NON-DOMINANCE OF THE MELNIKOV FUNCTION: AN EXAMPLE

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Abstract. We present, for the first time, an equation of high frequency Hamiltonian perturbations for which the classical Melnikov function *fails* to dominate the splitting distance of the stable and unstable manifold of a perturbed saddle. We prove that, for the equation studied in this paper, the splitting distance is *not* dominated by the zeroth order term, i.e. the classical Melnikov function, but by the first order term of [3]. Our study also produces a lower bound estimate on the splitting angle in terms of the dominating first order term, by which we assert the existence of a transversal intersection of the stable and unstable manifold.

When H. Poincare discovered homoclinic tangles in his study of the restricted three-body problem, he also presented a computational proof to confirm the existence of homoclinic tangles in a particular time-periodic equation [15], [16]. A general computational scheme that calculates the zeroth order term of the splitting distance of the stable and unstable manifold of a perturbed saddle was then developed at a later time by Melnikov [14] for Hamiltonian equations. Historically, this computational scheme has served as a main venue, through which the chaos theory – in particular, the theory of Smale horseshoe – is applied to the study of ordinary differential equations [7].

Let ε be the magnitude of a time periodic perturbation and t_0 be the initial time. Also let

$$D(t_0, \varepsilon) = E_0(t_0) + \varepsilon E_1(t_0) + \cdots + \varepsilon^n E_n(t_0) + \cdots$$

be the power series expansion of the splitting distance of stable and unstable manifold of a perturbd saddle. The gist of Poincare-Melnikov method resides in an integral formula derived for $E_0(t_0)$. Not only $E_0(t_0)$ is written explicitly as a definite integral, but also this integral can be routinely evaluated by using the residue theorem for given equations.

Poincare-Melnikov method works well in asserting the existence of a transversal homoclinic intersection when $E_0(t_0)$ is the dominating term in $D(t_0, \varepsilon)$. There are, however, two main degenerate cases in which $E_0(t_0)$ fails to dominate. The first degenerate case is when the perturbation is such that $E_0(t_0) \equiv 0$. In this case, $E_0(t_0)$ would do nothing for us in finding non-tangential solutions of $D(t_0, \varepsilon) = 0$. We need to move up to compute $E_1(t_0)$. In a recent paper [3], Chen and Wang introduced a new theory on $E_1(t_0)$. Not only they derived an explicit integral formula for $E_1(t_0)$, but also they introduced a long analytic scheme to calculate $E_1(t_0)$. This computational scheme was applied to a given time-periodic equation of $E_0(t_0) \equiv 0$ to evaluate $E_1(t_0)$, asserting the existence of transversal homoclinic intersections.

The second degenerate case is when ω, the frequency of perturbation, is large and ε is in integer power of ω^{-1} (see [17]). In this case, it is easy to confirm that $E_0(t_0)$ is exponentially small in magnitude so it fails to dominate *a priori* the rest of the power series expansion of $D(t_0, \varepsilon)$. However, a failure to dominant *a priori* is not quite the same as a true failure to dominant. It is entirely possible that this case is not really degenerate in the sense that the apparent non-dominance of $E_0(t_0)$ might be caused only by our lack of proper handling

of $R(t_0) := D(t_0) - E_0(t_0)$. The remainder $R(t_0)$ might also be exponentially small, and further, remain to be dominated by $E_0(t_0)$.

In contrast to the first degenerate case, on which published work are scarce, there has been substantial literature on the second. Elaborated theories have been developed by a number of authors ([4], [5], [6], [11], [12], [8], [1], etc.). This literature strives to prove that, for Hamiltonian equations subjected to high frequency perturbation, $R(t_0)$ remain to be dominated by $E_0(t_0)$. Their conclusions are derived through sophisticated analysis of solutions of the Hamilton-Jacobi equation by using techniques borrowed from complex Fourier analysis. We refer the reader to [1] and the references therein for a thorough review on this literature. A rough reformulation of the main conclusions of [1] and its predecessors, to me, are as follows. Let

$$g_{\pm 1} = \int_0^{2\pi\omega^{-1}} E_0(t_0) e^{\pm i\omega t_0} dt_0$$

be the first order Fourier coefficient of $E_0(t_0)$. Under the assumption that

$$g_1^2 + g_{-1}^2 \neq 0,$$

the C^1-norm of $E_0(t_0)$ is much larger than that of the remainder $R(t_0)$ provided that ω is sufficiently large and ε is in integer power of ω^{-1}.

