Dynamical Systems: A Brief Introduction

1. Objects of Study

Phase Space M: A geometric object (e.g. sphere, tori, open set in R^n).

A System T: A mapping $M \to M$.

Orbits: $x \to T(x) \to T(T(x)) \to \cdots$.

Ex1: $M = R^1$, $T : R^1 \mapsto R^1$ defined by

$$T(x) = 1 - 2x^2.$$

$x = 0.1$, $x_1 = 0.98$, $x_2 = -0.9208$, $\cdots$.

Ex2: $M = S^1$, $T(\theta) = \theta + \pi$.

$\theta = 1$, $\theta_1 = 1 + \pi$, $\theta_2 = 1$, $\cdots$.

Ex3: Example: $M = R^2, T : (x, y) \to (x_1, y_1)$

$$T : \begin{cases} x_1 &= 2x \\ y_1 &= \frac{1}{2}y \end{cases}$$

$z = (1,1), z_1 = (2, \frac{1}{2}), z_2 = (2^2, \frac{1}{2^2}), \cdots.$

Ex4: $M = R^2$, $\quad T : (r, \theta) \to (r_1, \theta_1)$

$$T : \begin{cases} r_1 & = & r \\ \theta_1 & = & \theta + r \end{cases}$$

$z = (1, 0), z_1 = (1, 1), \cdots, z_n = (1, n \mod (2\pi)), \cdots.$

– For $T : M \to M$, and $x_0 \in M$ given, the orbit started from x_0 is denoted as $\{x_n\}_{n=0}^{\infty}$.

– If T^{-1} exists, we say that T is invertible. We then are able to talk about backward orbit from x_0: $x_{-1} = T^{-1}x_0$ and so on.

– Ex1 in the above is not invertible. Ex2, Ex3 and Ex4 are invertible.

2. Fundamental Questions

Q1: Behave of individual orbits.

– Fixed points: $T(x) = x$.

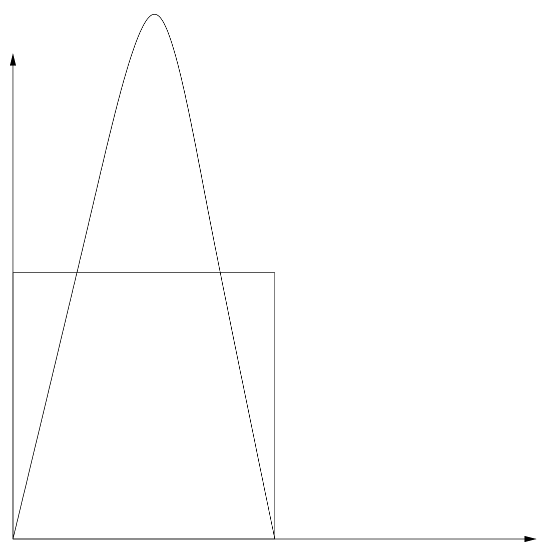
Ex: $T(x) = \mu x(1-x)$

$x = \mu x(1-x) \to x_1 = 0, \quad x_2 = 1 - \mu^{-1}$.

– Periodic orbits: $T^n(x) = x$.

The smallest n is the period.

Ex: $T(x) = 7x(1-x)$



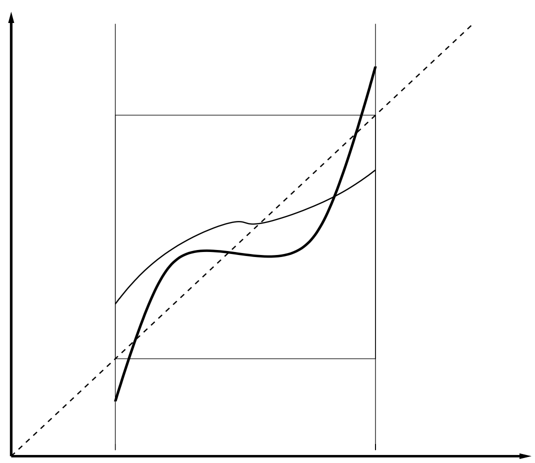
Claim: Periodic orbit of all period exists.

