ and $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geq c_0 e^{\frac{1}{3}\lambda_0 n}$.

Proof: Let x be such that $f^i(x) \notin C_\delta$ for $i \in [0, n)$. We divide $[0, n]$ into maximal time intervals $[i, i+k]$ such that $f^{i+j}(x) \notin C_{\delta_0}$ for $0 < j < k$, and estimate $|(f^k)'(f^i(x))|$ as follows:

Case 1. $f^i(x), f^{i+k}(x) \in C_{\delta_0}$. $|(f^k)'(f^i(x))| \geq e^{\frac{1}{3}\lambda_0 k}$ by Definition 1.1(a)(ii) and (b)(iii).

Case 2. $f^i(x) \notin C_{\delta_0}, f^{i+k}(x) \in C_{\delta_0}$. The estimate is given by Definition 1.1(a)(ii).

Case 3. $f^i(x), f^{i+k}(x) \notin C_{\delta_0}$. If $k \geq M_0$, then $|(f^k)'(f^i(x))| > e^{\lambda_0 k}$ by Definition 1.1(a)(i). If $k < M_0$, we let $\hat{k}$ be the smallest integer $> k$ such that $f^{i+\hat{k}}(x) \in C_{\delta_0}$. Using Definition 1.1(a)(i) for $\hat{k} \geq M_0$ and Definition 1.1(a)(ii) for $\hat{k} < M_0$, we conclude that $|(f^k)'(f^i(x))| > c_0(\max|f'(x)|)^{-M_0} e^{\lambda_0 k}$.

Case 4. $f^i(x) \in C_{\delta_0}, f^{i+k}(x) \notin C_{\delta_0}$. As in Case 3, with extra factor $(\min_{y \in C_{\delta_0}} |f''(y)|)\, \delta$.

Cases 3 and 4 are relevant only for part (a); each appears at most once in the estimate on $|(f^n)'(x)|$. ☐

In the interest of carrying as few constants around as possible, we write $c_1 = \min\{c_0, c_0', c_0''\}$.

1.2 Examples

Example 1. Let $f \in C^3(I, I)$ be such that
(i) $S(f) < 0$ where $S(f)$ denotes the Schwarzian derivative of f,[3]
(ii) $f''(\hat{x}) \neq 0$ for all $\hat{x} \in C$,
(iii) if $f^n(x) = x$, then $|(f^n)'(x)| > 1$, and
(iv) for all $\hat{x} \in C$, $\inf_{n>0} d(f^n(\hat{x}), C) > 0$.
Then $f \in \mathcal{M}$. For a proof of this fact, see Lemma 2.5 of [WY1].

We note that (i) and (ii) above are satisfied by all members of the quadratic family $f_a(x) = 1 - ax^2, a \in (0, 2]$, and (iii) and (iv) are satisfied by an uncountable number of a including $a = 2$.

Example 2. Another situation where maps in $\mathcal{M}$ arise naturally is through scaling. The following is a slight generalization of Lemma 5.3 in [WY3] and has the same proof: Let $f_a : S^1 \to S^1$ be given by

$$f_a(\theta) \ = \ \theta \ + \ a + \ L\Phi(\theta)$$

[3]We have elected to replace this condition by an explicit description of the dynamics in Definition 1.1 because (1) that is exactly what is used and (2) we have found that maps that arise in applications often do not have negative Schwarzian derivative.

where $a, L \in \mathbb{R}$ and $\Phi : S^1 \to S^1$ is an arbitrary function with nondegenerate critical points (and the right side is to be read mod 1). Then there exists $L_0 > 0$ such that for all $L \geq L_0$, there exists an $\mathcal{O}(\frac{1}{L})$-dense set of a for which $f_a \in \mathcal{M}$.

Reference: Maps of the type in Example 1 are introduced and studied in [M]. Maps of the type in Example 2 appear naturally in [WY3] and [WY4].

2 The Class $\mathcal{G}$: 3 Basic Properties

Condition (b)(ii) in Definition 1.1 severely limits the scope of $\mathcal{M}$ as a subset of $C^2(I, I)$. We now introduce in a neighborhood of each $f_0 \in \mathcal{M}$ an admissible set of perturbations $\mathcal{G}(f_0)$. Our set of "good maps" $\mathcal{G}$ is then defined to be $\bigcup_{f_0 \in \mathcal{M}} \mathcal{G}(f_0)$.

Throughout this section, let $f_0 \in \mathcal{M}$ be fixed, and let δ_0, λ_0, c_1 *etc.* be the constants in Sect. 1.1 associated with f_0.

2.1 Definition of $\mathcal{G}(f_0)$ and basic properties

For $\lambda, \alpha, \varepsilon > 0$ and $f \in C^2(I, I)$, we say $f \in \mathcal{G}(f_0; \lambda, \alpha, \varepsilon)$ if $\|f - f_0\|_{C^2} < \varepsilon$ and the following hold for all $\hat{x} \in C = C(f)$ and $n > 0$:

(G1) $d(f^n(\hat{x}), C) > \min\{\frac{1}{2}\delta_0, e^{-\alpha n}\}$;
(G2) $|(f^n)'(f(\hat{x}))| \geq c_1 e^{\lambda n}$.

Note that with $\lambda < \lambda_0$, (G1) and (G2) are relaxations of the conditions on critical orbits for f_0 (see Remark 2 in Sect. 1.1). The main result of this section is

Proposition 2.1 *Given $f_0 \in \mathcal{M}$, $\lambda < \frac{1}{4}\lambda_0$ and $\alpha < \frac{1}{100}\lambda$, there exists $\delta = \delta(f_0, \lambda, \alpha)$ and $\varepsilon = \varepsilon(f_0, \lambda, \alpha, \delta) > 0$ such that (P1)-(P3) below hold for all $f \in \mathcal{G}(f_0; \lambda, \alpha, \varepsilon)$.*

Here $\delta < \frac{1}{2}\delta_0$ is an auxiliary constant. For simplicity, we assume ε is small enough that $d(f^n(\hat{x}), C) > \frac{1}{2}\delta_0$ for all $\hat{x} \in C$ and $1 \leq n \leq n_0$ where n_0 is a large integer satisfying $e^{-\alpha n_0} << \delta$. Consequently, (G1) can be violated only when $f^n(\hat{x}) \in C_\delta$. Precise requirements on δ and ε will become clear in the proofs. In general, $\varepsilon << \delta << 1$. The arguments are perturbative; some of them will require that ε be taken very close to 0.

The set $\mathcal{G}(f_0)$ is defined to be the union of $\mathcal{G}(f_0; \lambda, \alpha, \varepsilon)$ as $(\lambda, \alpha, \varepsilon)$ ranges over all triples satisfying the conditions in Proposition 2.1.

We now state (P1)–(P3), introducing some useful language along the way.

(P1) Outside of C_δ: (i) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$, then $|(f^n)'(x)| \geq c_1 \delta e^{\frac{1}{4}\lambda_0 n}$;
(ii) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$ and $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geq c_1 e^{\frac{1}{4}\lambda_0 n}$.

Let $\hat{x} \in C$, and let $C_\delta(\hat{x}) := (\hat{x} - \delta, \hat{x} + \delta)$. For $x \in C_\delta(\hat{x}) \setminus \{\hat{x}\}$, we define $p(x)$, the **bound period** of x, to be the largest integer such that $|f^i(x) - f^i(\hat{x})| \leq e^{-2\alpha i}$ $\forall i < p(x)$.

(P2) Partial derivative recovery for $x \in C_\delta \setminus C$: For $x \in C_\delta(\hat{x}) \setminus \{\hat{x}\}$,
(i) $\frac{1}{3\ln(\max|f'|)} \log \frac{1}{|x - \hat{x}|} \leq p(x) \leq \frac{3}{\lambda} \log \frac{1}{|x - \hat{x}|}$;

(ii) $|(f^{p(x)})'(x)| > e^{\frac{\lambda}{3}p(x)}$.

(P2) leads to the following general description of orbits:

Decomposition into "bound" and "free" states: For $x \in I$ such that $f^i(x) \notin C$ for all $i \geq 0$ (for example, $x = f(\hat{x})$ for $\hat{x} \in C$), let

$$t_1 < t_1 + p_1 \leq t_2 < t_2 + p_2 \leq \cdots$$

be defined as follows: t_1 is the smallest $j \geq 0$ such that $f^j(x) \in C_\delta$. For $k \geq 1$, let p_k be the bound period of $f^{t_k}(x)$, and let t_{k+1} be the smallest $j \geq t_k + p_k$ such that $f^j(x) \in C_\delta$. (Note that an orbit may return to C_δ during its bound periods, i.e. t_i are not the only return times to C_δ.) This decomposes the orbit of x into segments corresponding to time intervals $(t_k, t_k + p_k)$ and $[t_k + p_k, t_{k+1}]$, during which we describe the orbit of x as being in "bound" and "free" states respectively; t_k are called times of *free returns.*

(P3) is about comparisons of derivatives for nearby orbits. To state what it means for two points to be close to each other, we introduce a partition $\mathcal{P}$ on I. First let $\mathcal{P}_0 = \{I_{\mu j}\}$ be the following partition on $(-\delta, \delta)$: Assume $\delta = e^{-\mu_*}$ for some $\mu_* \in \mathbb{Z}^+$. For $\mu \geq \mu_*$, let $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$; for $\mu \leq -\mu_*$, let I_μ be the reflection of $I_{-\mu}$ about 0. Each I_μ is further subdivided into $\frac{1}{\mu^2}$ subintervals of equal length called $I_{\mu j}$. For $\hat{x} \in C$, let $\mathcal{P}_0^{\hat{x}}$ be the partition on $C_\delta(\hat{x})$ obtained by shifting the center of $\mathcal{P}_0$ from 0 to $\hat{x}$. The partition $\mathcal{P}$ is defined to be $\mathcal{P}_0^{\hat{x}}$ on $C_\delta(\hat{x})$; on $I \setminus C_\delta$, its elements are intervals of length $\approx \delta$.

The following shorthand is used: We refer to $\pi \in \mathcal{P}$ corresponding to (translated) $I_{\mu j}$ intervals in $\mathcal{P}_0^{\hat{x}}$ simply as "$I_{\mu j}$". For $\pi \in \mathcal{P}$, π^+ denotes the union of π and the two elements of $\mathcal{P}$ adjacent to it. For an interval $\gamma \subset I$, we say $\gamma \approx \pi$ if $\pi \subset \gamma \subset \pi^+$. For practical purposes, π^+ intersecting ∂C_δ can be treated as "inside C_δ" or "outside C_δ".[4] For $\gamma \subset I_{\mu j}^+$, we define the bound period of γ to be $p(\gamma) = \min_{x \in I_{\mu j}^+}\{p(x)\}$. For $x, y \in I$, $[x, y]$ denotes the segment connecting x and y. We say x and y in I have *the same itinerary* (with respect to $\mathcal{P}$) *through time* $n-1$ if there exist $t_1 < t_1 + p_1 \leq t_2 < t_2 + p_2 \leq \cdots \leq n$ such that for every k, $f^{t_k}[x, y] \subset \pi^+$ for some $\pi \subset C_\delta$, $p_k = p(f^{t_k}[x, y])$, and for all $i \in [0, n) \setminus \cup_k [t_k, t_k + p_k)$, $f^i[x, y] \subset \pi^+$ for some $\pi \in \mathcal{P}$ with $\pi \cap C_\delta = \emptyset$.

(P3) Distortion estimate: There exists $K_0 > 1$ (depending only on f_0 and on λ) such that if x and y have the same itinerary through time $n-1$, then

$$\left|\frac{(f^n)'(x)}{(f^n)'(y)}\right| \leq K_0.$$

(P1)-(P3) are proved in next subsection. We finish by recording the following corollary of Proposition 2.1.

Corollary 2.1 *There exists K_1 (depending only on f_0 and on λ) such that for all $x \in I$ with $f^i(x) \notin C$ for all $0 \leq i < n$,*

$$|(f^n)'(x)| > K_1^{-1} d(f^j(x), C)\ e^{\frac{1}{4}\lambda n}$$

[4] In particular, if π is one of the outermost $I_{\mu j}$ in C_δ, then π^+ contains an interval of length δ.

where j is the time of the last free return before n. The factor $d(f^j(x), C)$ may be replaced by δ if $f^n(x)$ is free.

Proof: Let $0 \le t_1 < t_1 + p_1 \le t_2 < t_2 + p_2 \le \cdots$ be as in the paragraph following (P2). The derivatives on time intervals $[t_k, t_k + p_k)$ and $[t_k + p_k, t_{k+1})$ are given by (P2)(ii) and (P1)(ii) respectively, provided these intervals are completed before time n. We assume δ is sufficiently small so that the constant c_1 in (P1)(ii) is absorbed into the exponential estimate from the proceeding bound period. If $f^n(x)$ is in a bound period initiated at time j, then $|(f^{(n-j)})'(f^j(x))| \ge |f'(f^j(x))| K_0^{-1} c_1 e^{\lambda(n-j-1)}$; see (G2) and (P3). If $t_k + p_k \le n < t_{k+1}$ for some k, then the derivative between time $t_k + p_k$ and n is given by (P1)(i). $\square$

Remarks on the use of constants: In this article, $K, K_1, K_2, \cdots$, are reserved for use as *system constants*, which in Part I are constants that are allowed to depend only on (1) f_0, by which we included also the constants in Sect. 1.1 associated with f_0, and (2) our choice of λ. The more important of these constants, such as K_0 in (P3), carry a subscript; all others are referred to by the generic name K. The value of K, therefore, may vary from expression to expression.

