Notes on Shaking Pendulum

A. Equation of a shaking pendulum: Let (x, y) be a fixed coordinate frame. The end of a pendulum moving in the y-direction periodically. The movement is with a magnitude A and a frequency ω. Let ℓ be the length of the pendulum and ϕ be the angle in between the pendulum and the y-axis. For the pendulum we have

$$y = \ell \cos\phi + Acos\omega t, \qquad x = \ell \sin\phi.$$

From which we obtain

$$\begin{aligned}
\dot{y} &= -\ell \sin\phi\dot{\phi} - A\omega \sin\omega t \\
\ddot{y} &= -\ell \cos\phi(\dot{\phi})^2 - \ell \sin\phi\ddot{\phi} - A\omega^2 \cos\omega t \\
\dot{x} &= \ell \cos\phi\dot{\phi} \\
\ddot{x} &= -\ell \sin\phi(\dot{\phi})^2 + \ell \cos\phi\ddot{\phi}.
\end{aligned}$$

Use the second law of mechanics $\mathbf{F} = m\mathbf{a}$ and decompose into x and y direction, we obtain

$$\begin{aligned}
g - \frac{T}{m}cos\phi &= -\ell \cos\phi(\dot{\phi})^2 - \ell \sin\phi\ddot{\phi} - A\omega^2 \cos\omega t \\
-\frac{T}{m}\sin\phi &= -\ell \sin\phi(\dot{\phi})^2 + \ell \cos\phi\ddot{\phi}
\end{aligned}$$

where T is the magnitude of the force of tension on the pendulum. Multiply the first equation by $\sin\phi$, the second by $\cos\phi$. Subtract the second from the first, we obtain

$$\ddot{\phi} + (\frac{g}{\ell} + \frac{A\omega^2}{\ell}\cos\omega)\sin\phi = 0 \qquad (1)$$

It is important to remember that we are considering the case in which

$$A << \ell << \omega$$

This is to say, we regard ℓ, g as constants of normal size, comparing to which A is very small and ω is very large.

B. Analysis: We guess that a solution of equation (1) is roughly the sum of two functions: one is fast shaking with a small magnitude, which we denote as $\varepsilon(t)$ and the other is slowly oscillating, which we denote as $f(t)$. We write

$$\phi = f(t) + \varepsilon(t).$$

We put it into equation (1) to obtain

$$\begin{aligned}\ddot{f} + \ddot{\varepsilon} &= -(\frac{g}{\ell} + \frac{A\omega^2}{\ell}\cos\omega)\sin(f+\varepsilon) \\ &\approx -(\frac{g}{\ell} + \frac{A\omega^2}{\ell}\cos\omega)(\sin f + \varepsilon\cos f) \\ &= -\frac{g}{\ell}\sin f - \frac{A\omega^2}{\ell}\cos\omega t\sin f - \varepsilon\frac{g}{\ell}\cos f \\ &\quad - \varepsilon\frac{A\omega^2}{\ell}\cos\omega t\cos f\end{aligned}$$

Let us now try to split this equation into two as follows: we let

$$\begin{aligned}\ddot{\varepsilon} &= -\frac{A\omega^2}{\ell}\cos\omega t\sin f \\ \ddot{f} &= -\frac{g}{\ell}\sin f - \varepsilon\frac{g}{\ell}\cos f \\ &\quad - \varepsilon\frac{A\omega^2}{\ell}\cos\omega t\cos f\end{aligned} \tag{2}$$

First let us see if we can make sense of the first equation. Since $\varepsilon(t)$ changes real fast, and $f(t)$ slowly, we can then regard f as a constant when consider $\varepsilon(t)$. Regarding f as a constant, we can easily obtain a solution of the first equation as

$$\varepsilon(t) = \frac{A}{\ell} \sin f cos\omega t$$

This function is a fast shake with a small magnitude.

Putting $\varepsilon(t)$ back into the second equation we obtain

$$\begin{aligned} \ddot{f} = -\frac{g}{\ell} \sin f - \frac{gA}{\ell^2} cos\omega t \sin f \cos f \\ - \frac{A^2\omega^2}{\ell^2} cos^2\omega t \sin f \cos f \end{aligned} \tag{3}$$

C. Result of averaging: Equation (3) defines a 2D vector field that is time dependent.

To use the analysis of 2D systems we first replace the time-dependent vector field with an averaged vector field. Recall that for a function $P(t)$, the average is

$$\bar{P} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P(t)dt$$

Taking the average on the right of equation (3), we obtain

$$\begin{aligned}\ddot{f} &= -\frac{1}{2T} \int_{-T}^{T} \frac{g}{\ell} \sin f dt \\ &\quad - \frac{1}{2T} \int_{-T}^{T} \frac{gA}{\ell^2} cos\omega t \sin f \cos f dt \\ &\quad - \frac{1}{2T} \int_{-T}^{T} \frac{A^2\omega^2}{\ell^2} cos^2\omega t \sin f \cos f dt \\ &\quad \to -\frac{g}{\ell} \sin f - \frac{A^2\omega^2}{2\ell^2} \sin f \cos f\end{aligned}$$

as $T \to \infty$. So the averaged equation for f is

$$\ddot{f} + \frac{g}{\ell} \sin f + \frac{A^2\omega^2}{2\ell^2} \sin f \cos f = 0 \qquad (4)$$

At $f = \pi$, the eigenvalues of equation (4) are

$$\lambda = i\sqrt{\frac{A\omega^2}{2\ell^2} - \frac{g}{\ell}}.$$

Therefore $f = \pi$ is stable if

$$A\omega > \sqrt{2g\ell}.$$

Homework: (a) Write equation (4) as a system of two first order equations and find all equilibrium solution.

(b) Compute the eigenvalues of the Jacobi matrix at all equilibrium solutions of equation (4) to determine their type.

(c) In equation (4) let $A = \ell = 1$. Compute the potential function of equation (4) and use its graph to depict the phase portrait. Consider all cases regarding ω as a parameter.