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STRANGE ATTRACTORS IN A MULTISECTOR BUSINESS CYCLE MODEL

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Three coupled oscillating sectors in a multisector Kaldor-type business cycle model can give rise to the occurrence of chaotic motion. If the sectors are linked by investment demand interdependencies, this coupling can be interpreted as a perturbation of a motion on a three-dimensional torus. A theorem by Newhouse, Ruelle and Takens implies that such a perturbation may possess a *strange attractor* with the consequence that the flow of the perturbed system may become irregular or chaotic. A numerical investigation reveals that parameter values can be found which indeed lead to chaotic trajectories in this cycle model.

1. Introduction

Recent work on chaos and strange attractors in non-linear dynamical systems has raised the question whether a 'route to turbulence' can be traced in a continuous-time dynamical system by means of a steady increase of a control parameter and successive bifurcations with a change of the topological nature of the trajectories. A famous example for such a route to turbulence was provided by Ruelle and Takens (1971)[1]: Starting with an asymptotically stable fixed point for low values of the parameter, the system undergoes a Hopf bifurcation if the control parameter is sufficiently increased. While a second Hopf bifurcation implies a bifurcation of the generated limit cycle to a two-dimensional torus, a third Hopf bifurcation can lead to the occurrence of a strange attractor and hence of chaos.

In the following I show how a very simple multisector Kaldor-type business cycle model can be constructed which is compatible with this scenario. It was demonstrated in the pioneering work by Goodwin (1947) that the coupling of sectors can imply a dynamic behavior of an economy which is essentially different from the behavior of isolated sectors. While Goodwin showed that the coupling of two sectors can decrease the stability

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[1]See Haken (1983) and Eckmann (1981) for surveys of other routes.

of the sectoral equilibria and may imply persistent fluctuations the present paper illustrates that regularly oscillating sectors may display chaotic behavior if they are appropriately coupled.

Section 2 presents a short description of the relation between toroidal motion and strange attractors. The model is introduced in section 3, and section 4 contains results of a numerical simulation.

2. Toroidal motion and the occurrence of strange attractors

The concept of *strange attractors* was introduced by Ruelle and Takens in 1971. Ruelle (1979) defines a strange attractor as follows.

Definition. Consider the n-dimensional dynamical system

$$\dot{x} = f(x, \mu), \qquad x \in \boldsymbol{R}^n, \quad \mu \in \boldsymbol{R}. \tag{1}$$

A bounded set $A \subset \boldsymbol{R}^n$ is a *strange attractor* for (1) if there is a set $\boldsymbol{U}$ with the following properties:

(i) U is an n-dimensional neighborhood of A.
(ii) If $x(0) \in U$, then $x(t) \in U \forall t > 0$ and $x(t) \to A$, i.e., any trajectory approaches and remains arbitrarily close to A for t large enough.
(iii) There is a sensitive dependence on initial conditions when $x(0)$ is in U, i.e., small variations of the initial value $x(0)$ lead to essentially different time paths of the system after a short time.
(iv) The attractor is indecomposable.

It is usually difficult to detect the existence of a strange attractor directly even in the simplest case of a 3-dimensional continuous-time dynamical system.[2] However, it is possible to relate the occurrence of strange attractors to the so-called *quasi-periodic* motion on an n-dimensional torus with $n \geqq 3$. The time evolution of a dynamical system (1) is said to be quasi-periodic if the solution can be written as[3]

$$x(t) = F_k(\omega_1 t, \omega_2 t, \ldots, \omega_k t), \qquad x \in \boldsymbol{R}^n, \tag{2}$$

with ω_i as independent frequencies and F_k as a periodic function of period 2π in each of the arguments.

Consider the n-dimensional system (1) and suppose that a stationary equilibrium, i.e., $x_i(t) = x_i^*$ with $\dot{x}_i = 0 \, \forall i$, exists. Furthermore, suppose that for μ sufficiently low the equilibrium is locally asymptotically stable, i.e., the real parts of the characteristic roots are all negative. As is well-known the fixed

[2]Compare, e.g., Arneodo et al. (1982).
[3]Cf. Ruelle (1979, p. 134).

point looses its stability and bifurcates into a closed orbit if an increase of the control parameter μ leads to the case in which at $\mu_1 > 0$ a pair of conjugate complex roots with zero real parts occurs and all other eigenvalues are different from zero.[4] Provided that the dimension of the system is large enough and that the orbit is stable for increasing μ, a further increase of the parameter μ may give rise to the case that another bifurcation takes place such that the limit cycle bifurcates into a so-called *torus*. The torus in fig. 1 is called a two-dimensional torus because it can be constructed by an appropriate glueing of a two-dimensional area. Write $\boldsymbol{T}^2$ for the two-dimensional torus $\boldsymbol{T}^2 = \boldsymbol{S}^1 \times \boldsymbol{S}^1$ with $\boldsymbol{S}^1$ as the unit cycle. Further bifurcations may lead to the tori $\boldsymbol{T}^3$, $\boldsymbol{T}^4$ etc.