2.2 Proofs of (P1)–(P3)

Proof of (P1): First we deduce from Lemma 1.1(a) that there exists $N = N(\delta)$ such that for all $y \in I$, if $y, f_0(y), \cdots, f_0^{N-1}(y) \notin C_{\frac{1}{2}\delta}(f_0)$, then $|(f_0^N)'(y)| > e^{\frac{1}{3.5}\lambda_0 N}$. We then choose ε small enough that f is sufficiently close to f_0 for N iterates in the sense below:

(i) if x and n are as in (P1) and $n \le N$, then $|(f^n)'(x) - (f_0^n)'(x)|$ is small enough that the conclusions of (P1) follow from Lemma 1.1;

(ii) if $f^i(y) \notin C_\delta$ for $0 \le i < N$, then $|(f^N)'(y)| > e^{\frac{1}{4}\lambda_0 N}$.

If n in (P1) is $> N$, we let k be such that $kN \le n < (k+1)N$, and estimate $|(f^n)'(x)|$ by the chain rule, comparing $(f^N)'(f^{iN}(x))$ with $(f_0^N)'(f^{iN}(x))$ for $i \le k$ using (ii) above, and $(f^{n-kN})'(f^{kN}(x))$ with $(f_0^{n-kN})'(f^{kN}(x))$ using (i). $\square$

Lemma 2.1 *The following holds if δ and ε are sufficiently small and suitably related: Let $\hat{x} \in C$, and let $x \in C_\delta(\hat{x})$. Then for all $y \in [\hat{x}, x]$ and $k < p(x)$,*

$$\frac{1}{2} \le \frac{(f^k)'(y_1)}{(f^k)'(\hat{x}_1)} \le 2.$$

Proof: First,

$$\log \frac{(f^k)'(y_1)}{(f^k)'(\hat{x}_1)} \le \sum_{j=1}^{k} \frac{|f'(y_j) - f'(\hat{x}_j)|}{|f'(\hat{x}_j)|} \le K \sum_{j=1}^{k} \frac{|y_j - \hat{x}_j|}{d(\hat{x}_j, C)}.$$

We choose h_0 large enough that (i) $\sum_{i>h_0}^{\infty} e^{-\alpha j} << 1$ and (ii) $e^{-\alpha h_0} < \frac{1}{2}\delta_0$ (so that $d(\hat{x}_j, C) > e^{-\alpha j}$ for $j > h_0$). Next we choose δ small enough that $\delta \sum_{j=1}^{h_0} \frac{2}{\delta_0} (\max |f'|)^j << 1$. Finally, let ε be small enough that $d(\hat{x}_j, C) > \frac{1}{2}\delta_0$ for all $j \le h_0$. Then

$$\sum_{j=1}^{k} \frac{|y_j - \hat{x}_j|}{d(\hat{x}_j, C)} < \sum_{j=1}^{h_0} \frac{2}{\delta_0} (\max |f'|)^j \delta + \sum_{j=h_0+1}^{k} \frac{e^{-2\alpha j}}{e^{-\alpha j}} << 1.$$

$\square$

Proof of (P2): Suppose $|x-\hat{x}|=e^{-h}$. Then (G2) together with Lemma 2.1 implies that

$$|x_p-\hat{x}_p| \geq \frac{1}{2}|(f^{p-1})'(\hat{x}_1)||x_1-\hat{x}_1| \geq K^{-1}e^{\lambda(p-1)}(x-\hat{x})^2.$$

From this we deduce that $p<\frac{3}{\lambda}h$, assuming that h is sufficiently large (or δ is sufficiently small). The lower bound on p is obtained by comparing the inequalities $|x_p-\hat{x}_p| < (\max|f'|)^{p-1}e^{-2h}$ and $|x_p-\hat{x}_p| \geq e^{-2\alpha p}$ (definition of bound period).

To prove (P2)(ii), taking square root of $|x_p-\hat{x}_p| < K|(f^{p-1})'(\hat{x}_1)|(x-\hat{x})^2$, we obtain

$$K|(f^{p-1})'(\hat{x}_1)|^{\frac{1}{2}}\ |x-\hat{x}| > e^{-\alpha p}. \tag{1}$$

Then writing $|(f^p)'(x)|$ as

$$|(f^p)'(x)| = |(f^{p-1})'(x_1)||f'(x)| > (K^{-1}|(f^{p-1})'(\hat{x}_1)|^{\frac{1}{2}}\ |x-\hat{x}|)\cdot|(f^{p-1})'(\hat{x}_1)|^{\frac{1}{2}}$$

and substituting in (1), we see that $|(f^p)'(x)| > K^{-1}c_1^{\frac{1}{2}}e^{\frac{1}{2}\lambda(p-1)}e^{-\alpha p}$, which we may assume is $> e^{\frac{1}{3}\lambda p}$ if p is sufficiently large, or equivalently, δ is sufficiently small. $\square$

Proof of (P3): We write $\sigma_0=[x,y]$, $\sigma_k=f^{t_k}\sigma_0$, and assume for definiteness that $\sigma_0\subset C_\delta$ and $t_q+p_q\leq n\leq t_{q+1}$. Then

$$\log\frac{(f^n)'(x)}{(f^n)'(y)} \ \leq\ \sum_{j=0}^{n-1}\frac{|f'(y_j)-f'(x_j)|}{|f'(y_j)|}\ \leq\ K\sum_{k=1}^{q}(S'_k+S''_k)$$

where

$$S'_k=\sum_{j=t_k}^{t_k+p_k-1}\frac{|y_j-x_j|}{d(y_j,C)} \quad\text{and}\quad S''_k=\sum_{t_k+p_k}^{t_{k+1}-1}\frac{|y_j-x_j|}{d(y_j,C)}$$

except for S''_q, which ends at index $n-1$. Observe that for $k<q$, it follows from (P2)(ii) and (P1)(ii) that $|\sigma_{k+1}|\geq c_1e^{\frac{1}{3}\lambda(t_{k+1}-t_k)}|\sigma_k|$, which we may assume is $\geq\tau|\sigma_k|$ for some $\tau>1$ (the factor c_1 having been absorbed into the exponential assuming δ is sufficiently small).

I. Bound on $\sum_{k=1}^{q}S'_k$

First we estimate S'_k. Suppose $y_{t_k}\in C_\delta(\hat{x})$. For $t_k<j<t_k+p_k$ we write

$$\frac{|y_j-x_j|}{d(y_j,C)}=\frac{|y_j-x_j|}{|y_j-\hat{x}_{j-t_k}|}\cdot\frac{|y_j-\hat{x}_{j-t_k}|}{d(y_j,C)}.$$

By Lemma 2.1 and the usual estimates near $\hat{x}$, the first factor on the right is

$$<\ K\frac{|y_{t_k+1}-x_{t_k+1}|}{|y_{t_k+1}-\hat{x}_1|}\ <K\frac{|f'(x_{t_k})|\ |y_{t_k}-x_{t_k}|}{|y_{t_k}-\hat{x}|^2}\ <\ K\frac{|\sigma_k|}{d(y_{t_k},C)}.$$

Thus

$$S'_k\ \leq\ K\frac{|\sigma_k|}{d(y_{t_k},C)}\left(1+\sum_{j=t_k+1}^{t_k+p_k-1}\frac{|y_j-\hat{x}_{j-t_k}|}{d(\hat{x}_{j-t_k},C)}\right)\ \leq\ K\frac{|\sigma_k|}{d(y_{t_k},C)},$$

treating the sum inside the parentheses as in Lemma 2.1.

Now let $\mathcal{K}_\mu = \{k \le q : \ \sigma_k \subset I_{\mu j} \text{ for some } j\}$. We claim that

$$\sum_{k\in\mathcal{K}_\mu} S'_k \ < \ K \sum_{k\in\mathcal{K}_\mu} \frac{|\sigma_k|}{e^{-|\mu|}} \ < K\frac{1}{\mu^2}.$$

The first inequality is from the estimate above. The second follows from (i) $|\sigma_{k+1}| \ge \tau|\sigma_k|$ and (ii) the term with the largest index is bounded above by $|I^+_{\mu j}|$, which is $\le \frac{3}{\mu^2}e^{-|\mu|}$. To finish, we sum over all μ, $|\mu| \ge \log\frac{1}{\delta}$, to obtain $\sum S'_k < K$.

II. Bound on $\sum_{k=1}^q S''_k$

For $k < q$ and $t_k + p_k \le j \le t_{k+1} - 1$, we have, by (P1)(ii), $|\sigma_{k+1}| \ge c_1 e^{\lambda(t_{k+1}-j)}|x_j - y_j|$, so $S''_k \le K\frac{|\sigma_{k+1}|}{\delta}$. This together with $|\sigma_{k+1}| \ge \tau|\sigma_k|$ gives $\sum_{k=0}^{q-1} S''_k \le K\frac{|\sigma_q|}{\delta} < K$. If $n < t_{q+1}$, S''_q needs to be estimated separately: Assume the orbit in question visits C_{δ_0} during the time interval $[t_q + p_q, n)$ (otherwise the estimate is trivial). Let $\hat{n}$ be the time of the last such visit. Then

$$S''_q = \sum_{j=t_q+p_q}^{\hat{n}-1} K\frac{|y_j - x_j|}{d(y_j, C)} \ + \ K\frac{|y_{\hat{n}} - x_{\hat{n}}|}{d(y_{\hat{n}}, C)} \ + \sum_{j=\hat{n}+1}^{n-1} K\frac{|y_j - x_j|}{d(y_j, C)}.$$

The first sum is estimated using $|y_{\hat{n}-1} - x_{\hat{n}-1}| > c_1 e^{\frac{1}{4}\lambda_0(\hat{n}-1-j)}|y_j - x_j|$ for $t_q + p_q \le j < \hat{n}$ ((P1)(ii)). The last sum is estimated using $|y_{n-1} - x_{n-1}| > c_1\delta_0 e^{\frac{1}{4}\lambda_0(n-j-1)}|y_j - x_j|$ for $\hat{n} < j < n-1$ ((P1)(i) with δ_0 in the place of δ). It follows that

$$S''_q \le K\frac{|y_{\hat{n}-1} - x_{\hat{n}-1}|}{\delta} + K\frac{|y_{\hat{n}} - x_{\hat{n}}|}{\delta} + K\frac{|y_{n-1} - x_{n-1}|}{\delta_0^2} < K.$$

□

References: Conditions of the type (G1) are first used in [BC1] and [BC2]. (G2) is introduced in [CE]. A version of the material in this section for $f_0(x) = 1 - 2x^2$ first appeared in [BC1] and Section 2 of [BC2].

3 Absolutely Continuous Invariant Measures

The goal of this section is to prove

Theorem 1 *Every $f \in \mathcal{G}$ has an absolutely continuous invariant probability measure.*

3.1 Growth of segments with bounded distortion

Let $f \in \mathcal{G}$. To prove that f admits an absolutely continuous invariant probability measure (*acipm*), we need to show that the forward images of Lebesgue measure have certain regularity properties. Expansion is conducive to such regularity, but distortion bounds are also essential. Since (P3) guarantees uniform distortion bounds only on intervals *with the same* *itinerary*, we need to show that intervals of this type grow with sufficient regularity. This is the mission of the present subsection.

We begin by introducing what will be referred to as a **canonical subdivision by itinerary** on an interval $\omega \subset I$. This consists of an increasing sequence of partitions $\mathcal{Q}_0 < \mathcal{Q}_1 < \mathcal{Q}_2 < \cdots$ on ω defined as follows: Let us say an interval is "short" if it is strictly contained in an element of $\mathcal{P}$. $\mathcal{Q}_0$ is defined to be $\mathcal{P}|_\omega$ except that the end intervals are attached to their neighbors if they are short. We assume inductively that all $\hat{\omega} \in \mathcal{Q}_i$ are intervals and all points in $\hat{\omega}$ have the same itinerary through time i. To go from $\mathcal{Q}_i$ to $\mathcal{Q}_{i+1}$, we consider one $\hat{\omega} \in \mathcal{Q}_i$ at a time:

- If $f^{i+1}(\hat{\omega})$ is in a bound period, then it is automatically put into $\mathcal{Q}_{i+1}$. (Observe that if $f^{i+1}(\hat{\omega}) \cap C_\delta \neq \emptyset$, then $f^{i+1}(\hat{\omega}) \subset I^+_{\mu' j'}$ for some μ', j', i.e. no cutting is needed during bound periods. This is an easy exercise.)
- If $f^{i+1}(\hat{\omega})$ is not in a bound period, but all points in $\hat{\omega}$ have the same itinerary through time $i+1$, we again put $\hat{\omega} \in \mathcal{Q}_{i+1}$.
- If neither of the last two cases holds, then we partition $\hat{\omega}$ into segments $\{\hat{\omega}'\}$ according to their itineraries through time $i+1$, requiring that $f^{i+1}(\hat{\omega}') \approx \gamma$ for some $\gamma \in \mathcal{P}$ (*i.e.*, no cuts are made that lead to short intervals). The resulting partition is $\mathcal{Q}_{i+1}|_{\hat{\omega}}$.

For $x \in \omega$ and $i \geq 0$, let $\mathcal{Q}_i(x)$ denote the element of $\mathcal{Q}_i$ containing x. We introduce the following stopping time on ω: For $x \in \omega$, $S(x)$ is the smallest $i > 0$ such that $f^i(\mathcal{Q}_{i-1}(x))$ is not in a bound period and has length $> \delta$. The main result of this subsection is

Proposition 3.1 *There exists $K_2 > 0$ such that for any $\omega \subset I$ with $\delta < |\omega| < 3\delta$,*

$$|\{S > n\}| < e^{-K_2^{-1} n} |\omega| \qquad \text{for} \quad n > K_2 \log \delta^{-1}.$$

Here $|\cdot|$ denotes the Lebesgue measure of the set. The proof of this proposition follows from a series of lemmas.