Proof: Let $I_1 = [0, \frac{1}{2}], I_2 = [\frac{1}{2}, 1]$. $\forall n > 0$ given, $\exists$ an interval I, such that

(i) $T^i(I) \subset I_1, i = 0, 1, \cdots, n-2$;

(ii) $T^{n-1}I \subset I_2$, $T^nI = I_1$.

Fact: Let $T : I \to R$ be continuous such that $T(I) \subset I$ (or $T(I) \supset I$). The T has a fixed point in I.

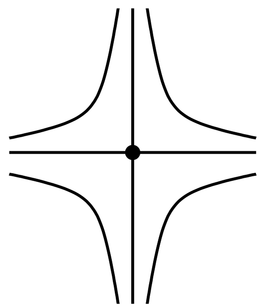


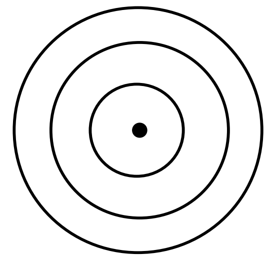
Note that the period of the orbit constructed is n by design.

Q2: Organizations of Orbits.

– Phase portrait: (orbit structure)

Example A: $T : \begin{cases} x_1 = 2x \\ y_1 = \frac{1}{2}y \end{cases}$





Example B: $T : \begin{cases} r_1 = r \\ \theta_1 = \theta + r \end{cases}$

– Local Stability:

Let x_0 be a fixed point. An open neighborhood of x_0 is denoted as $U(x_0)$, $V(x_0)$, etc.

(a) x_0 is **stable** if for every $U(x_0)$ given, there exists an $V(x_0)$ such that $T^iV(x_0) \subset U(x_0)$ for all $i \geq 1$.

(b) x_0 is **asymptotically stable** (a sink, or an attracting fixed point) if there exists $U(x_0)$, such that for all $x \in U(x_0)$, $T^ix \to x_0$ as $i \to \infty$.

$\{x \in M : T^i x \to x_0\}$ is the **attracting basin** for x_0.

(c) Let $x_0 \in M$ be such that $T^n x_0 = x_0$, then x_0 is a fixed point of T^n. (a) and (b) apply to define respectively **stable** and **asymptotically stable** periodic orbits.

Example B is Stable but not asymptotically stable.

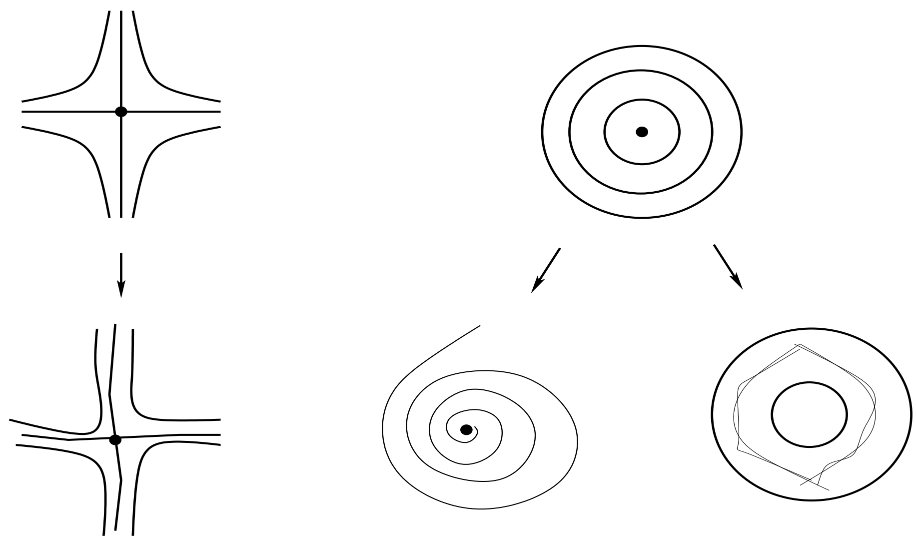
Example A is not stable.

No asymptotically stable fixed point for measure preserving maps.