It has been conjectured that turbulences can be described by quasi-periodic motions on high-dimensional tori in which many different and independent frequencies are involved.[5] However, a quasi-periodic motion is not sensitive to initial conditions: a minor variation of the initial state only varies the frequencies minimally.[6] The following theorem by Newhouse, Ruelle and Takens (1978)[7] establishes another explanation of turbulence by means of the notion of strange attractors mentioned above.

Theorem. *Let $a = (a_1, \ldots, a_n)$ be a constant vector field on the torus $\boldsymbol{T}^n$. If $n = 3$, in every C^2 neighborhood of a there exists an open vector field with a strange attractor. If $n \geqq 4$, in every C^∞ neighborhood of a there exists an open vector field with a strange attractor.*

In other words, as soon as a dynamical system is quasi-periodic on a 3-torus, it is thus *possible* that a *perturbation* of this system faces a strange attractor. It may therefore be possible that in some dynamical systems chaotic behavior already occurs after three subsequent Hopf-bifurcations.

In this context a serious problem usually arises: if an arbitrary n-dimensional system is considered, the first Hopf-bifurcation is established by means of an investigation of the eigenvalues of the Jacobian evaluated at equilibrium. If the system moves on a limit cycle and the bifurcation into a torus is considered, the Jacobian of this oscillating system is time periodic and the eigenvalues change over time. Generally, the bifurcation behavior of this time-period motion can be studied only with the help of Poincaré-maps and related concepts.[8] However, this problem does not arise if the dynamical system consists of coupled non-linear oscillators, i.e., if an n-dimensional system can be written as, e.g.,

[4]See Guckenheimer and Holmes (1983, p. 150ff) or Marsden and McCracken (1976) for this so-called 'Hopf-bifurcation'.

[5]See Haken (1983, p. 264) for a description of this 'Landau-Hopf'-route to turbulence.

[6]Cf. Ruelle (1979, p. 135) and Haken (1983, p. 264ff).

[7]Newhouse, Ruelle and Takens (1978) make use of Smale's so-called 'Axiom A' in their original formulation of the theorem. Compare also Ruelle and Takens (1971, p. 188).

[8]For an extensive treatment of this subject see Iooss and Joseph (1981).

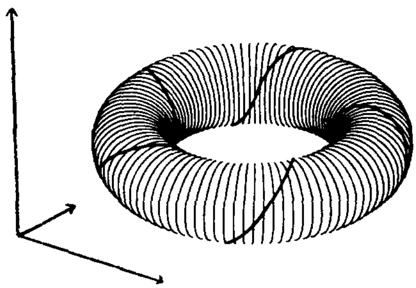

Fig. 1. The two-dimensional torus T^2 [*Source*: Haken (1983, p. 29)}.

$$\ddot{x}_i + f_i(x_i)\dot{x}_i + g_i(x_i) = h_i(x_i, x_j), \qquad x \in \boldsymbol{R}^2, \quad i \neq j, \quad i \in [1, n/2], \tag{3}$$

with $f_i(x_i)$ as appropriate non-linear functions. The coupling is called weakly linear if $h_i(x_i, x_j)$ is linear and the coefficients are small. In this case, a torus arises immediately as the product $(\boldsymbol{S} \times \boldsymbol{S} \times \cdots)$, of the cycles $\boldsymbol{S}$ of the involved uncoupled oscillators, i.e., with $h_i = 0\,\forall i$, and the introduction of the coupling terms can be considered as a perturbation of the motion on the torus.[9] The following model will make use of this geometric description of uncoupled limit cycle oscillators.

3. A multisector busines cycle model

Consider the well-known non-linear macroeconomic business cycle model of Kaldor[10]

$$\dot{Y} = \alpha(I(Y, K) - S(Y)), \qquad \dot{K} = I(Y, K) - \delta K, \tag{4}$$

with Y as the real gross product, K as the capital stock, I and S as gross investment and savings, respectively, and δ as the depreciation rate. The parameter α is the adjustment coefficient in the goods market, savings depends positively on output, and $I_K < 0$. Further, investment depends on output in the well-known sigmoid form.

The model has been analyzed extensively in the literature. Chang and Smyth (1971) investigated the limit cycle behavior of the model by means of the Poincaré–Bendixson theorem, while Cugno and Montrucchio (1982) provided an elaborated discussion of the bifurcation behavior of the continuous-time version (4). It was shown by Dana and Malgrange (1984) that a discrete version of the Kaldor-model is able to display chaotic dynamics.[11] They demonstrated that an increase of the adjustment coefficient α generates a dynamic behavior which is characterized by succeedingly stable fixed points, closed orbits, and finally chaos. Because of the impressive

[9] Cf. Guckenheimer and Holmes (1983), p. 59f. and Rand and Holmes (1980) for this procedure.

[10] Cf. Kaldor (1940). Kaldor also assumes a dependence of savings on the capital stock, which is not essential to the model and which is consequently dropped in the following.