Lemma 3.1 *Let $\omega \approx I_{\mu j}$. Then $|f^p(\omega)| > e^{-\frac{7\alpha}{\lambda}|\mu|}$ where $p = p(\omega)$.*

Proof: It follows from Lemma 2.1 that for $I_{\mu j} \in \mathcal{P}^{\hat{x}}$,

$$\begin{aligned} |f^p(I_{\mu j})| &= \frac{|f^p(I_{\mu j})|}{|f^p([\hat{x}, \hat{x} \pm e^{-|\mu|}])|} \, |f^p([\hat{x}, \hat{x} \pm e^{-|\mu|}])| \\ &\geq K^{-1} \frac{|f(I_{\mu j})|}{|f([\hat{x}, \hat{x} \pm e^{-|\mu|}])|} \, |f^p([\hat{x}, \hat{x} \pm e^{-|\mu|}])| \\ &\geq K^{-1} \frac{1}{\mu^2} e^{-2\alpha p} . \end{aligned}$$

By (P2)(i), $p < \frac{3}{\lambda}|\mu|$. The lemma then follows assuming δ is sufficiently small. $\square$

For $\omega \approx I_{\mu j}$, we define the *extended bound period* of ω to be the largest n such that all points in ω have the same itinerary (in the sense of (P3)) for $n-1$ iterates. The next lemma follows immediately from Corollary 2.1.

Lemma 3.2 *There exists K such that the extended bound period of $I_{\mu j}$ is $< K|\mu|$.*

Lemma 3.3 *Assume δ is sufficiently small. Then for $\omega \approx I_{\mu_0 j_0}$,*

$$|\{x \in \omega : S(x) > n\}| \ < \ e^{-\frac{1}{2}K^{-1}n} \ |\omega| \qquad \text{for all } n > K\mu_0$$

where K is the constant in Lemma 3.2.

Proof: Let $x \in \omega$ be such that $S(x) > n$. We define the *essential return times* and *addresses* of x before n as follows: Let t be the extended bound period of ω. Then either $S = t$, or $f^t(\omega) \subset C_\delta$ (more precisely $\cup I^+_{\mu j}$). In the latter case, we say $t_1 = t$ is the first essential return time of x, and if $f^{t_1}(\mathcal{Q}_{t_1}(x)) \approx I_{\mu_1 j_1} \subset C_\delta$, then we say $I_{\mu_1 j_1}$ is its first essential return address. If S has not been reached, we continue iterating. Let t be the extended bound period of $f^{t_1}(\mathcal{Q}_{t_1}(x))$. Then either $S(x) = t_1 + t$, or we define the second essential return time to be $t_2 = t_1 + t$ and second return address to be $I_{\mu_2 j_2}$ if $f^{t_2}(\mathcal{Q}_{t_2}(x)) \approx I_{\mu_2 j_2} \subset C_\delta$, and so on.

Let $A_q = \{x \in \omega : S(x) > n, \ f^i(x)$ makes q but not $q+1$ essential returns before time $n\}$. Then $|\{S > n\}| \ = \sum_q |A_q|$. We write $A_q = \cup_R A_{q,R}$ where $A_{q,R} = \{x \in A_q :$ if $(\mu_1, \cdots, \mu_q)$ are the μ-coordinates of its first q return addresses, then $|\mu_1| + |\mu_2| + \cdots + |\mu_q| = R\}$, and further decompose $A_{q,R}$ into intervals σ consisting of points whose first q return addresses are identical. Each such σ is equal to $\mathcal{Q}_{t_q}(x)$ for some x, since the extended bound period of $f^{t_q}(\mathcal{Q}_{t_q}(x))$ is not completed before time n (see above). Writing $Q_{t_k} = \mathcal{Q}_{t_k}(x)$, we have

$$\begin{aligned}
|\sigma| \ &= \ \frac{|Q_{t_q}|}{|Q_{t_{q-1}}|} \frac{|Q_{t_{q-1}}|}{|Q_{t_{q-2}}|} \cdots \frac{|Q_{t_1}|}{|\omega|} \, |\omega| \\
&\le \ K_0^q \ \frac{|f^{t_{q-1}+p_{q-1}}(Q_{t_q})|}{|f^{t_{q-1}+p_{q-1}}(Q_{t_{q-1}})|} \ \frac{|f^{t_{q-2}+p_{q-2}}(Q_{t_{q-1}})|}{|f^{t_{q-2}+p_{q-2}}(Q_{t_{q-2}})|} \cdots \frac{|f^{p(\omega)}(Q_{t_1})|}{|f^{p(\omega)}(\omega)|} \, |\omega| \\
&\le \ c_1^{-q} K_0^q \ \frac{|f^{t_q}(Q_{t_q})|}{|f^{t_{q-1}+p_{q-1}}(Q_{t_{q-1}})|} \ \frac{|f^{t_{q-1}}(Q_{t_{q-1}})|}{|f^{t_{q-2}+p_{q-2}}(Q_{t_{q-2}})|} \cdots \frac{|f^{t_1}(Q_{t_1})|}{|f^{p(\omega)}(\omega)|} \, |\omega| \\
&\le \ K^q \ \frac{e^{-|\mu_q|}}{e^{-\frac{7}{\lambda}\alpha|\mu_{q-1}|}} \ \frac{e^{-|\mu_{q-1}|}}{e^{-\frac{7}{\lambda}\alpha|\mu_{q-2}|}} \cdots \frac{e^{-|\mu_1|}}{e^{-\frac{7}{\lambda}\alpha|\mu_0|}} \, |\omega| \ .
\end{aligned}$$

Here (P3) is used in the first inequality, (P1)(ii) is used in the second and Lemma 3.1 in the third. With $\alpha < \frac{1}{100}\lambda$, the estimate above gives

$$|\sigma| \ < \ K^q e^{-\sum_{k=1}^q \frac{9}{10}|\mu_k| + \frac{1}{10}|\mu_0|} |\omega| \ = \ K^q e^{-\frac{9}{10}R + \frac{1}{10}|\mu_0|} |\omega| \ := \ |\sigma|_R.$$

(This estimate is valid only if $f^i(\omega)$ has completed its first bound period, which is not a problem since $n > K|\mu_0|$.) We estimate $|\{S > n\}|$ by

$$|\{S > n\}| \ = \ \sum_{q,R} |A_{q,R}| \ \le \ \sum_R \ (\text{number of } \sigma \text{ in } \cup_q A_{q,R}) \cdot |\sigma|_R \ .$$

There are $\begin{pmatrix} R-1 \\ q \end{pmatrix}$ ways of decomposing R into a sum of $q+1$ integers. For a fixed q-tuple $(\mu_1, \cdots, \mu_q)$, $\mu_i > 0$, we claim there are $\le 2^q \mu_1^2 \mu_2^2 \cdots \mu_q^2$ possibilities for σ with the

μ-coordinates of their essential free return addresses being $(\pm\mu_1, \cdots \pm \mu_q)$. This is because $f^{t_k}(\mathcal{Q}_{t_k}(\sigma))$ is short enough that it can meet at most one $C_\delta(\hat{x})$, which contains $\leq 2\mu_k^2$ intervals of the form $I_{\pm\mu_k j}$. Furthermore, for $(\mu_1, \cdots, \mu_q)$ with $|\mu_1| + |\mu_2| + \cdots + |\mu_q| = R$, we have $\mu_1^2\mu_2^2\cdots\mu_q^2 \leq (\frac{R}{q})^{2q}$.

There is one other piece of information that is crucial to us, namely that all bound periods are $\geq \Delta := K^{-1}\log\frac{1}{\delta}$. This means that for a given R, the only feasible q are $\leq \frac{R}{\Delta}$. For a fixed R, then, the number of σ in $\cup_q A_{q,R}$ is

$$\leq \sum_q \binom{R-1}{q} \cdot 2^q \left(\frac{R}{q}\right)^{2q} \leq \frac{R}{\Delta} \cdot \binom{R}{\frac{R}{\Delta}} \cdot 2^{\frac{R}{\Delta}}\, \Delta^{2\frac{R}{\Delta}},$$

which, by Sterling's formula, is

$$\sim \frac{R}{\Delta}\left(e^{\epsilon(\frac{1}{\Delta})}\, 2^{\frac{1}{\Delta}}\, \Delta^{\frac{2}{\Delta}}\right)^R \quad \text{where} \quad \epsilon\left(\frac{1}{\Delta}\right) \to 0 \text{ as } \delta \to 0.$$

Calling the expression above $(1+\eta(\delta))^R$, we have $\eta(\delta) \to 0$ as $\delta \to 0$.

To finish, we note that $n \leq K(|\mu_0| + R)$ where K is as in Lemma 3.2, since the essential bound period following the qth essential return expires before time $K(|\mu_0| + R)$. Thus

$$\begin{aligned} |\{S > n\}| &< \sum_{R \geq K^{-1}n - |\mu_0|} K^q (1+\eta(\delta))^R e^{-\frac{9}{10}R + \frac{1}{10}|\mu_0|}|\omega| \\ &< e^{-\frac{4}{5}K^{-1}n + \frac{1}{10}|\mu_0|}|\omega| \;<\; e^{-\frac{1}{2}K^{-1}n}|\omega| \end{aligned}$$

provided that $n > K|\mu_0|$. □

Proof of Proposition 3.1: First there is the trival case where for some $i > 0$, all points in ω have the same itinerary through time $i-1$ and $|f^i(\omega)| > \delta$, so that $S|_\omega = i$. This case aside, let $t_0 \geq 0$ be the first time $\mathcal{Q}_{t_0}$ contains more than one element. Clearly, $t_0 < K\log\frac{1}{\delta}$, and $|f^{t_0}(\omega)| > K^{-1}\delta$ by (P1). Let $n > 0$ be an arbitrary integer. For each $\omega' \in \mathcal{Q}_{t_0}$ such that $f^{t_0}(\omega') \approx I_{\mu j}$, $|\mu| < K^{-1}n$, we have $|\omega' \cap \{S > t_0 + n\}| < K_0 e^{-\frac{1}{2}K^{-1}n}|\omega'|$ by Lemma 3.3, K_0 here being the distortion constant for f^{t_0}. The measure of the union of $\omega' \in \mathcal{Q}_{t_0}$ with $|\mu| > K^{-1}n$ is $\leq Ke^{-K^{-1}n}$. It follows therefore that

$$|\{x \in \omega : S(x) > t_0 + n\}| < Ke^{-\frac{1}{2}K^{-1}n}|\omega| + K\delta^{-1}e^{-K^{-1}n}|\omega| < e^{-K_2^{-1}(t_0+n)}|\omega|$$

provided K_2 is sufficiently large and $n + t_0 > K_2 \log\delta^{-1}$. □

3.2 Proof of Theorem 1

Let m denote Lebesgue measure on I, and let $f_*^i(m)$ be the Borel measure with $f_*^i(m)(E) = m(f^{-i}(E))$. Fix $l := I_{\mu_0 j_0}$ for some μ_0, j_0. For $n = 1, 2, \cdots$, let

$$\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i(m|_l).$$

Clearly, any limit point of ν_n in the weak* topology is f-invariant. As we will see, it suffices to show that a positive fraction of these measures is absolutely continuous with respect to m (written "$<< m$"). The next lemma helps us "catch" this fraction:

Lemma 3.4 *There exist (i) an interval $L \subset I$, (ii) a number $c > 0$, (iii) a sequence of integers $n_1 < n_2 < \cdots$, and (iv) for each $i = 0, 1, 2, \cdots$, a collection of subsegments $\{\omega_j^{(i)}\}$ of l, with the property that the following hold for each i, j:*

(a) $f^i(\omega_j^{(i)}) = L$;

(b) $\frac{(f^i)'(x)}{(f^i)'(y)} < K_0$ *for all* $x, y \in \omega_j^{(i)}$;

(c) $\frac{1}{n_k} \sum_{i=0}^{n_k - 1} m(\cup_j \omega_j^{(i)}) \geq\ c\ m(l)$.

We first finish the proof assuming the conclusion of Lemma 3.4: Let

$$\hat{\nu}_{n_k} = \frac{1}{n_k} \sum_{i=0}^{n_k - 1} f_*^i(m|_{(\cup_j \omega_j^{(i)})}),$$

and let $\hat{\nu}$ be a limit point of $\hat{\nu}_{n_k}$. It follows from Lemma 3.4(c) that $\hat{\nu}(L) > cm(l) > 0$. From Lemma 3.4(a) and (b), we see that if ρ_k is the density of $\hat{\nu}_{n_k}$, then $\rho_k(x)/\rho_k(y) \leq K_0$ for all $x, y \in L$. These bounds are passed to the limit measure $\hat{\nu}$. In particular, $\hat{\nu} << m$.

Now let ν be a limit point of ν_{n_k}. Then $\nu \geq \hat{\nu}$, meaning $\nu - \hat{\nu}$ is a nonnegative measure. We decompose ν into $\nu_{ac} + \nu_\perp$ where $\nu_{ac} << m$ and $\nu_\perp$ is singular with respect to m. Since $f_*(\nu_{ac}) << m$ and $f_*(\nu_\perp) \perp m$, it follows that both ν_{ac} and $\nu_\perp$ are f-invariant. It remains to argue that $\nu_{ac}(I) > 0$, which is true since $\nu_{ac} \geq \hat{\nu}$ and $\hat{\nu}(L) > 0$.