$T(x) = \frac{1}{2}x$ has a unique fixed point $x_0 = 0$ that is asymptotically stable. Its attracting basin is $\mathbb{R}$.

– Structure stability: Does the orbit structure of a given T change under small perturbation?

Hyperbolic structure and Elliptic structure



3. Source of Inspirations:

Differential Equation $\Rightarrow$ Dynamical Systems

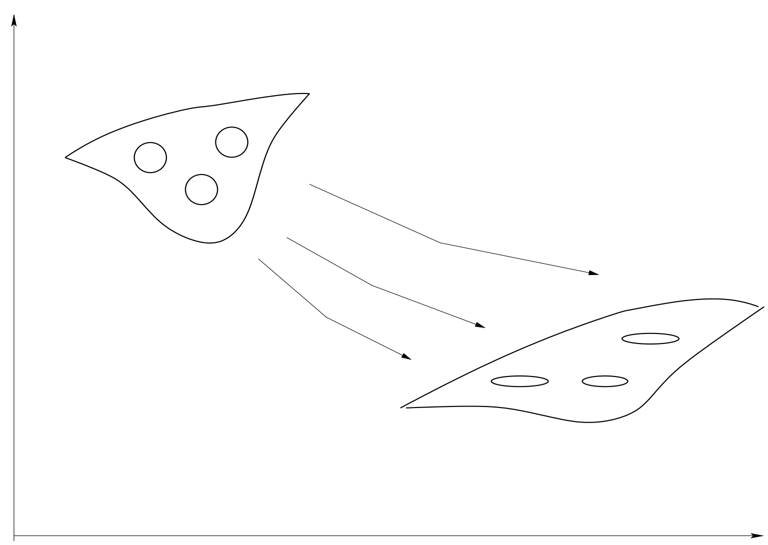
$$\frac{dx}{dt} = f(x,t) \qquad x \in \mathbb{R}^n.$$

(a) Autonomous Equations: $f(x,t) = f(x)$

Solution: $x = x(t, x_0)$;

Map: $T(x_0) = x(1, x_0)$; $T^n(x_0) = x(n, x_0)$.

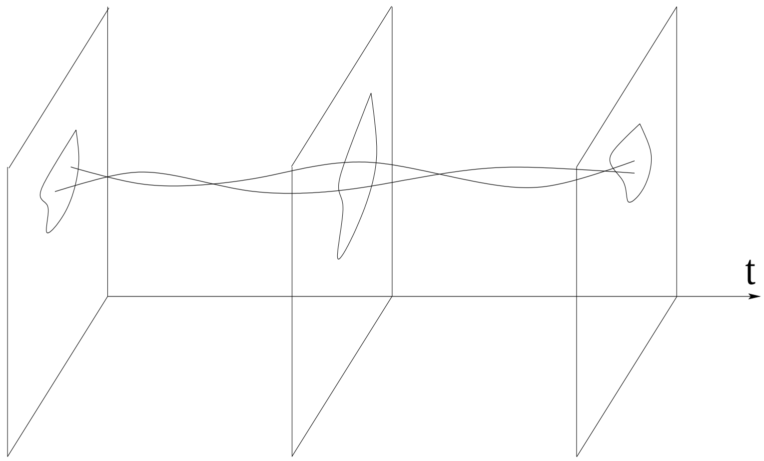
Arnold's cat: $U \to TU$.



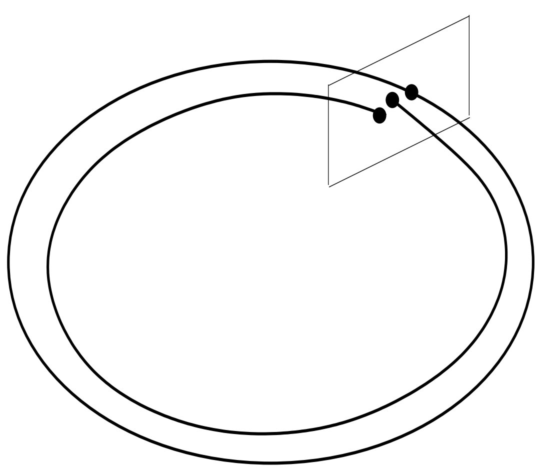
(b) Non-auto Equations: $f(x,t) = f(x,t+T)$

Solution: $x(t, x_0)$;

Map: $T(x_0) = x(T, x_0)$; $T^n(x_0, T) = x(nT, x_0)$.