[11] For a similar simulation that makes use of another investment function see Lorenz (1985).

demonstration of the influence of α on the dynamic behavior of the Kaldor-model, the following multisector model will also use the adjustment coefficient as the bifurcation parameter.

Consider an economy with the three sectors $i=1,2,3$. Let Y_i and K_i denote gross output and capital in each of the sectors. As before, let gross investment in each sector be dependent on output and capital in that sector in the Kaldorian way

$$I_i = I_i(Y_i, K_i), \qquad i=1,2,3. \tag{5}$$

The capital stocks in all three sectors thus develop according to the equations

$$\dot{K}_i = I_i(Y_i, K_i) - \delta_i K_i, \qquad i=1,2,3. \tag{6}$$

Suppose that a part of the investment demand of sector i does not consist of produced goods of sector i but is delivered by other sectors. Write $I_i^i(Y_i, K_i)$ for the part of total investment demand of sector i for goods produced in i, and I_i^j as the part which is effective in sector j. Assume for the sake of simplicity that I_i^j depends on Y_i only and that it can be expressed as a linear function, i.e., $dI_i^j/dY_i = b_{ji} = \text{const.}$[12] Total investment demand of sector i, which is effective in sector i or j, is therefore given by

$$I_i(Y_i, K_i) = I_i^i(Y_i, K_i) + \sum_j I_i^j(Y_i), \qquad j \neq i. \tag{7}$$

Denote the actual investment demand for the produced goods of sector i as I_i^d, consisting of the internal investment demand $I_i^i(Y_i, K_i)$ and the investment demand of the sectors for goods of i, i.e., $I_j^i(Y_j)$

$$\begin{aligned} I_i^d(Y_1, Y_2, Y_3) &= I_i^i(Y_i, K_i) + \sum_j I_j^i(Y_j) \\ &= I_i^i(Y_i, K_i) + \sum_j b_{ij} Y_j, \qquad j \neq i. \end{aligned} \tag{8}$$

Assume further that these investment demand interdependencies are unidirectional in the sense that every sector demands investment goods only from sectors with indexes lower than its own, i.e.,[13]

[12]This simplifying assumption does not alter the basic analytical result presented below but is convenient in the numerical simulation of section 4.

[13]This triangular form of the matrix $\{b_{ij}\}$ is not as pathological as it may seem. In the input–output literature, it is usually attempted to triangulate the I–O-matrixes by a re-ordering of the sectors such that the investment goods industries and the consumption goods industries constitute the first and the last sectors, resp., in the ordering. For details on this triangulation see Helmstädter (1962), Chenery and Watanabe (1958) and Aujac (1960). In face of the empirically observable evidence of a tendency toward a triangular form of I–O-matrixes, Aujac (1960) even proposed to replace the common expression 'interdependencies' by 'dependencies' in empirical I–O-studies. The ordering $i=1,2,3$ with the matrix $\{b_{ij}\}$ may thus be considered as the outcome of such a triangulation procedure. Note that this triangular form corresponds with Goodwin's unilateral coupling, cf., Goodwin (1947).

$$b_{ij}\begin{cases}\geqq 0 & \text{if} \quad i<j \\ =0 & \text{if} \quad i>j.\end{cases} \tag{9}$$

With $I_i^d(Y_1, Y_2, Y_3)$ as the demand for investment goods produced in i, excess demand in sector i is given as $Y_i - C_i(Y_i) - I_i^d(Y_1, Y_2, Y_3)$ such that the output adjustment equation in sector i reads

$$\begin{aligned}\dot{Y}_i &= \alpha_i(I^d(Y_1, Y_2, Y_3) - S_i(Y_i)) \\ &= \alpha_i\left(I_i^i(Y_i, K_i) - S_i(Y_i) + \sum_j b_{ij} Y_j\right),\end{aligned} \tag{10}$$

with $i=1,2,3$ and $i \neq j$. Eqs. (10) and (6) together with (9) constitute the six-dimensional continuous-time dynamical system (11)

$$\begin{aligned}\dot{Y}_i &= \alpha_i\left(I_i^i(Y_i, K_i) - S_i(Y_i) + \sum_j b_{ij} Y_j\right), \\ \dot{K}_i &= I_i(Y_i, K_i) - \delta_i K_i = I_i^i(Y_i, K_i) + \sum_j b_{ji} Y_i - \delta_i K_i,\end{aligned} \tag{11a}$$

with $i, j = 1,2,3$ and $i \neq j$, or, explicitly written,

$$\begin{aligned}\dot{Y}_1 &= \alpha_1(I_1^1(Y_1, K_1) - S_1(Y_1) + b_{12} Y_2 + b_{13} Y_3), \\ \dot{K}_1 &= I_1^1(Y_1, K_1) - \delta_1 K_1, \\ \dot{Y}_2 &= \alpha_2(I_2^2(Y_2, K_2) - S_2(Y_2) + b_{23} Y_3), \\ \dot{K}_2 &= I_2^2(Y_2, K_2) + b_{12} Y_2 - \delta_2 K_2, \\ \dot{Y}_3 &= \alpha_3(I_3^3(Y_3, K_3) - S_3(Y_3)), \\ \dot{K}_3 &= I_3^3(Y_3, K_3) + b_{13} Y_3 + b_{23} Y_3 - \delta_3 K_3.\end{aligned} \tag{11b}$$

Consider the case in which no intersectoral trade takes place, i.e.,

$$\begin{aligned}\dot{Y}_i &= \alpha_i(I_i^i(Y_i, K_i) - S_i(Y_i)), \\ \dot{K}_i &= I_i^i(Y_i, K_i) - \delta_i K_i,\end{aligned} \tag{12}$$

and let J_i be the Jacobian matrixes of each sector i of (12).