Proof of Lemma 3.4: We introduce a sequence of stopping times $S_0 < S_1 < S_2 < \cdots$ on l as follows: Let $S_0 = 0$ and $S_1 = S$ where S is as defined in Sect. 3.1. For $k \geq 1$, let $x \in \omega$ be such that $S_k(x)$ has been defined. For definiteness, suppose $x \in \omega \in \mathcal{Q}_{i-1}$ with $S_k|_\omega = i$. We define $S_{k+1}|_\omega = S_k + S|_{f^{S_k}(\omega)} \circ f^{S_k}$, *i.e.*, $S_{k+1}(x)$ is defined to be the smallest $j > i$ such that $f^j(\mathcal{Q}_{j-1}(x))$ is free and has length $> \delta$. Let $\tilde{\mathcal{Q}}_{i-1} = \{\omega \in \mathcal{Q}_{i-1}$ such that $S_k|_\omega = i$ for some $k\}$. It follows from Proposition 3.1 that for all such ω,

$$\int_\omega (S_{k+1} - S_k) dm\ \leq M|\omega|$$

for some M possibly dependent on δ but independent of ω, k or i. Keeping k fixed while summing over ω and i, we obtain $\int_l (S_{k+1} - S_k) dm \leq M\ m(l)$. Summing over k then gives

$$\int_l S_k dm \leq Mk\ m(l).$$

By Chebychev's Inequality,

$$m\{S_{[\frac{1}{2M}N]} > N\}\ \leq\ \frac{1}{2} m(l).$$

Hence

$$\frac{1}{N} \sum_{i \leq N} m(\cup\{\omega \in \tilde{\mathcal{Q}}_{i-1}\})\ \geq\ \frac{1}{4M}\ m(l). \tag{2}$$

To finish, we partition I into intervals $L_1, L_2, \cdots, L_{\frac{3}{\delta}}$ of length $\frac{1}{3}\delta$ each. For each $\omega \in \tilde{\mathcal{Q}}_{i-1}$, since $|f^i(\omega)| > \delta$, there exists $n = \psi(\omega)$ such that $f^i(\omega) \supset L_n$. Let $\hat{\omega} = \omega \cap f^{-i} L_n$.

By (P3), there exists K' such that $m(\hat\omega) > K'^{-1} m(\omega)$. Together with (2), this implies that for each N, there exists $n = n(N)$ such that

$$\frac{1}{N} \sum_{i \le N} m(\cup\{\omega \in \tilde{\mathcal{Q}}_{i-1} : \psi(\omega) = n\}) \;\ge\; \frac{\delta}{12MK'}\, m(l).$$

Let n^* be such that $n^* = n(N)$ for infinitely many N. We let $L = L_{n^*}$, and for each i, let $\{\omega_j^{(i)}\} := \{\hat\omega : \omega \in \tilde{\mathcal{Q}}_{i-1},\ \psi(\omega) = n^*\}$. □

References: A version of the material in Sect. 3.1 is used in [BC2] (in a different context). Sect. 3.2 follows the construction of SRB measures in [BY].

Part II PARAMETER ISSUES

4 Admissible One-parameter Families

4.1 Statement of Theorem 2

We say a 1-parameter family of 1D maps $\{f_a \in C^2(I, I), a \in (a_1, a_2)\}$ is C^2 if the map $(x, a) \mapsto f_a(x)$ is C^2. If I is an interval, we assume $f_a(I)$ is contained in the interior of I; small modifications of some statements are needed otherwise. Assuming $a_1 < 0 < a_2$ and $f_0 \in \mathcal{M}$, certain orbits of f_0 have natural *continuations* to a near 0. For example:

(i) Continuations $a \mapsto \hat{x}(a)$ of every $\hat{x} \in C(f_0)$ is clearly well defined.

(ii) Let $\Lambda \subset I$ be a closed subset with the property that $f_0(\Lambda) \subset \Lambda$ and $\Lambda \cap C(f_0) = \emptyset$. Then Λ has a natural continuation $a \mapsto \Lambda(a)$ to a small interval containing 0. Moreover, for each $x \in \Lambda$, $a \mapsto x(a)$ is differentiable. (For more detail, see Sect. 4.2).

Definition 4.1 *Let $\{f_a\}$ be a 1-parameter family with $f_0 \in \mathcal{M}$. We say $\{f_a\}$ satisfies the* **parameter transversality condition (PT)** *at f_0 if for every $\hat{x} \in C(f_0)$ and $q = f_0(\hat{x})$,*

$$\hat{c}(\hat{x}) := \frac{d}{da}(f_a(\hat{x}(a)) - q(a))\Big|_{a=0} \neq 0.$$

The notation "$q(a)$" in the displayed formula above is to be interpreted as follows: Since $f_0 \in \mathcal{M}$, q is contained in a closed subset Λ of the kind in (ii) above. By $q(a)$, we refer to the continuation of q in the sense of (ii).

As a varies, $f_a(\hat{x}(a))$ moves with a, as does the set $\Lambda(a)$. Roughly speaking, (PT) stipulates that the two move at different speeds. In general, Λ moves more slowly, and the trajectory of $a \mapsto f_a(\hat{x}(a))$ moves "through" Λ. This is why we think of (PT) as a transversality condition.

Theorem 2 *For every C^2 family $\{f_a\}$ satisfying (PT) at $f_0 \in \mathcal{M}$, the set $\{a : f_a \in \mathcal{G}\}$ has positive Lebesgue measure.*

The rest of this paper is devoted to the proof of Theorem 2.

To simplify slightly the discussion, we assume from here on that $C(f_a) = C(f_0) := C$ for all a. This is easily arranged via a-dependent changes of coordinates that do not affect the content of the theorem. As before, critical points will be denoted by "hats" (e.g., $\hat{x} \in C$, while x is an arbitrary point in I).

Standing assumptions for Part II:

- *(a_1, a_2) is an interval with $a_1 < 0 < a_2$;*
- *$\{f_a, a \in (a_1, a_2)\}$ is a C^2 family with $C(f_a) = C(f_0)$ for all a;*
- *$f_0 \in \mathcal{M}$, and (PT) is satisfied at f_0.*

System constants K for Part II are allowed to depend only on (i) f_0 (including the constants in Sect. 1.1 associated with f_0), (ii) the C^2 norm of the family $\{f_a\}$, (iii) $\hat{c} := \min_{\hat{x} \in C} |\hat{c}(\hat{x})|$ and (iv) our choice of λ.

4.2 Alternate formulation of (PT)

We begin with some simple facts about the symbolic dynamics of $f = f_0 \in \mathcal{M}$. Let $\mathcal{J} = \{J_1, \cdots, J_q\}$ be the components of $I \setminus C$. For $x \in I$ such that $f^i x \notin C$ for all $i \geq 0$, let $\phi(x) = (\iota_i)_{i=0,1,\cdots}$ be given by $\iota_i = k$ if $f^i x \in J_k$.

Lemma 4.1 *For $f \in \mathcal{M}$, there exists an increasing sequence of compact sets $\Lambda^{(n)}$ such that*
(a) $\Lambda^{(n)} \cap C = \emptyset$, $f(\Lambda^{(n)}) \subset \Lambda^{(n)}$, and $f|_{\Lambda^{(n)}}$ is conjugate to a shift of finite type;
(b) $\cup_n \Lambda^{(n)}$ is dense in I;
(c) if $\inf_{i \geq 0} d(f^i(x), C) > 0$, then $x \in \Lambda^{(n)}$ for some n.

Proof: First we argue that $\cup_{i \geq 0} f^{-i} C$ is dense in I. If not, there would be an interval ω with the property that $\phi(x)$ is identical for all $x \in \omega$. Let ω be a maximal interval of this type. Then either (i) $f^{n+k}(\omega) \subset f^n(\omega)$ for some n, k, or (ii) $f^k(\omega), k = 0, 1, \cdots$, are pairwise disjoint. Case (i) cannot happen since it implies the presence of a periodic point x with $|(f^k)'x| \leq 1$. Case (ii) is equally absurd, for it implies the existence of $\{k_i\}$ where $f^{k_i}(\omega)$ are arbitrarily short. We leave it as an easy exercise to see that this is incompatible with the definition of $\mathcal{M}$.

For definiteness, consider $I = S^1$. Let $l_n(\hat{x})$ and $r_n(\hat{x})$ be the two points in $\cup_{0 < i \leq n} f^{-i} C$ closest to $\hat{x} \in C$, and let $\Lambda^{(n)} = \{x \in I : f^i x \notin \cup_{\hat{x} \in C} (l_n(\hat{x}), r_n(\hat{x}))\ \forall i \geq 0\}$. Then for each n, $\Lambda^{(n)}$ is compact and $f(\Lambda^{(n)}) \subset \Lambda^{(n)}$. That $\cup_n \Lambda^{(n)}$ is dense in I follows from (i) $\cup_{0 < i \leq n} f^{-i} C \subset \Lambda^{(n)}$ for all n large enough that $(l_n(\hat{x}), r_n(\hat{x})) \subset (\hat{x} - \delta_0, \hat{x} + \delta_0)$, and (ii) $\cup_{i > 0} f^{-i} C$ is dense in I. Assertion (c) follows immediately from this construction.

To show that $f|_{\Lambda^{(n)}}$ is conjugate to a shift of finite type, let $\mathcal{J}^{(n)} = \{J_i^{(n)}\}$ be the partition of I by $\cup_{0 \leq i \leq n} f^{-i} C$. For $J_i^{(n)} \neq (l_n(\hat{x}), \hat{x})$ or $(\hat{x}, r_n(\hat{x}))$, observe that by construction, $f(J_i^{(n)})$ is equal to the union of a finite number of elements of $\mathcal{J}^{(n)}$. Let $\Lambda_i^{(n)} = \Lambda^{(n)} \cap J_i^{(n)}$. Then the alphabet of the shift in question is $\{i : \Lambda_i^{(n)} \neq \emptyset\}$, and the transition $i \to j$ is admissible if $f(\Lambda_i^{(n)}) \supset \Lambda_j^{(n)}$.

Where I is an interval, $\Lambda^{(n)}$ is as above but restricted to the interval $[z_n^1, z_n^2]$ where z_n^1 and z_n^2 are the two points in $\cup_{0<i\leq n} f^{-i}C$ closest to the endpoints of I. The end intervals are excluded because they do not have the required Markov property. $\square$

Our next result guarantees that $q(a)$ in Definition 4.1 is well defined.

Corollary 4.1 *For $f \in \mathcal{M}$, let $q \in I$ be such that $\delta_1 := \inf_{n\geq 0} d(f^n(q), C) > 0$. Then for all g with $\|g - f\|_{C^2} < \varepsilon$ where $\varepsilon = \varepsilon(\delta_1)$, there is a unique point $q_g \in I$ with $\phi_g(q_g) = \phi_f(q)$.*

Proof: Fix n large enough that for all $i \geq 0$, $f^i(q) \notin (l_n(\hat{x}), r_n(\hat{x}))$ for all $\hat{x} \in C$, and let $\Lambda = \Lambda^{(n)}$. Let $B = \cup_i \partial \bar{\Lambda}_i$ where $\bar{\Lambda}_i$ is the shortest interval containing Λ_i. Since B is a finite set with $f(B) \subset B$, it consists of preperiodic points. From Lemma 1.1, the periodic points in question are repelling. Thus if g is sufficiently near f, there is a unique set B_g with $g(B_g) \subset B_g$ such that $g|_{B_g}$ is conjugate to $f|_B$. Using B_g, we recover a set Λ_g on which g is conjugate to $f|_\Lambda$. The uniqueness of q_g follows from the expanding property of g away from C (Lemma 1.1). $\square$

We return now to the C^2 family $\{f_a\}$ with $f_0 \in \mathcal{M}$. Let $\hat{x} \in C$ be fixed, and let $q = f_0(\hat{x})$. We write $F(x, a) := f_a(x)$, reserving $(\cdot)'$ for x-derivatives, *i.e.*, $\partial_x F(x, a) = (f_a)'(x)$.

Proposition 4.1 *There is an interval ω in a-space containing 0 in its interior on which $q(a)$ is defined, $a \mapsto q(a)$ is differentiable, and*

$$\frac{d}{da} q(a) \;=\; -\sum_{i=1}^{\infty} \frac{\partial_a F(f_a^{i-1}(q(a)), a)}{(f_a^i)'(q(a))}. \tag{3}$$

Proof: Let q be as in Corollary 4.1. We assume ω is short enough that for all $a \in \omega$, $q(a)$ is well defined, $q(a) \cap C_{\frac{1}{2}\delta_0} = \emptyset$, and that (P1) holds for f_a outside of $C_{\frac{1}{2}\delta_0}$ with uniform bounds. In the computations below, we suppress the dependence on a, writing $f = f_a, q = q(a)$, $\partial_a F(\cdot) = \partial_a F(\cdot, a)$, and so on.

Continuing to use the notation in Corollary 4.1, we let $\Lambda_{i_0, i_1, \cdots, i_m} = \{x \in I : f^j(x) \in \Lambda_{i_j}, 0 \leq j \leq m\}$, and let $\Lambda_{i_0, i_1, \cdots, i_m}(q)$ be the cylinder set containing q. For each m, choose $q^{(m)} \in \partial \bar{\Lambda}_{i_0, i_1, \cdots, i_m}(q)$. Then $q^{(m)} \to q$. It suffices to show that as functions of a, $\frac{d}{da} q^{(m)}$ converges uniformly to the right side of (3). Let $p^{(m)} = f^m(q^{(m)})$. Differentiating, we obtain

$$\frac{d}{da} p^{(m)} \;=\; \sum_{i=1}^{m} (f^{m-i})'(f^i(q^{(m)}))\; \partial_a F(f^{i-1}(q^{(m)})) \;+\; (f^m)'(q^{(m)})\; \frac{d}{da} q^{(m)}.$$

This gives

$$\frac{d}{da} q^{(m)} \;=\; \frac{\frac{d}{da} p^{(m)}}{(f^m)'(q^{(m)})} \;-\; \sum_{i=1}^{m} \frac{\partial_a F(f^{i-1} q^{(m)})}{(f^i)'(q^{(m)})}\;. \tag{4}$$

We stress that all the action below takes place outside of $C_{\frac{1}{2}\delta_0}$, where $|(f^n)'|$ grows exponentially (with prefactor $\frac{1}{2}\delta_0 c_1$).