(c) Poincáre section.



1D Dynamics: Periodic Orbits

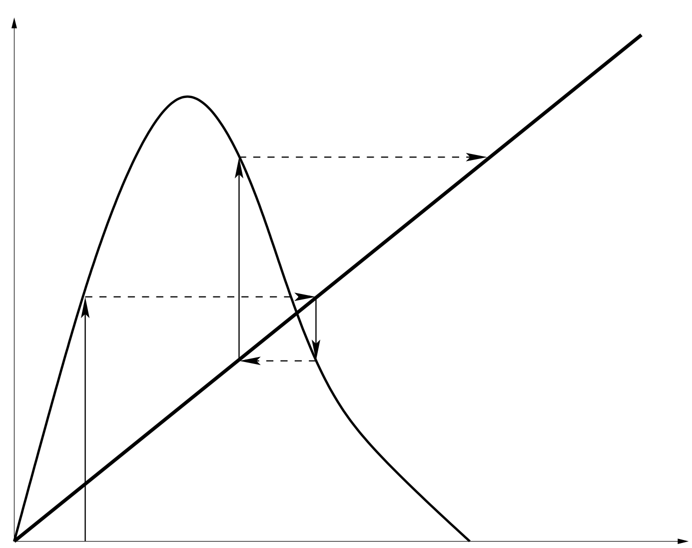
Maps of study: $T(x) : \mathbb{R} \to \mathbb{R}$.

$T(x)$ is as smooth as we need along the way.

1. Graph of $T(x)$ and orbits

– Fixed points: Intersection of the graph $y = T(x)$ and $y = x$.

– A given orbit: Trace the graph.



2. Stability of a fixed point

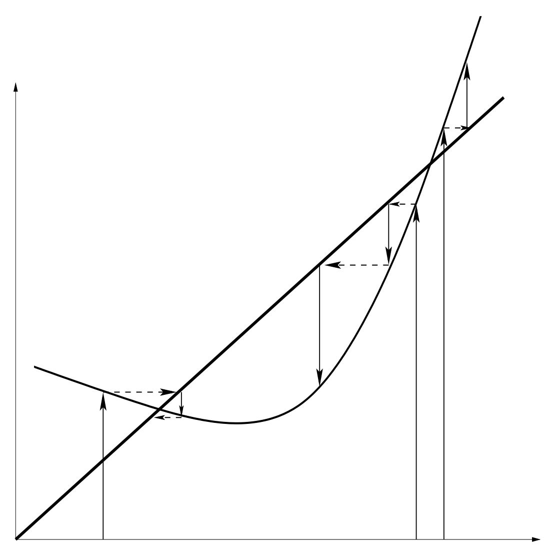
Claim: Let x_0 be a fixed point. x_0 is asymptotically stable if $|T'(x_0)| < 1$. It is unstable if $|T'(x_0)| > 1$.

Proof: If $|T'(x_0)| < 1$, then by continuity there exist $I(x_0)$ (an interval contains x_0), such that $|T'(x)| < \lambda < 1$ for all $x \in I(x_0)$. By mean value theorem then,

$$|T(x) - x_0| = |T(x) - T(x_0)| < \lambda |x - x_0|$$

for all $x \in I(x_0)$. This implies $|T^n(x) - x_0| < \lambda^n |x - x_0| \to 0$. The other half is similar.

A demonstration using graph



3. $|T'(x_0)| = 1$: Degenerate case

Ex: $T(x) = x + x^3$: unstable at $x = 0$.

Proof: $T'(x) = 1 + 3x^2 > 1$ around $x = 0$. So $|T(x) - 0| > x$ for all $x \neq 0$.

Ex: $T(x) = x - x^3$: asymptotically stable at $x = 0$.