Assumption. (i) There exists a Hopf-bifurcation value of α_i in each sector i

described by (12), i.e., $\operatorname{tr} J_i = 0, \det J_i > 0$, and $d(\operatorname{tr} J_i)/d\alpha_i > 0$, evaluated at the bifurcation value of α_i. (ii) The bifurcations are supercritical, i.e., the emerging closed orbits are stable for increasing α_i.

If the assumptions are fulfilled every sector will be oscillating for appropriate parameter values. Note that this implies the possibility of economically self-sustaining sectors.[14]

It is now possible to suggest the following corollary to the Newhouse, Ruelle and Takens theorem:

Proposition. If all three sectors in the uncoupled system (12) are oscillating the introduction of intersectoral trade may imply the occurrence of a strange attractor.

The three different uncoupled two-dimensional limit cycle oscillators in (12) constitute a three-dimensional torus, i.e., a three-dimensional object in six-dimensional space. The coupled system (11) with intersectoral trade can be interpreted as a perturbation of the uncoupled system (12). The Newhouse, Ruelle and Takens theorem therefore implies the possible occurrence of a strange attractor in the coupled system (11).

Whether or not the coupled system (11) indeed possesses a strange attractor depends on the assumed structural forms of the involved functions and the specific numerical parameter values. Any model which attempts to establish the existence of chaotic dynamics via the Newhouse, Ruelle and Takens scenario must therefore finally refer to a numerical simulation.

4. A numerical simulation

The continuous-time dynamical system (11) has been numerically simulated by the use of a standard Runge–Kutta algorithm. The algebraic specifications of the involved functions have arbitrarily been chosen and do not represent a limitation of the results.

All three sectors are assumed to be identical with numerically identical investment functions which feature the properties of Kaldorian functions. The adjustment coefficients α_i are the same in all three sectors, and $\delta_3 > \delta_2 > \delta_1$ such that the Hopf bifurcation values are different in the three sectors.[15]

[14]While no explicit production function has been introduced in the present rudimentary multisector model, this oscillation property implies that external investment goods are not exclusively necessary in the production process and that they can be substituted by sector-specific goods, i.e., every sector produces goods which can be consumed or invested.

[15]The notation of the simulated model differs slightly from that of the model presented above: the ratio s in the output adjustment equation of the simulation is defined as the sum of the marginal savings rate and that part of sector i's investment demand which is effective in other markets. A constant ratio s in the simulation therefore implies varying savings rates when the b_{ij}'s are changing.

The analytical considerations above indicate that for increasing values of the adjustment coefficient α the equilibria in the three sectors become unstable and that the system oscillates. For values of α greater than the bifurcation value of the third sector, the dynamic behavior of the first two sectors is characterized either by quasi-periodic or by chaotic behavior,[16] while the third sector can only oscillate in a limit cycle manner.

The results of the simulation are illustrated in figs. 2–6.[17] The parameters of the investment function, the savings rate, the depreciation rates and the initial values were held constant in every simulation run in order to allow a comparison of the effects of a varied adjustment coefficient α and the coupling terms b_{ij}.

While the uncoupled system is characterized by convergence to the stationary equilibria for low values of α, further increases of α beyond $\alpha \approx 1.7$ in this numerically specified example lead to the occurrence of cyclical motions in all three sectors. If positive but small coupling terms $b_{ij} > 0$ are introduced and if the adjustment parameter α is increased the dynamic behavior of the first two sectors appears to be irregular. Figs. 2 and 3 show the result for $\alpha = 5.0$ and identical numerical values of the involved coupling terms of 0.015. In addition to the illustrations of the motion of output and capital in the three sectors, an extract of the development of Y_1 vs. time and a projection of the co-movements of output in all three sectors is plotted in fig. 3. The third sector is of course characterized by a limit cycle behavior, because this sector is not influenced by the developments in the other two sectors. In figs. 2 and 3, the trajectories switch irregularly and seemingly arbitrarily between inner and outer regions of the attractor, suggesting the presence of chaotic motion. However, these graphical illustrations of the motion are not sufficient to prove the existence of chaos numerically. Fig. 4 represents the power spectra[18] of the generated time series for the parameter constellation in figs. 2 and 3. While the sharp peaks in the power spectra of Y_3 and K_3 indicate harmonic oscillation with associated harmonics, the power spectra of Y_1 and (though less distinct) of K_1 are similar to broad band noise for low frequencies, implying that the motion is indeed chaotic. As has to be expected, the power spectra of Y_2 and K_2 still exhibit sharp peaks at certain frequencies of approximately equal distance such that the motion is not chaotic.