To estimate (4), observe that since $p^{(m)} \in B$ (see the proof of Corollary 4.1), and B is a finite set, $\frac{d}{da} p^{(m)}$ is uniformly bounded for all m. With $|(f^m)'(q^{(m)})|$ growing exponentially, the first term on the right is exponentially small. It remains to check that the second term converges uniformly to the right side of (3). Since the tail of the sum in (3) decreases exponentially (uniformly bounded numerator and exponentially increasing denominator), it suffices to verify that

$$\begin{aligned} A &:= \left| \sum_{i=1}^{m} \frac{\partial_a F(f^{i-1}(q^{(m)}))}{(f^i)'(q^{(m)})} - \sum_{i=1}^{m} \frac{\partial_a F(f^{i-1}(q))}{(f^i)'(q)} \right| \\ &\le \sum_{i=1}^{m} \frac{|\partial_a F(f^{i-1}(q^{(m)})) - \partial_a F(f^{i-1}(q))|}{|(f^i)'(q^{(m)})|} + \sum_{i=1}^{m} \frac{|\partial_a F(f^{i-1}(q))|}{|(f^i)'(q^{(m)})|} \cdot \left| \frac{(f^i)'(q^{(m)})}{(f^i)'(q)} - 1 \right| \end{aligned}$$

converges to 0 uniformly. Consider the ith term in the first sum. By the expanding property of f out of $C_{\frac{1}{2}\delta_0}$, the numerator is $<$ const $e^{-\frac{1}{4}\lambda_0(m-i)}\ |f^m(q^{(m)}) - f^m(q)|$, while the denominator is $>$ const $e^{\frac{1}{4}\lambda_0 i}$. The second sum is estimated similarly. Together they give $A <$ const $me^{-\frac{1}{4}\lambda_0 m}$. $\square$

Let $\hat{x}_i(a) := f_a^i(\hat{x}(a))$, and recall the definition of $\hat{c}(\hat{x})$ in Definition 4.1.

Corollary 4.2

$$\hat{c}(\hat{x}) = \frac{d\hat{x}_1}{da}(0) + \sum_{i=1}^{\infty} \frac{\partial_a F(\hat{x}_i(0), 0)}{(f_0^i)'(\hat{x}_1(0))} .$$

We have thus obtained an equivalent formulation of (PT) that involves only $\partial_a F(\cdot,\cdot)$ and properties of f_0.

4.3 Comparability of x- and a-derivatives

For f_0, λ, α and ε as in Proposition 2.1, we define

$$\mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon) := \{f : \|f - f_0\|_{C^2} < \varepsilon \text{ and } (G1), (G2) \text{ hold for all } \hat{x} \in C \text{ and } n \le N\}.$$

Proposition 4.2 *Let λ, α and ε be fixed. Then there exist $\hat{\varepsilon} > 0$ and $\hat{i} \in \mathbb{Z}^+$ such that the following holds for all $N \in \mathbb{Z}^+$: Let $\Omega_N \subset (-\hat{\varepsilon}, \hat{\varepsilon})$ be such that $f_a \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$ for all $a \in \Omega_N$. Then for every $a \in \Omega_N$ and $\hat{x} \in C$,*

$$\frac{1}{2}|\hat{c}(\hat{x})| < \frac{|\frac{d}{da}\hat{x}_i(a)|}{|(f_a^{i-1})'(\hat{x}_1)|} < 2|\hat{c}(\hat{x})| \qquad \text{for } \hat{i} < i \le N.$$

Proof: Writing

$$\frac{d}{da}\hat{x}_i(a) = (f_a)'(\hat{x}_{i-1})\frac{d}{da}\hat{x}_{i-1}(a) + \partial_a F(\hat{x}_{i-1}, a),$$

we obtain inductively

$$\frac{\frac{d}{da}\hat{x}_i(a)}{(f_a^{i-1})'(\hat{x}_1)} = \frac{d}{da}\hat{x}_1(a) + \sum_{j=1}^{i-1} \frac{\partial_a F(\hat{x}_j, a)}{(f_a^j)'(\hat{x}_1)} .$$

Letting $I(a,i)$ denote the expression on the right side above, we choose $\hat{\imath}$ large enough that (i) $I(0,\hat{\imath}) \approx \hat{c}(\hat{x})$ and (ii) for $i > \hat{\imath}$,

$$\left| \sum_{j=\hat{\imath}}^{i-1} \frac{\partial_a F(\hat{x}_j, a)}{(f_a^j)'(\hat{x}_1)} \right| << |\hat{c}(\hat{x})| \qquad \text{uniformly for all } a \in J.$$

(i) makes sense because $\hat{c}(\hat{x}) \neq 0$ by (PT). (ii) is because $|\partial_a F(\hat{x}_j, a)| < K$ and $|(f_a^j)'(\hat{x}_1)| > c_1 e^{\lambda j}$ from (G2). Since only a finite number of iterates are involved, we may now shrink $\hat{\varepsilon}$ sufficiently so that $|I(a,\hat{\imath}) - I(0,\hat{\imath})| << |\hat{c}(\hat{x})|$ for all $a \in (-\hat{\varepsilon}, \hat{\varepsilon})$. □

References: The comparability of x- and a-derivatives is first used in [BC1]; the unimodal case of the material in Sects. 4.2 and 4.3 is in [TTY].

5 Evolution of Critical Curves

5.1 Logistics

Much of the rest of the analysis revolves around evolutions of the type

$$a \mapsto \hat{x}_i(a), \quad i = 0, 1, 2, \cdots, \quad \text{for } \hat{x} \in C, \quad a \in J,$$

where J is an interval in $(-\hat{\varepsilon}, \hat{\varepsilon})$. Via the geometry of these curves, we seek to determine what *fraction* of J corresponds to "bad" parameters, that is to say, what fraction of these curves comes too close to the critical set (therefore violating (G1)), or visits the critical set too often (thereby violating (G2)). Herein lies the dilemma: In order for the curves $a \mapsto \hat{x}_i(a)$ to have controlled geometry, the maps f_a corresponding to the a's involved must (individually) be good to start with. On the other hand, by studying curve segments corresponding to good parameters only, how are we to determine what fraction of parameters are bad?

The following discussion motivates our answer to this logistical dilemma.

Properties of $f \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$ up to time $\frac{1}{\alpha^*}N$, $\alpha^* := \frac{3}{\lambda}\alpha$

Assume $f \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$, and consider $\hat{x} \in C$.

Suppose for some $n \leq \frac{1}{\alpha^*}N$, $\hat{x}_n \in C_\delta$ with $d(\hat{x}_n, C) > e^{-\alpha n}$. We claim that (P2) in Sect. 2.1 holds for this return even though n may be greater than N. This is because the critical point that will guide $\hat{x}_n$ through its partial derivative recovery obeys (G1) and (G2) up to time N, and by the proof of (P2), the time it takes to complete this recovery is $< \frac{3}{\lambda}\alpha n \leq N$.

Indeed if we *assume* (G1) holds for $\hat{x}$ up to time n, $n \leq \frac{1}{\alpha^*}N$, then on the time interval $[0,n]$, the orbit of $\hat{x}_1$ has the bound/free behavior described in Sect. 2.1. Moreover, by an argument identical to that for Corollary 2.1, we have $|(f^j)'(\hat{x}_1)| > K^{-1}e^{\frac{1}{4}\lambda j}$ for $j \leq n$.

We remark that beyond time $\frac{1}{\alpha^*}N$, the dynamical description of $\hat{x}$ in the last paragraph ceases to be valid as soon as a bound period $> N$ is encountered. Conversely, the behaviors of other critical orbits beyond time N do not impact the properties of $\hat{x}$ up to time $\frac{1}{\alpha^*}N$.

In view of the discussion above, we modify Proposition 4.2 slightly as follows:

Proposition 4.2' *In addition to the hypotheses in Proposition 4.2, we assume that for some $\hat{x} \in C$ and $n \in (N, \frac{1}{\alpha^*}N]$, (G1) holds for $\hat{x}$ up to time n. Then the conclusion of Proposition 4.2 holds for this $\hat{x}$ for all $i \leq n$.*

5.2 Duality between phase-space and parameter-space dynamics

Setting. Let $\lambda < \frac{1}{4}\lambda_0$ be as before. To establish the above-mentioned duality, new upper bounds are imposed on α and ε (or equivalently $\hat{\varepsilon}$). Let $\Omega_N = \{a \in (-\hat{\varepsilon}, \hat{\varepsilon}) : f_a \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)\}$. For the rest of Section 5, we fix $\hat{x} \in C$. All parameters considered are assumed to be in Ω_N; all indices considered are assumed to be $\le \frac{1}{\alpha^*}N$, and (G1) is assumed to hold for $\hat{x}$ for all the indices in question. We use the notation $\tau_i(a) := \frac{d}{da}\hat{x}_i(a)$.

Our main results are (P1')–(P3'), three properties of $a \mapsto \hat{x}_i(a)$ that are the analogs of (P1)–(P3) in Sect. 2.1. We state also two lemmas that lie at the heart of these properties. To avoid disrupting the flow of ideas, proofs are postponed to Sect. 5.3.

Lemma 5.1 *Let $n > \hat{i}$ where $\hat{i}$ is as in Proposition 4.2. Then*

$$(1 - Ke^{-\frac{1}{4}\lambda n})\ |(f_a^i)'(\hat{x}_n)|\ \le\ \frac{|\tau_{n+i}|}{|\tau_n|}\ \le\ (1 + Ke^{-\frac{1}{4}\lambda n})\ |(f_a^i)'(\hat{x}_n)|.$$

(P1') (Outside of C_δ): There exists $i_0 \ge \hat{i}$ such that the following hold for $n \ge i_0$:

(i) If $\hat{x}_n$ is free, and $\hat{x}_{n+j} \notin C_\delta\ \forall\ 0 \le j < j_0$, then $|\tau_{n+j}| > \frac{1}{2}c_1\delta e^{\frac{1}{4}\lambda_0 j}|\tau_n|$ for $j \le j_0$;

(ii) if in addition $\hat{x}_{n+j_0} \in C_{\delta_0}$, then $|\tau_{n+j_0}| > \frac{1}{2}c_1 e^{\frac{1}{4}\lambda_0 j_0}|\tau_n|$.

The reader should think of i_0 as the time after which x- and a-derivatives are sufficiently close in the sense of Lemma 5.1. We assume $\hat{x}_i \notin C_{\frac{1}{2}\delta_0}$ for all $i < i_0$.

Consider next an interval $\omega \subset \Omega_N$ with $\hat{x}_n(\omega) \subset I^+_{\mu j}$. To establish the desired relationship between phase-space and parameter-space dynamics during the bound period, we impose the following additional upper bound on α: Let L be a Lipschitz constant of the map $G : (x, a) \mapsto (f_a(x), a)$. We assume α is small enough that

$$L^{\frac{3}{\lambda}\alpha}\ <\ e^{\frac{1}{8}\lambda}. \tag{5}$$

For each $a \in \omega$, let p_a denote the bound period of $I_{\mu j}$ with respect to f_a, and let $\mathrm{HD}(\cdot,\cdot)$ denote the Hausdorff distance between two sets.

Lemma 5.2 *Let ω and α be as above. Then the following hold for all $a, a' \in \omega$:*

$$\mathrm{HD}(\hat{x}_{n+j}(\omega), f_a^j(\hat{x}_n(\omega))) << e^{-2\alpha j} \quad \text{for all } j < p_a.$$

We define the *bound period* $\hat{p}_n(\omega)$ of $\hat{x}_n(\omega)$ *in parameter-space dynamics* to be

$$\hat{p}_n(\omega) := \min\{p_a : a \in \omega\}.$$

(P2') (Partial derivative recovery): *Suppose $\hat{x}_n(\omega) \subset I^+_{\mu j}$, and let $\hat{p} = \hat{p}_n(\omega)$. Then*

(a) $\frac{1}{3\ln(max|f'|)}|\mu| \le \hat{p} \le \frac{3}{\lambda}|\mu|$;

(b) for $a, a' \in \omega$ and $j < \hat{p}$, $|\hat{x}_{n+j}(a) - \hat{x}_{n+j}(a')| < 2e^{-2\alpha j}$;

(c) $|\tau_{n+\hat{p}}(a)| > e^{\frac{\lambda\hat{p}}{4}}|\tau_n(a)|$ for all $a \in \omega$;

(d) if $\hat{x}_n(\omega) \approx I_{\mu j}$, then $|\hat{x}_{n+\hat{p}}(\omega)| \ge e^{-\frac{8\alpha}{\lambda}|\mu|}$.

To state (P3'), we divide each orbit in the time interval $[i_0, n]$ into bound and free periods as in Sect. 2.1, and say all $a \in \omega$ have *the same itinerary* up to time n if (i) their bound and free periods coincide and (ii) whenever $\hat{x}_i(\omega)$ is free, it is $\subset \pi^+$ for some $\pi \in \mathcal{P}$.

(P3') (Global distortion): *There exist $i_1 > i_0$ and $K_3 > 1$ such that if $n \geq i_1$ and all points in ω have the same itinerary through step $n-1$, then for all $a, a' \in \omega$,*

$$K_3^{-1} < \frac{\tau_n(a)}{\tau_n(a')} < K_3.$$

Shrinking ε if necessary, we assume from here on that $\hat{x}_i \notin C_{\frac{1}{2}\delta_0}$ for all $i < i_1$.