Proof: $T'(x) = 1 - 3x^2 < 1$ around $x = 0$. So $|T(x) - 0| < |x - 0|$. Starting from, say, $x \neq 0$, $\{x_n\}$ is a in creasing sequence. So $x_n \to \hat{x}$. $\hat{x}$ must be a fixed point. So $\hat{x} = 0$ and $x_n \to 0$.

Ex: $T(x) = x^3 \sin \frac{1}{x} + x$: Stable but not asymptotically stable at $x = 0$.

Proof: Fixed points of this $T(x)$ is defined by

$$x^3 \sin \frac{1}{x} = 0.$$

This is a case in which $T(x)$ has infinitely many fixed points accumulating at $x = 0$.

4. Existence of periodic orbits

Claim: Let $T : \mathbb{R} \to \mathbb{R}$. If T has a periodic orbit of period three, then it has periodic orbit of all periods.

Proof: Assume $a < b < c$ is such that $f(a) = b$, $f(b) = c$ and $f(c) = a$. Let $I_0 = [a, b]$, $I_1 = [b, c]$, we have $T(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$.

Basic observation: For any interval A such that $T^i(A) \supset I_1$, there are two sub-intervals $A^0, A^1 \subset A$, such that $T^{i+1}(A^0) = I_0$, $T^{i+1}(A^1) = I_1$.

Let n be fix, we will be able to find an sub-interval A in I_1 such that

(a) $T^i(A) \subset I_1$ for all $i < n - 1$;

(b) $T^{n-1}(A) = I_0$.

Since $T^n(A) = T(I_0) = I_1 \supset A$. T^n has a fixed point, which is a periodic orbit of T of period n. The period of this orbit can not be less than n by design.

5. Sarkovskii's Theorem

Sarkaovskii order

$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \cdots$

$\triangleright 2^m \cdot 3 \triangleright 2^m \cdot 5 \triangleright \cdots \triangleright 2^n \triangleright 2^{n-1} \triangleright \cdots \triangleright 2 \triangleright 1.$

Remark: We can always write an integer n in the form $n = p2^m$ where $p \geq 1$ is odd and $m \geq 0$ be positive.

Theorem: Assume that $T : \mathbb{R} \to \mathbb{R}$ is continuous. If T has a periodic orbit of period n, then for all n' such that $n \triangleright n'$, T has a periodic orbit of period n'.

– The previous claim (period three implies all period) is a special case of this claim.

– If a 1D map has only finitely many periodic solutions, their period has to be multiples of 2.

– This claim is true only for interval maps (not even true for maps from S^1 to S^1).

Homework

1. Find T^{-1} for Ex. 2-4 in the first two pages.

2. Let x_0 be a periodic point of period n in $[0,1]$ for $T(x) = 7x(1-x)$. Is x_0 stable? Why?

3. Let $x \in \mathbb{R}^n$, and $A = (a_{ij})_{n\times n}$ be a constant matrix. Find the time-1 map of the equation $\frac{dx}{dt} = Ax$.

4. For the given set of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + y - x(x^2+y^2)^{\frac{1}{2}} \\ \frac{dy}{dt} &= -x + y - y(x^2+y^2)^{\frac{1}{2}} \end{aligned}$$

(a) Find all periodic solutions of this equation.

(b) Let S be the y-axis. Find the Poincare map induced by this equation around the indicated periodic solutions.

5. Discuss the stability of the fixed points of $T(x) = \mu x(1-x)$ for $2 < \mu < 5$.

6. Let $T(x) = x^3 - \lambda x$ for $\lambda > 0$.

(a) Find all periodic points and discuss their stabilities for $0 < \lambda < 1$.

(b) Prove that, if $|x|$ is sufficiently large, then $|f^n(x)| \to \infty$.

7. Suppose $A_0, A_1, \cdots, A_n$ are closed intervals and $T(A_i) \supset A_{i+1}$ for $i = 0, \cdots, n-1$. Prove that there is a point $x \in A_0$ such that $T^i(x) \in A_i$ for all $i \leq n$.