This dynamic behavior prevails for decreasing values of the involved b_{ij}'s. If the coupling terms are increased in comparison to the parameter constellation in figs. 2–4, the dynamic behavior of the first sector can become quasi-periodic for certain parameter values. While the trajectories in Y_1, K_1-space

[16]It has been stressed that pure quasi-periodic motions on a 3-torus are usually unobservable in experimental investigations of dynamical systems in various disciplines. Compare Swinney (1983), p. 8.

[17]A more complete list of results is available on request.

[18]Cf. Granger and Hatanaka (1964) for the concept of power spectra in time series analysis.

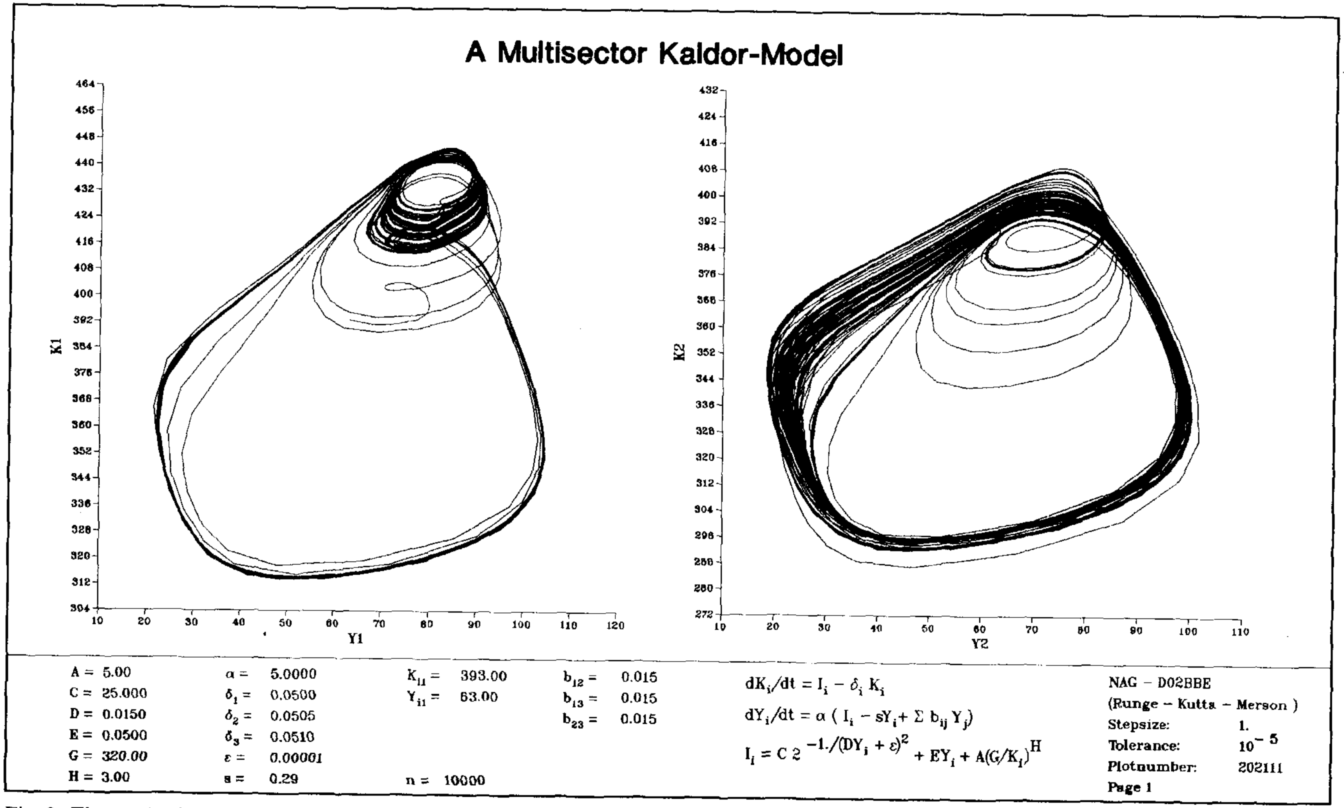

Fig. 2. The projection of the dynamic motion onto the (Y_1, K_1) plane (left) and the (Y_2, K_2) plane (right). The trajectories switch irregularly between inner and outer regions of the attractor.