5.3 Proofs of (P1')–(P3')

Proof of Lemma 5.1: Let $\xi = \hat{x}_n$. From

$$\tau_{n+i} = (f_a)'(\xi_{i-1})\tau_{n+i-1} + \partial_a F(\xi_{i-1}, a),$$

we deduce inductively that

$$\begin{aligned} \tau_{n+i} &= (f_a^i)'(\xi)\tau_n + \sum_{j=0}^{i-1} (f^{i-j-1})'(\xi_{j+1})\partial_a F(\xi_j, a) \\ &= (f_a^i)'(\xi)\tau_n \left(1 + \sum_{j=0}^{i-1} \frac{\partial_a F(\xi_j, a)}{(f_a^{j+1})'(\xi)\tau_n}\right). \end{aligned}$$

Proposition 4.2' and Corollary 2.1 then give

$$\left|\sum_{j=0}^{i-1} \frac{\partial_a F(\xi_j, a)}{(f_a^{j+1})'(\xi)\tau_n}\right| \;\leq\; 2|\hat{c}(\hat{x})| \sum_{j=0}^{i-1} \left|\frac{\partial_a F(\xi_j, a)}{(f_a^{n+j})'(\hat{x}_1(a))}\right| \;\leq\; K e^{-\frac{1}{4}\lambda n}.$$

$\square$

Proof of (P1'): (P1') follows immediately from (P1) via Lemma 5.1. With i_0 large enough, the x- and a-derivatives are as close as need be. $\square$

Proof of Lemma 5.2: It suffices to consider the end points of the segments to be compared. Suppose $\omega = [\bar{a}_1, \bar{a}_2]$. Then for $i = 1, 2$,

$$|\hat{x}_{n+j}(\bar{a}_i) - f_a^j(\hat{x}_n(\bar{a}_i))| \;<\; |G^j(\hat{x}_n(\bar{a}_i), \bar{a}_i) - G^j(\hat{x}_n(\bar{a}_i), a)| \;\leq\; L^j|\bar{a}_i - a| \;=\; L^j|\omega|. \quad (6)$$

By Proposition 4.2 and Corollary 2.1, $|\omega| \leq K e^{-\frac{1}{4}\lambda n}$. Also, $j \leq p_a$, which, by (P2), is $\leq \frac{3}{\lambda}\alpha n$. Thus $L^j|\omega| << e^{-2\alpha j}$ if α satisfies (5). $\square$

Proof of (P2'): (a) is true because $\hat{p} = p_{\bar{a}}$ for some $\bar{a} \in \omega$. (b) and (d) are immediate consequences of Lemma 5.2, and (c) follows from Lemma 5.1. $\square$

Turning now to the setting of (P3'), we let $a, a' \in \omega$, and for some k with $i_0 \leq k < n$, let $\xi = \hat{x}_k(a)$ and $\xi' = \hat{x}_k(a')$.

Lemma 5.3 *There exists $K>0$ such that for all i such that $k+i\le n$,*

$$\frac{(f_a^i)'(\xi)}{(f_{a'}^i)'(\xi')} \le \exp\left\{\sum_{j=0}^{i-1} K\frac{|\xi_j-\xi_j'|}{d(\xi_j,C)}\right\}.$$

Proof: First we write

$$\begin{aligned}\log\frac{(f_{a'}^i)'(\xi')}{(f_a^i)'(\xi)} &\le \sum_{j=0}^{i-1}\frac{|(f_a)'(\xi_j)-f_{a'}'(\xi_j')|}{|(f_a)'(\xi_j)|} < \sum_{j=0}^{i-1}K\frac{|\xi_j-\xi_j'|+|a-a'|}{d(\xi_j,C)}\\ &< K\sum_{j=0}^{i-1}\left(1+\frac{|a-a'|}{|\xi_j-\xi_j'|}\right)\frac{|\xi_j-\xi_j'|}{d(\xi_j,C)}.\end{aligned}$$

Then we use Proposition 4.2' and Corollary 2.1 to estimate the quantity in parenthesis: Assuming for definiteness that $a'<a$,

$$|\xi_j-\xi_j'| = \int_{a'}^a |\tau_{k+j}(s)|ds \ge \frac{1}{2}|\hat{c}(\hat{x})|\int_{a'}^a |(f_s^{k+j})'(\hat{x}_1(s))|ds \ge \frac{1}{2}|\hat{c}(\hat{x})|e^{\frac{1}{4}\lambda k}|a-a'|. \qquad (7)$$

$\square$

Proof of (P3'): By Proposition 4.2', it suffices to show

$$\frac{(f_{a'}^{n-1})'(\hat{x}_1(a'))}{(f_a^{n-1})'(\hat{x}_1(a))} < K,$$

the proof of which follows closely that of (P3) in Sect. 2.2. Let $i_0<t_1<t_1+p_1\le t_2< t_2+p_2\le\cdots$ where t_i are free return times and p_i are the associated bound periods. For definiteness we assume $t_q+p_q\le n\le t_{q+1}$. Then

$$\log\frac{(f_{a'}^{n-1})'(\hat{x}_1(a'))}{(f_a^{n-1})'(\hat{x}_1(a))} \le \log\frac{(f_{a'}^{t_1-1})'(\hat{x}_1(a'))}{(f_a^{t_1-1})'(\hat{x}_1(a))} + \sum_{k=1}^q(S_k'+S_k'')$$

where

$$S_k' = \log\frac{|(f_{a'}^{p_k})'(\hat{x}_{t_k}(a'))|}{|(f_a^{p_k})'(\hat{x}_{t_k}(a))|} \quad\text{and}\quad S_k'' = \log\frac{|(f_{a'}^{t_{k+1}-(t_k+p_k)})'(\hat{x}_{t_k+p_k}(a'))|}{|(f_a^{t_{k+1}-(t_k+p_k)})'(\hat{x}_{t_k+p_k}(a))|},$$

except for S_q'', which ends at time $n-1$ instead of t_{q+1}.

Let $\sigma_k=[\hat{x}_{t_k}(a),\hat{x}_{t_k}(a')]$. It follows from (P2')(c) and (P1')(ii) that, for $k<q$, $|\sigma_{k+1}|\ge \frac{1}{2}c_1e^{\frac{1}{4}\lambda(t_{k+1}-t_k)}|\sigma_k|$, which we may assume is $\ge\hat{\tau}|\sigma_k|$ for some $\hat{\tau}>1$ (the factor $\frac{1}{2}c_1$ is again absorbed into the exponential assuming δ is sufficiently small).

In the proof of (P3), the derivative estimates in S_k' and S_k'' are converted to distance estimates. This is exactly what we have done in Lemma 5.3. More precisely, if $\xi=\hat{x}_{t_k}(a)$ and $\xi'=\hat{x}_{t_k}(a')$, then by Lemma 5.3,

$$S_k' \le K\sum_{j=0}^{p_k-1}\frac{|\xi_j-\xi_j'|}{d(\xi_j,C)} \le K\sum_{j=0}^{p_k-1}\frac{|f_a^j(\xi)-f_a^j(\xi')|}{d(\xi_j,C)} + K\sum_{j=0}^{p_k-1}\frac{|f_a^j(\xi')-f_{a'}^j(\xi')|}{d(\xi_j,C)}.$$

The first sum on the right involves only the map f_a and is $\leq K\frac{|\sigma_k|}{d(\xi,C)}$ by the corresponding argument in the proof of (P3). For the second sum, we combine the estimate in (6) with the result of (7) to show that it is

$$\leq \; K\left(\sum_{j=0}^{p_k-1}(Le^{\alpha})^j\right)|a-a'| \;\leq\; K(Le^{\alpha})^{\frac{3}{\lambda}\alpha t_k}e^{-\frac{1}{4}\lambda t_k}|\sigma_k| \;<<\; |\sigma_k|.$$

After this, the rest of the proof is as before. The sums S_k'' (including $k=q$) are estimated similarly, the only difference being that the exponential growth of $|\xi_j-\xi_j'|$ here is derived from (P1').

It remains to treat the initial stretch. The estimate from time i_0 to time t_1 is identical to that for S_k'', and

$$\log\frac{(f_{a'}^{i_0-1})'(x_1(a'))}{(f_a^{i_0-1})'(x_1(a))} \;\leq\; K^{i_0}|a-a'|$$

where K is related to the C^2 norm of the 1-parameter family. Since $|a-a'|<Ke^{-\frac{1}{4}\lambda i_1}$, we know that the contribution of the first i_0 iterates is <1 if i_1 is sufficiently large.

This completes the proof of (P3'). $\square$

Reference: A version of the material in Sects. 5.2 and 5.3 is contained in [BC1].

6 Two Parameter Estimates

6.1 Processes defined by critical orbits

Let $\lambda,\alpha,\varepsilon$ and $\hat\varepsilon$ be as in Sect. 5.2. Associated with each $\hat x\in C$ we now introduce the idea of a process $\{\gamma_i\}$ describing the dynamics of $a\mapsto\hat x(a)$ combined with a deletion process. The domain of definition of this process is Δ_0, a subinterval of $(-\hat\varepsilon,\hat\varepsilon)$. For each $i=0,1,2,\cdots$ and $a\in\Delta_0$, $\gamma_i(a)$ is equal to either $\hat x_i(a)$ or $*$, the meaning of the latter being that the parameter in question will not be considered further, *i.e.*, it is *deleted*. In particular, if $\gamma_i(a)=*$, then $\gamma_{i+k}(a)=*$ for all $k>0$,

Roughly speaking, we seek to identify a decreasing sequence of subsets $\{\gamma_i\neq *\}$ of Δ_0 and a sequence of partitions $\mathcal{Q}_i$ defined on $\{\gamma_i\neq *\}$ representing the canonical subdivision associated with $a\mapsto\hat x(a)$. This subdivision is carried out in a manner analogous to that in Sect. 3.1. As the reader will recall, the construction in Sect. 3.1 relies on the decomposition of orbits into bound and free periods. We have seen in Section 5 that bound/free notions are well defined for $a\mapsto\hat x(a)$ under suitable circumstances. The idea is that *whenever we are unable to guarantee these circumstances, we will delete the parameter.* Later on, we will see that it is useful also to make deletions for other purposes (but will not concern ourselves with that for now). This is the motivation for the definition below.

For $N\leq\infty$, we say $\{\gamma_i, i<N\}$ is *a process associated with* $\hat x$ if the following hold: Let $\Delta_0\subset(-\hat\varepsilon,\hat\varepsilon)$ be an interval containing 0, and let i_1 be as in Sect. 5.2.

(1) (a) We assume $\gamma_i(\Delta_0) \cap C_{\frac{1}{2}\delta_0} = \emptyset$ for all $i \le i_1$. In this time range, γ_i has no meaning. We set $\gamma_i = \hat{x}_i$ and $\mathcal{Q}_i = \{\Delta_0\}$. No deletions are permitted.

(b) A subdivision must occur before $\gamma_i(\Delta_0)$ meets C_δ.

(2) At time $i > i_1$, we assume that for all $j < i$, γ_j are defined, as are $\mathcal{Q}_j$, representing a canonical subdivision. We assume also that the notion of bound/free makes sense on each $\omega \in \mathcal{Q}_j$. Consider now $\omega \in \mathcal{Q}_{i-1}$ (on which $\gamma_{i-1} \neq *$). We first put on it the canonical partition $\mathcal{Q}_i$ as defined in Sect. 3.1. On each $\hat{\omega} \in \mathcal{Q}_i|_\omega$, there are 2 options: we either let $\gamma_i = \hat{x}_i$ on all of $\hat{\omega}$, or we let it $= *$ on all of $\hat{\omega}$. The rules are as follows:

(a) We are free to set $\gamma_i = *$ or $\hat{x}_i$ on any $\hat{\omega}$ outside of C_δ.

(b) For $\hat{\omega} \subset C_\delta$, the following conditions must be met if we wish to set $\gamma_i|_{\hat{\omega}} = \hat{x}_i$:

(i) $\hat{x}_i(\hat{\omega}) \cap \{d(\cdot, C) < e^{-\alpha i}\} = \emptyset$;

(ii) $\hat{\omega} \subset \Omega_{\alpha^* i}$, *i.e.*, $f_a \in \mathcal{G}_{\alpha^* i}(f_0; \lambda, \alpha, \varepsilon)$ for all $a \in \hat{\omega}$.

Finally, we set $\gamma_i = *$ on $\{\gamma_{i-1} = *\}$. This completes our definition at the ith step. Paragraph (2) is then repeated with $i+1$ in the place of i.

We observe that the process above is well defined. It is clearly well defined initially. When $\gamma_i(\hat{\omega}) \subset C_\delta$, (2)(b) guarantees that for all $a \in \hat{\omega}$ for which $\gamma_i(a) \neq *$, the ensuing bound period is meaningful (see Sect. 5.1). Once this is taken care of, Sect. 5.2 gives the desired resemblance to phase-space dynamics until the next free return.

We remark that even though $\{\gamma_i\}$ is associated with a particular $\hat{x} \in C$, condition (2)(b)(ii) demands good behavior of *all* critical orbits up to time $\alpha^* i$. The fact that this requirement is only up to time $\alpha^* i$, which is $<< i$, is crucial for us. Observe also that on each Δ_0, there is a *maximal* process, referring to the one in which the only deletions are those in (2)(b). All other processes are subordinate to this one, meaning they are defined by rules that demand further deletions.

6.2 Deletions due to (G1)

We fix $\hat{x} \in C$ and Δ_0, and let $\{\gamma_i, i < n\}$ be a process associated with $\hat{x}$.

Lemma 6.1 *There exists K such that for all $\omega \in \mathcal{Q}_{n-1}$, if ω_1 is the part of ω deleted on account of (G1) at step n, then*

$$|\omega_1| \; < \; K e^{-\frac{1}{2}\alpha n} |\omega| \; .$$

Proof: Suppose $\omega_1 \neq \emptyset$. Let j_0 be the largest $j < n$ such that (i) $\omega \in \mathcal{Q}_j$, (ii) $\gamma_j(\omega)$ is free and "long". (Such a j_0 exists by condition (1)(b) in Sect. 6.1; one may have to go back to the time when ω is created as a result of a subdivision.) There are two possibilities:

Case 1. $\gamma_{j_0}(\omega)$ is outside of C_δ, *i.e.*, $\gamma_{j_0}(\omega) \approx \pi$ for some π with $\pi \cap C_\delta = \emptyset$. Then $|\gamma_{j_0}(\omega)| \ge \delta$, and $|\gamma_n(\omega)| > K^{-1}\delta$. Not knowing the location of $\gamma_n(\omega)$, we assume the worst-case scenario, namely that $\gamma_n(\omega)$ crosses entirely a forbidden region $\{d(\cdot, \hat{y}) < e^{-\alpha n}\}$ for some $\hat{y} \in C$. Thus the fraction of ω with $d(\hat{x}_n, C) < e^{-\alpha n}$ is $< 2e^{-\alpha n} \cdot K\delta^{-1}$, which we may assume is $< Ke^{-\frac{1}{2}\alpha n}$ (see the paragraph following Proposition 2.2). Here (P3') is used to transfer the ratio of lengths on $\gamma_n(\omega)$ back to ω.

Case 2. $\gamma_{j_0}(\omega) \approx I_{\mu j}$. Let p be the bound period initiated at time j_0. Observe first that $|\gamma_n(\omega)| > K^{-1}|\gamma_{j_0+p}(\omega)| > K^{-1}e^{-\frac{1}{10}|\mu|}$: Since $\gamma_n(\omega)$ is free (otherwise there would be no deletion), $n \geq j_0 + p$. The first inequality follows from (P1')(ii) combined (possibly) with (P2')(c); the second follows from (P2')(d). Observe also that by design, $|\mu| \leq \alpha j_0 < \alpha n$, so the fraction of ω being estimated is again $< Ke^{-\alpha n}e^{\frac{1}{10}\alpha n} < Ke^{-\frac{1}{2}\alpha n}$. $\square$

6.3 Deletions on account of (G2)

We begin with an estimate on derivative growth in terms of the time an orbit spends in bound periods initiated at returns to $C_{\hat{\delta}}$ for arbitrary $\hat{\delta} < \delta$. Consider $f \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$ and $n \leq \frac{1}{\alpha^*}N$. Let $x \in I$ be such that $d(x_i, C) \geq \min\{\frac{1}{2}\delta_0, e^{-\alpha i}\}$ for all $0 \leq i < n$. By the reasoning in Sect. 5.1, the usual bound/free decomposition makes sense for the orbit of x up to time n. Let $B(\hat{\delta}; n)$ denote the total number of i, $0 \leq i \leq n$, such that $x_i \in C_{\hat{\delta}}$ or it is in a bound period initiated from a visit to $C_{\hat{\delta}}$.

Lemma 6.2 *Let f and x be as above. Given $\hat{\delta} \leq \delta$ and $\sigma > 0$, if $B(\hat{\delta}; n) \leq \sigma n$, then*

$$|(f^n)'(x)| > K^{-1}\hat{\delta}\ e^{[(1-\sigma)\frac{1}{4}\lambda_0 - \alpha]n}\ .$$

Proof: Consider first the case where x_n is free. Let $\hat{t}_1 < \hat{t}_1 + \hat{p}_1 \leq \hat{t}_2 < \hat{t}_2 + \hat{p}_2 \leq \cdots \leq \hat{t}_k + \hat{p}_k \leq n$ where $\hat{t}_1, \cdots, \hat{t}_k$ are the consecutive free return times to $\{d(\cdot, C) < \hat{\delta}\}$. Then

$$(f^n)'(x) = (f^{n-\hat{t}_k-\hat{p}_k})'(x_{\hat{t}_k+\hat{p}_k}) \cdot (f^{\hat{p}_k})'(x_{\hat{t}_k}) \cdot (f^{\hat{t}_k-\hat{t}_{k-1}-\hat{p}_{k-1}})'(x_{\hat{t}_{k-1}+\hat{p}_{k-1}}) \cdots (f^{\hat{t}_1})'(x).$$

We use (P1)(i) for $(f^{n-\hat{t}_k-\hat{p}_k})'(x_{\hat{t}_k+\hat{p}_k})$, (P1)(ii) for $(f^{t_i-\hat{t}_{i-1}-\hat{p}_{i-1}})'(x_{\hat{t}_{i-1}+\hat{p}_{i-1}})$, and the trivial bound $|(f^{\hat{p}_i})'(x_{\hat{t}_i})| > c_1^{-1}$ from (P2)(ii). This gives $|(f^n)'(x)| > K^{-1}\hat{\delta}e^{\frac{1}{4}\lambda_0(1-\sigma)n}$ since $\hat{p}_1 + \cdots + \hat{p}_k \leq \sigma n$ by assumption. The factor $-\alpha n$ is needed if n is not free; see Corollary 2.1. $\square$

Corollary 6.1 *Let the hypotheses be as in Lemma 6.2, with $x = \hat{x}_1$ for some $\hat{x} \in C$. We assume further that $d(\hat{x}_i, C) > \frac{1}{2}\delta_0$ for all $i \leq n_0$ where n_0 is sufficiently large depending on $\hat{\delta}$. Then $B(\hat{\delta}; 0, n) < \sigma n$ implies $|(f^n)'(\hat{x}_1)| > c_1 e^{[(1-\sigma)\frac{1}{4}\lambda_0 - \alpha]n}$.*

Proof: The factor $\hat{\delta}$ is absorbed into the initial growth if $n > n_0$. $\square$

For $f \in \mathcal{G}$, it can be deduced from properties of the invariant measure that $\int \frac{1}{n}B(\hat{\delta}; n)d\mu$ decreases with $\hat{\delta}$. In light of the duality in Sect. 5.2, one may expect a similar phenomenon for $a \mapsto \gamma_i(a)$. We formulate below a large deviation estimate useful for estimating the measure of parameters deleted on account of (G2).

Let $\{\gamma_i, i < n\}$ be as in Sect. 6.2. For a such that $\gamma_n(a) \neq *$, let $B(a, \hat{\delta}; n)$ be the number $B(\hat{\delta}; n)$ defined above with $f = f_a$ and $x = \hat{x}$.

Proposition 6.1 *Given any $\sigma > 0$, there exist positive numbers $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(\sigma)$ and $\hat{\delta} = \hat{\delta}(\sigma)$ such that*

$$|\{a \in \Delta_0 : \gamma_n \neq * \text{ and } B(a, \hat{\delta}; n) > \sigma n\}| < e^{-\hat{\varepsilon}_1 n}|\Delta_0|.$$

6.4 Large deviation estimate

We first state the analog of Lemma 3.3. Let $\hat\omega \in \mathcal{Q}_{j_0}$ be such that $\gamma_{j_0}(\hat\omega) \neq *$ and is free. On $\hat\omega$ we define $\hat S$, a stopping time starting from j_0, as follows: We extend the process on ω beyond time j_0, and for each $a \in \omega$, let $k = k(a) > j_0$ be the first time when $\gamma_k(\mathcal{Q}_{k-1}(a))$ is not in a bound period and has length $> \delta$. If such a k exists, we set $\hat S(a) = k - j_0$. If a is deleted before that happens, we set $\hat S(a) = 0$.

Lemma 6.3 *Let $\hat\omega \in \mathcal{Q}_{j_0}$ be such that $\gamma_{j_0}(\hat\omega)$ is free and $\approx I_{\mu j}$. Then*

$$|\{a \in \hat\omega : \hat S(a) > m\}| \ < \ e^{-\frac{1}{2}K^{-1}m}\ |\hat\omega| \quad \text{for all} \quad m > K \log|\mu|.$$

The proof is entirely parallel to that of Lemma 3.3 in Sect. 3.1.

Proof of Proposition 6.1: We take a probabilistic viewpoint, with underlying probability space (Δ_0, P), P being normalized Lebesgue measure on Δ_0. Let $\hat\delta > 0$ be a small number to be determined. Let n be fixed. The idea is to introduce X_i dominated by certain exponential random variables such that $B(a) := B(a, \hat\delta, n) \leq \sum X_i(a)$.

Step I. Formulation of problem as one involving $\sum X_i$

For each $a \in \Delta_0$, we define $t_0 < t_1 < \cdots$ and $S_1, S_2, \cdots$ via the following algorithm, with the understanding that the algorithm terminates as soon as $\gamma_i(a) = *$ or time n is reached. To get started, let t_0 be the smallest $j > 0$ such that $\gamma_j(\mathcal{Q}_{j-1}(a)) \cap C_{\hat\delta} \neq \emptyset$.

(i) After t_i is defined, we define S_{i+1}: If $\mathcal{Q}_{t_i}(a) \cap C_{\hat\delta} = \emptyset$, set $S_{i+1} = 0$; if $\mathcal{Q}_{t_i}(a) \subset C_{\hat\delta}$, let $S_{i+1} = \min(\hat S, n - t_i)$ where $\hat S$ is the stopping time above starting from t_i.

(ii) If $S_{i+1} = 0$, let t_{i+1} be the smallest $j > t_i$ such that $\gamma_j(\mathcal{Q}_{j-1}(a)) \cap C_{\hat\delta} \neq \emptyset$; define t_{i+1} the same way if $S_{i+1} > 0$ except that j is taken $\geq t_i + S_{i+1}$.

Suppose $t_i(a)$ is defined. Let $Q = \mathcal{Q}_{t_i-1}(a)$. Assuming $\hat\delta << \delta$, we claim:

(1) $\gamma_{t_i}(Q)$ is free;

(2) $|\gamma_{t_i}(Q)| > \hat\delta^{\frac{1}{10}}$;

(3) for all $a', a'' \in Q$, $\tau_{t_i}(a')/\tau_{t_i}(a'') < K$.

(1) is true because trajectories of critical curves in bound periods initiated outside of $C_{\hat\delta}$ cannot meet $C_{\hat\delta}$. If $S_i > 0$, it may happen that $t_i = t_{i-1} + S_i$, in which case $|\gamma_{t_i}(Q)| > \delta$ by definition. Otherwise we back up to time t when Q was first created as an element of some $\mathcal{Q}_j$. Then $t_{i-1} + S_i \leq t < t_i$, and $\gamma_t(Q) \cap C_{\hat\delta} = \emptyset$. If $\gamma_t(Q)$ is outside of C_δ, then $|\gamma_{t_i}(Q)| > K^{-1}\delta$ by (P1'). If $\gamma_t(Q) \approx I_{\mu j}$ for some $I_{\mu j} \subset C_\delta \setminus C_{\hat\delta}$, then $|\gamma_{t_i}(Q)| > K^{-1}\hat\delta^{\frac{8\alpha}{\lambda}}$ by (P2')(d). In all cases, (2) holds assuming $\hat\delta << \delta$. (3) follows from (P3').

Because of (1)–(3), we think of t_i as times of *dynamical renewal.*

In preparation for Step II, we organize some of the information from above as follows. Let $X_0 = 0$. For $i = 1, 2, \cdots, n$, let $X_i : \Delta_0 \to \mathbb{Z}^+$ be such that $X_i(a) = S_i(a)$ where $S_i(a)$ is defined, 0 otherwise. Then $B \leq \sum_{i \leq n} X_i$. It suffices to show $P\{\sum_{i\leq n} X_i > \sigma n\}$ decreases exponentially with n. We define the following σ-algebras on ω: Let A_i be the set of a for which t_i is defined. Then $A_i \in \mathcal{F}_i$, and for $a \in A_i$, the atom of $\mathcal{F}_i$ containing a is $\mathcal{Q}_{t_i(a)-1}(a)$. For $a \notin A_i$, the atom of $\mathcal{F}_i$ containing a is $\mathcal{Q}_k(a)$ where k is the last step before the algorithm above is terminated. One verifies that $\mathcal{F}_i$ so defined is a σ-algebra, that $\mathcal{F}_0 < \mathcal{F}_1 < \cdots < \mathcal{F}_n$, and that X_i is measurable with respect to $\mathcal{F}_i$.

Step II. Large deviation estimate for $\sum_{1\le i\le n} X_i$

First we compute the conditional distribution of X_{i+1} given $\mathcal{F}_i$, $i \ge 0$. Consider $Q \in \mathcal{F}_i|_{A_i}$. (On $Q \in \mathcal{F}_i$ with $Q \cap A_i = \emptyset$, $X_{i+1} = 0$.) From (2) and (3) above, we have

(i) $P(X_{i+1} = 0 \mid Q) \ge 1 - K\hat{\delta}^{\frac{9}{10}}$.

For $I_{\mu j} \subset C_{\hat{\delta}}$, Lemma 6.3 together with (1) and (3) above give

(ii) $P(X_{i+1} > m \mid Q \cap \{\gamma_{t_i} \in I_{\mu j}\}) < Ke^{-\frac{1}{2}K^{-1}m}$ if $m \ge K|\mu|$; no information otherwise.

Combining the last two estimates, we obtain for all $m \ge 0$,

$$P(X_{i+1} > m \mid Q) < K\hat{\delta}^{-\frac{1}{10}} \min(\hat{\delta}, e^{-K^{-1}m}) + K\hat{\delta}^{\frac{9}{10}} e^{-\frac{1}{2}K^{-1}m}. \tag{8}$$

A simple computation then gives $E[e^{\rho X_{i+1}}|Q] < \infty$ if $\rho < \frac{1}{2}K^{-1}$ (where K is as in the exponents above). We note further that by decreasing $\hat{\delta}$ (keeping ρ fixed), $E[e^{\rho X_{i+1}}|Q]$ can be made arbitrarily close to 1. Let $\eta > 0$ be a number to be determined shortly, and choose $\hat{\delta} = \hat{\delta}(\eta)$ sufficiently small that $E[e^{\rho X_{i+1}}|Q] < e^{\eta}$. Observing that the upper bound in (8) and hence that for $E[e^{\rho X_{i+1}}|Q]$ do not depend on i or on Q, we conclude that with the choices of ρ, η and $\hat{\delta}$ above, $E[e^{\rho X_{i+1}}|\mathcal{F}_i] < e^{\eta}$ for every $i \ge 0$.