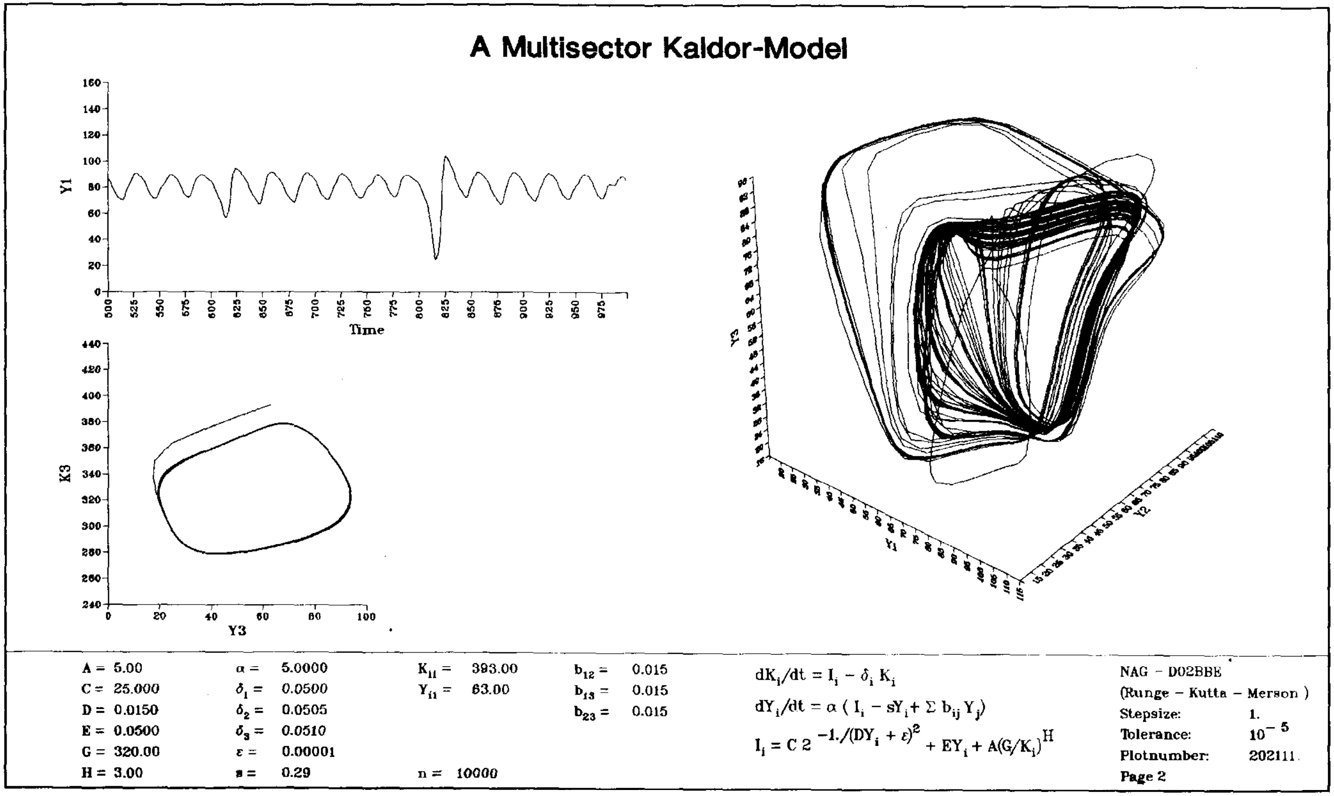

Fig. 3. The projection of the dynamic motion onto the (Y_3, K_3) plane (lower left) and a projection onto the (Y_1, Y_2, Y_3) space (right). The upper left plot shows the time path of Y_1 for an arbitrarily chosen time interval.

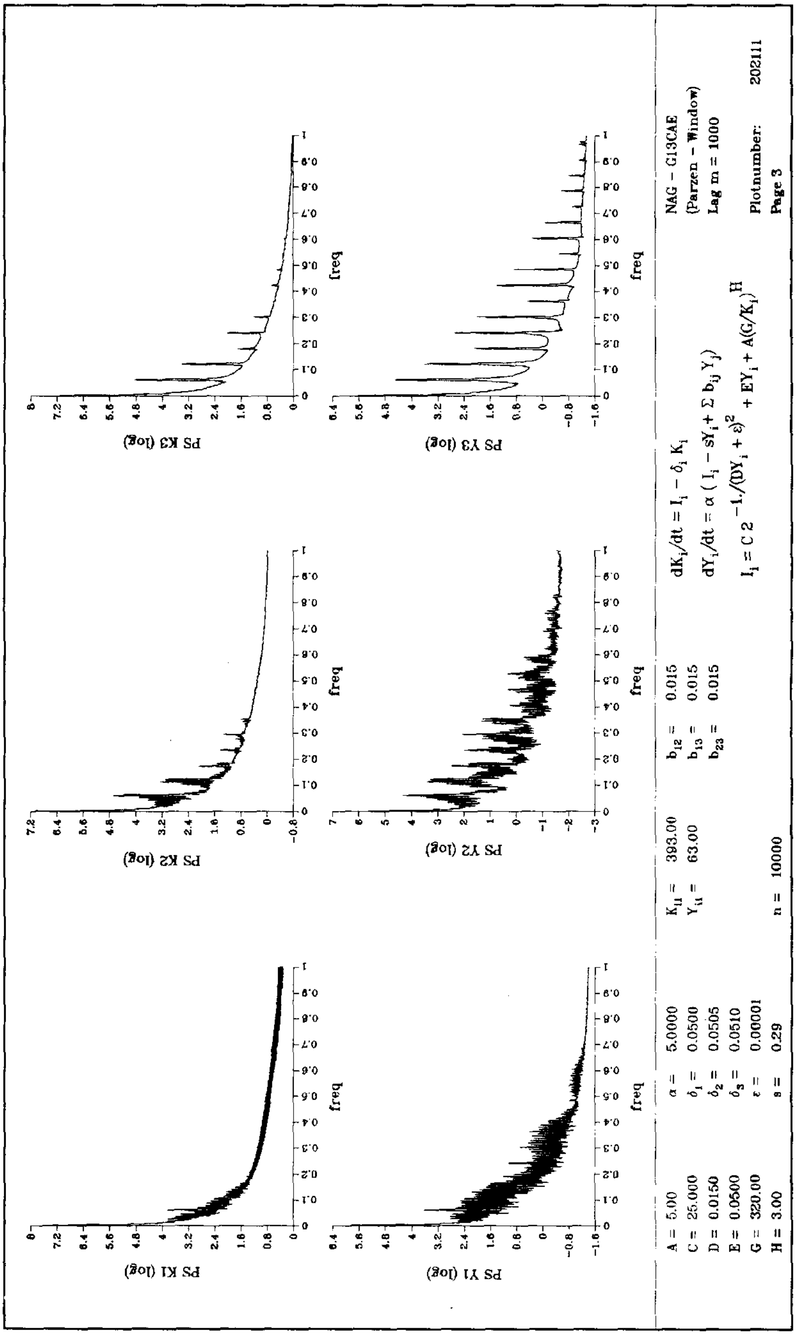

Fig. 4. Power spectra of the time series of Y_i (lower plots) and K_i (upper plots) in all three sectors (from left to right) for the parameter constellation in figs. 2 and 3.

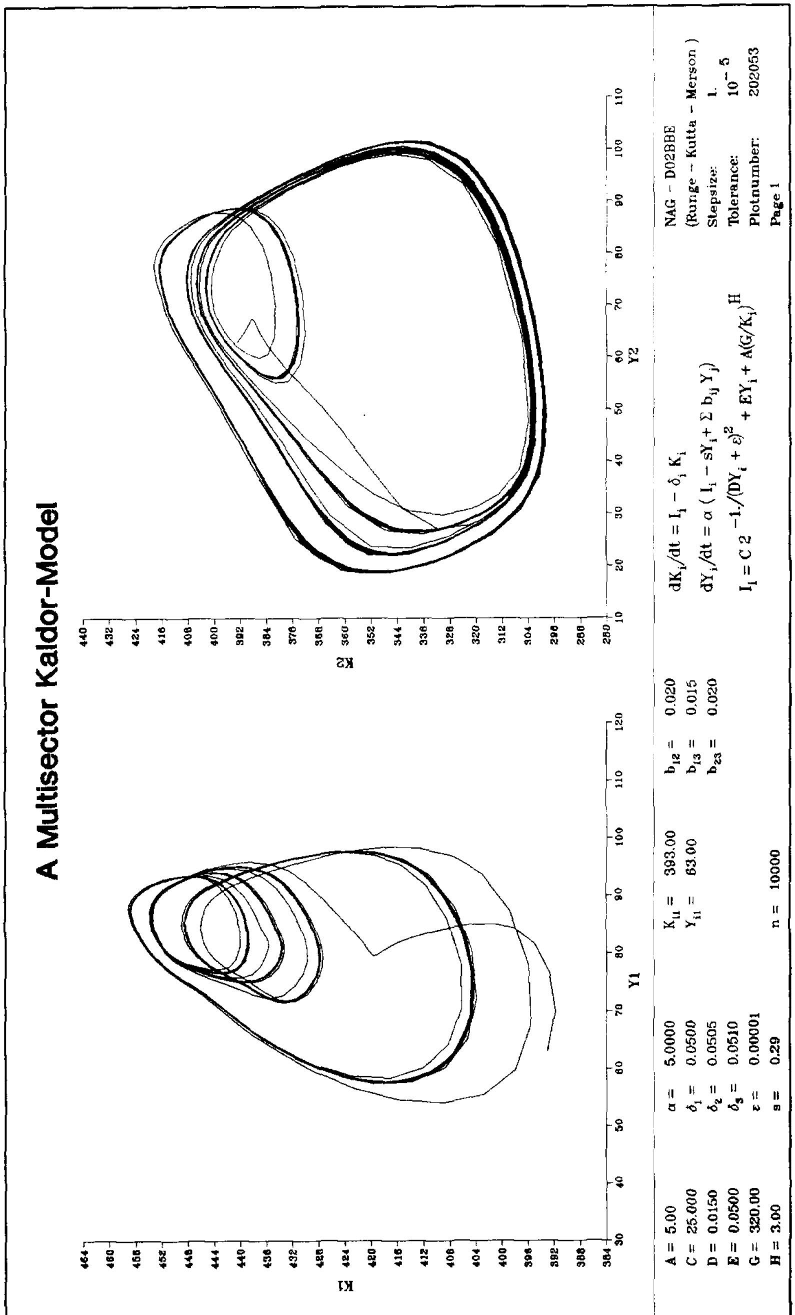

Fig. 5. Same constellation as in fig. 2 for increased values of the coupling terms.