To finish, we observe that

$$\begin{aligned} E\left[e^{\rho\sum_{1\le i\le n} X_i}\right] &= E\left[E[e^{\rho\sum_{1\le i\le n} X_i}|\mathcal{F}_{n-1}]\right] = E\left[e^{\rho\sum_{1\le i\le n-1} X_i}\ E[e^{\rho X_n}|\mathcal{F}_{n-1}]\right] \\ &\le e^{\eta}\ E\left[e^{\rho\sum_{1\le i\le n-1} X_i}\right], \end{aligned}$$

giving inductively $E[e^{\rho\sum_{1\le i\le n} X_i}] \le e^{n\eta}$. We arrive, therefore, at the estimate

$$P\{B > \sigma n\} \ < \ P\left\{\sum_{1\le i\le n} X_i > \sigma n\right\} \ < \ e^{\eta n - \rho\sigma n}.$$

This is $< e^{-\frac{1}{2}\rho\sigma n}$ if η is chosen $< \frac{1}{4}\rho\sigma$. $\square$

References: A version of Sects. 6.2 and 6.3 is used in [BC2]; Sect. 6.4 is taken from [WY2].

7 Positive measure sets of good parameters

7.1 Preliminary definitions and choices

1. We fix $\lambda \le \frac{1}{5}\lambda_0$.

2. *Augmented versions of (G1) and (G2).* For reasons to become clear, it will be advantageous to put our good maps "deeper inside" $\mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$. We say $\hat{x} \in C$ satisfies (G1)$^{\#}$ and (G2)$^{\#}$ up to time N if for all $1 \le i \le N$,

(G1)$^{\#}$ $d(\hat{x}_i, C) > \min(\frac{1}{2}\delta_0, 2e^{-\alpha i})$;

(G2)$^\#$ $|(f^i)'(\hat{x}_1)| > 2c_1 e^{\lambda_1 i}$ *where* $\lambda_1 = \lambda + \frac{1}{100}\lambda_0$,

and say $f \in \mathcal{G}_N^\#(f_0; \lambda, \alpha, \varepsilon)$ if all $\hat{x} \in C$ satisfy (G1)$^\#$ and (G2)$^\#$ up to time N. Clearly, $\mathcal{G}_N^\#(f_0; \lambda, \alpha, \varepsilon) \subset \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$. The proof of the following lemma is straightforward.

Lemma 7.1 *There exists* $K_4 > 1$ *for which the following holds: If* $f_{\hat{a}} \in \mathcal{G}_N^\#(f_0; \lambda, \alpha, \varepsilon)$, *then for all* $n \leq N$, $f_a \in \mathcal{G}_n(f_0; \lambda, \alpha, \varepsilon)$ *for all* $a \in [\hat{a} - K_4^{-n}, \hat{a} + K_4^{-n}]$.

3. *Choice of* α. In addition to $\alpha < \frac{1}{100}\lambda$, we impose two upper bounds on α: The first is introduced in (5) in Sect. 5.2; the second is (9) in Sect. 7.2. With λ and α fixed, $\mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$ and $\mathcal{G}_N^\#(f_0; \lambda, \alpha, \varepsilon)$ will be abbreviated as $\mathcal{G}_N$ and $\mathcal{G}_N^\#$ from here on.

4. Choices of σ and $\hat{\delta}$. We need σ to be small enough that the exponent in Corollary 6.1, namely $(1-\sigma)\frac{1}{4}\lambda_0 - \alpha$, is $> \lambda_1$. (For example, $\sigma = \frac{1}{100}$ will do.) We then let $\hat{\delta}$ be given by Proposition 6.1 with $\frac{1}{2}\sigma$ in the place of σ.

5 *The start-up interval* Δ_0. We choose $\Delta_0 \subset (-\hat{\varepsilon}, \hat{\varepsilon})$ to contain 0 and to be short enough that for some n_0 sufficiently large, $d(\hat{x}_i, C) > \frac{1}{2}\delta_0$ for all $i \leq n_0$, $\hat{x} \in C$ and $a \in \Delta_0$. A number of impositions on n_0 have been made; see, for example, Sects. 2.1 and 5.2, and Corollary 6.1. There will be more in the next two pages.

7.2 Inductive construction of Δ

We seek to construct a sequence of sets $\Delta_0 \supset \Delta_{n_0} \supset \Delta_{2n_0} \supset \Delta_{2^2 n_0} \supset \cdots$ in parameter space with the properties that

(i) for each ℓ, $\{f_a, a \in \Delta_{2^\ell n_0}\} \subset \mathcal{G}_{2^\ell n_0}^\#$ and

(ii) $\Delta := \cap_{\ell \geq 0} \Delta_{2^\ell n_0}$ has positive Lebesgue measure.

The rules of construction are detailed below; the measure estimate is given in Sect. 7.3.

Overview of procedure

Let $C := \{\hat{x}^1, \hat{x}^2, \cdots, \hat{x}^q\}$. Associated with each $\hat{x}^k$, we define a process $\{\gamma_i^k, i < \infty\}$ in the sense of Sect. 6.1 with the property that for every a such that $\gamma_{2^\ell n_0}^k(a) \neq *$, $\hat{x}^k(a)$ satisfies (G1)$^\#$ and (G2)$^\#$ up to time $2^\ell n_0$. We then let

$$\Delta_{2^\ell n_0}^k := \{\gamma_{2^\ell n_0}^k \neq *\} \qquad \text{and} \qquad \Delta_{2^\ell n_0} := \cap_{1 \leq k \leq q} \Delta_{2^\ell n_0}^k.$$

It follows that $f_a \in \mathcal{G}_{2^\ell n_0}^\#$ for every $a \in \Delta_{2^\ell n_0}$.

The processes γ_i^k are updated in N-to-$2N$ cycles, $N = 2^\ell n_0, \ell = 1, 2, \cdots$. Within each cycle, we first update each of the q processes individually, *i.e.*, extend γ_i^k from $i = N$ to $i = 2N$. At the end of this updating, we reset some of the values of γ_{2N}^k to $*$ to reflect the combined status of all q processes before moving to the next cycle.

Remarks. It is absolutely essential to take inventory of the global picture at regular time intervals (as we do at times $2^\ell n_0$). Other than that, the precise order in which γ_i^k is updated is unimportant. Also, the number "2" has little significance: all that is needed is a relation with α that gives (9) below.

Getting started: Let n_1^k be the smallest $i>0$ such that $\hat{x}_i^k(\Delta_0)\cap C_{\frac{1}{2}\delta_0}\neq\emptyset$. Then $n_1^k>n_0$, and $|\hat{x}^k_{n_1^k}(\Delta_0)|>\frac{1}{2}\delta_0$. Since $\delta<<\frac{1}{2}\delta_0$, the first subdivision occurs at or before time n_1^k. Condition (1)(b) in the definition of a process in Sect. 6.1 is met. Recall that we have assumed $e^{-\alpha n_0}<<\delta$, so that Lemma 6.1 applies to all deletions due to (G1)$^{\#}$.

Formal procedure from step $N=2^\ell n_0$ to step $2N$

At time N, we are handed q well defined processes $\{\gamma_i^k, i\leq N\}$, $k=1,2,\cdots,q$, with the property that for each $a\in\Delta_N^k=\{\gamma_N^k\neq *\}$,

(i) $f_a\in\mathcal{G}_{2\alpha^* N}$, and

(ii) $\hat{x}^k(a)$ satisfies (G1)$^{\#}$ and (G2)$^{\#}$ up to time N.

The procedure from N to $2N$ consists of the following:

Step 1. The following is carried out for each of the processes $\{\gamma_i^k\}$.

1a. We extend γ_i^k up to $i=2N$, deleting all $\omega\in\mathcal{Q}_i$ with $\gamma_i(\omega)\cap\{d(\cdot,C)<2e^{-\alpha i}\}\neq\emptyset$. (Deletions corresponding to (2)(b)(ii) in Sect. 6.1 are not needed because (i) above already puts $\{\gamma_N^k\neq *\}\subset\Omega_{2\alpha^* N}$.) We denote the resulting γ_{2N}^k by $\gamma^k_{2N,\mathrm{tmp1}}$ as its values will be reset momentarily.

1b. On each $\omega\in\mathcal{Q}_{2N}^k$, we let $\gamma^k_{2N,\mathrm{tmp2}}=*$ if $B(a,\hat{\delta};2N)>\sigma N$ for $a\in\omega$, and let it be $=\gamma^k_{2N,\mathrm{tmp1}}$ otherwise.

Step 2. On each $\omega\in\mathcal{Q}_{2N}^k$, we let $\gamma_{2N}^k=\gamma^k_{2N,\mathrm{tmp2}}$ if $\omega\cap\Delta_N\neq\emptyset$, otherwise let it $=*$.

It remains to show that these steps lead to (i) and (ii) above with N replaced by $2N$. For $a\in\Delta_{2N}^k=\{\gamma_{2N}^k\neq *\}$, Step 1a ensures that $\hat{x}_i^k(a)$ satisfies (G1)$^{\#}$ up to time $2N$. By Step 1b, $B(a,\hat{\delta};n)\leq\sigma N<\sigma n$. Our choice of σ (see Sect. 7.1) together with Corollary 6.1 then gives the lower bound for $|(f^{n-1})'(\hat{x}_1^k)|$ in (G2)$^{\#}$. It remains to prove $f_a\in\mathcal{G}_{4\alpha^* N}$ for $a\in\Delta_{2N}^k$. Let ω be the element of $\mathcal{Q}_{2N}^k$ containing a. Since ω survived the deletion in Step 2, there must be a point $\hat{a}\in\omega\cap\Delta_N$. By Lemma 7.1, it suffices to show that $\omega\subset[\hat{a}-K_4^{-4\alpha^* N},\hat{a}+K_4^{-4\alpha^* N}]$. By Proposition 4.2', $|\omega|<2(\hat{c}c_1)^{-1}e^{-2\lambda_1 N}$. (Observe that the hypotheses of Proposition 4.2' are met: $\omega\subset\Omega_{2\alpha^* N}$, and $\hat{x}_i^k$ obeys (G1) up to time $2N$.) We conclude that $\omega\subset\Omega_{4\alpha^* N}$ if

$$2(\hat{c}c_1)^{-1}e^{-2\lambda_1 N}<K_4^{-4\alpha^* N}. \tag{9}$$

This is one of the conditions imposed on α in item 3 of Sect. 7.1.

7.3 Lower bound on measure of Δ

For each N and k, we now estimate the contribution to $\Delta_N\setminus\Delta_{2N}$ (not $\Delta_N^k\setminus\Delta_{2N}^k$!) due to deletions in γ_i^k from $i=N$ to $i=2N$.

Step 1a: By Lemma 6.1, the total measure deleted is $<\sum_{N<i\leq 2N}Ke^{-\frac{1}{2}\alpha i}|\Delta_0|$.

Step 1b: By Proposition 6.1, the total measure deleted is $<e^{-\hat{\varepsilon}_1 N}|\Delta_0|$.

Step 2: The sets ω removed in this step do not meet Δ_N, *i.e.*, they have been deleted earlier. Thus the actions taken in this step do not contribute to $\Delta_N \setminus \Delta_{2N}$.

Summing over the q critical points, we obtain

$$|\Delta_N \setminus \Delta_{2N}| \ \le q \ \left(K \sum_{N<i\le 2N} e^{-\frac{1}{2}\alpha i} \ + \ e^{-\hat{\varepsilon}_1 N} \right) |\Delta_0|.$$

It follows that

$$|\Delta| \ge |\Delta_0| - \sum |\Delta_{2^\ell n_0} \setminus \Delta_{2^{\ell+1} n_0}| > \left(1 - q \sum_{i=n_0}^{\infty} (K e^{-\frac{1}{2}\alpha i} + e^{-\hat{\varepsilon} i}) \right) |\Delta_0|,$$

which is positive if n_0 is sufficiently large. This completes the proof of Theorem 2. □

Reference: This section follows [WY2], which, together with its precursor [WY1] contain the corresponding material for rank one attractors.

References

[BC1] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. Math. **122** (1985), 1-25.

[BC2] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133** (1991), 73-169.

[BY] M. Benedicks and L.S. Young, *Absolutely continuous invariant measures and random perturbation for certain one-dimensional maps* Ergod. Th. and Dynam. Sys. **12** (1992), 13-37.

[CE] P. Collet and J.P. Eckmann, *Positive Liapunov exponents and absolute continuity for maps of the interval*, Ergodic Theory and Dynamical Systems **3** (1983), 13-46.

[J] M. Jakobson, *Absolutely continues invariant measures for one-parameter families of one-dimensional maps*, Comm. Math. Phys. **81** (1981), 39-88.

[M] M. Misiurewicz, *Absolutely continues invariant measures for certain maps of an interval*, Publ. Math. IHES. **53** (1981), 17-51.

[NS] T. Nowicki and S. Van Strien, *Absolutely continuous invariant measures under a summability condition*, Invent. Math. **105** (1991), 123-136.

[R] M. Rychlik, *Another proof of Jakoson's theorem and related results*, Ergodic Theory and Dynamical Systems **8** (1988), 83-109.

[T] M. Tsujii, *A proof of Benedicks-Carleson-Jakobson theorem*, Tokyo J. Math. **16** (1993), 295-310.

[TTY] P. Thieullen, C. Tresser and L.-S. Young, *Positive exponent for generic 1-parameter families of unimodal maps*, C.R. Acad. Sci. Paris, t. 315 Série I (1992), 69-72; *J'Analyse* **64** (1994), 121-172.

[WY1] Q. Wang and L.-S. Young, *Strange attractors with one direction of instability*, Commun. Math. Phys. **218** (2001), 1-97.

[WY2] Q. Wang and L.-S. Young. *Toward a theory of rank one attractors*, 2004 preprint.

[WY3] Q. Wang and L.-S. Young, *From invariant curves to strange attractors*, Commun. Math. Phys. **225** (2002), 275-304.

[WY4] Q. Wang and L.-S. Young, *Strange attractors in periodically-kicked limit cycles and Hopf bifurcations*, Commun. Math. Phys. **240** (2002), 509-529.