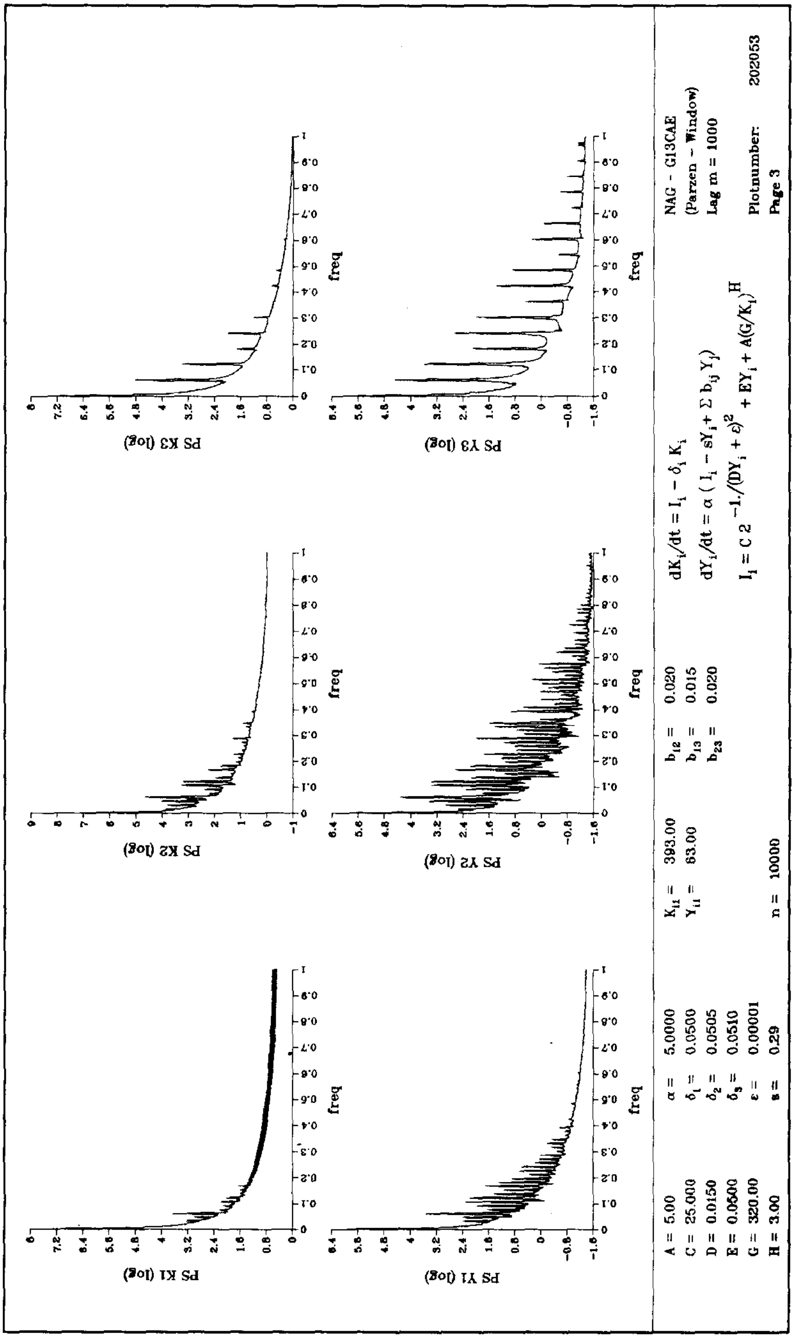

Fig. 6. Power spectra for the parameter constellation in fig. 5.

in the previous examples all describe broad bands with possibly additional loops as in fig. 2, fig. 5 shows an example of an increase in the coupling terms that leads to a relatively thin band of trajectories performing several loops. The associated power spectra in fig. 6 display distinguishable single peaks in the first sector and are similar to the spectra of the second sector. The same phenomenon can be observed when the adjustment coefficient α is considerably increased.

5. Conclusion

It has been demonstrated that it is possible to construct a simple multisector business cycle model in which chaotic dynamics may emerge. The example should only be considered as a demonstration of the underlying mechanisms, and the absence of production functions or the assumption of possibly autarchic sectors are clearly shortcomings which prevent a comparison with other multisector models. However, the results suggest that the method of detecting strange attractors via toroidal motions can be applied to various fields in economics: as soon as oscillators are coupled in the described way in, e.g., international trade theory,[19] the possible occurrence of strange attractors and chaos has to be taken into account.

[19]Cf. Lorenz (1987).

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