The open question is then whether $E_0(t_0)$ remains dominating when $g_1^2 + g_{-1}^2 = 0$. In this paper, we study the periodically perturbed Duffing equation

$$\frac{d^2x}{dt^2} = x - x^3 + \varepsilon x^2 \left(\cos 2\omega t + \cos 3\omega t\right)$$

where ε is the magnitude and ω is the frequency of a time periodic perturbation. We prove that the classic Melnikov function $E_0(t_0)$ indeed fails to dominate the remainder $R(t_0)$ assuming ε is in integer power of ω^{-1}. The lack of dominance of $E_0(t_0)$ is caused by the interactions of perturbations of different forcing frequencies. The effect of these interactions does not show up in $E_0(t_0)$, but they have a clear presence in $E_n(t_0)$ for all $n \geq 1$. Our study also produces a lower bound estimate on the splitting angle, asserting the existence of transversal homoclinic intersection of the perturbed saddle.

The analytic tools employed in this paper are entirely different from the method developed in [1] and its predecessors. Our analysis is based on the computational scheme of [3] on $E_1(t_0)$ and the refined analysis of [18] on $E_n(t_0)$ for all $n \geq 1$.

1. Statement of The Main Theorem

A slightly restricted version of the Poincare-Melnikov method is as follows. We study a periodically perturbed second order equation

$$\frac{dx}{dt} = y; \qquad \frac{dy}{dt} = f(x) + \varepsilon P(x,t) \tag{1.1}$$

where ε is a small parameter and $P(x,t) = P(x, t+T)$ is a periodic function in t of period $T = 2\pi\omega^{-1}$. The function $P(x,t)$ is expanded into a Fourier series as

$$P(x,t) = \sum_{m=-\infty}^{+\infty} P_m(x) e^{im\omega t}. \tag{1.2}$$

We assume that the unperturbed equation ($\varepsilon = 0$ in equation (1.1)) has a saddle fixed point, which we denote as p_0. We also assume p_0 has a homoclinic solution, which we denote as

$(x, y) = (a(t), b(t))$. Let $\ell = \{(a(t), b(t)),\ t \in (-\infty, +\infty)\}$ be the homoclinic loop and D_ℓ be a small neighborhood of ℓ in the space of (x, y). Let I be the projection of D_ℓ to the x-axis. We assume, in addition, that $f(x)$ and $P_m(x)$ for all m in (1.2) are real analytic and are uniformly bounded by a constant that is independent of ω on I.

For a sufficiently small $\varepsilon \neq 0$, let $p_\varepsilon(t)$ be the corresponding periodic solution of equation (1.1) of p_0. We call $p_\varepsilon(t)$ the perturbed saddle. Let $\ell^\perp$ be a small segment of the x-axis that crosses ℓ. For a given initial time $t_0 = [0, T)$, there is a unique solution $p^s(t, t_0, \varepsilon)$ in D_ℓ satisfying $p^s(t_0, t_0, \varepsilon) = p^+(t_0, \varepsilon) \in \ell^\perp$ such that $\lim_{t\to+\infty} |p^s(t, t_0, \varepsilon) - p_\varepsilon(t)| = 0$. There is also a unique solution $p^u(t, t_0, \varepsilon)$ in D_ℓ satisfying $p^u(t_0, t_0, \varepsilon) = p^-(t_0, \varepsilon) \in \ell^\perp$ such that $\lim_{t\to-\infty} |p^u(t, t_0, \varepsilon) - p_\varepsilon(t)| = 0$.

The splitting distance $D(t_0, \varepsilon)$ is defined by letting

$$D(t_0, \varepsilon) = \varepsilon^{-1}|E(p^+(t_0, \varepsilon)) - E(p^-(t_0, \varepsilon))|$$

where

$$E(p) = \frac{1}{2}y^2 - \int f(x)dx$$

is the unperturbed energy function.

Definition 1.1. *We say that the stable and unstable manifold of the perturbed saddle $p_\varepsilon(t)$ of equation (1.1) admits a transversal homoclinic intersection if there exists a $t_0 \in [0, T)$ such that*

$$D(t_0, \varepsilon) = 0, \qquad dD(t_0, \varepsilon)/dt_0 \neq 0. \tag{1.3}$$

If the perturbed saddle of equation (1.1) admits a transversal homoclinic intersection, then many of the conclusions of the chaos theory, including that of the Smale horseshoe, would be readily applied to equation (1.1).

To prove the existence of a transversal homoclinc intersection, we expand $D(t_0, \varepsilon)$ into a formal power series of ε as

$$D(t_0, \varepsilon) = E_0(t_0) + \varepsilon E_1(t_0) + \cdots + \varepsilon^n E_n(t_0) + \cdots . \tag{1.4}$$

Following the computational scheme introduced by Poincare and Melnikov, we obtain

$$E_0(t_0) = \int_{-\infty}^{+\infty} b(t)P(a(t), t + t_0)dt. \tag{1.5}$$

Proposition 1.1. (Poincare-Melnikov Method) *Under the assumption that there exist a $t_0 \in [0, T)$ so that*

$$E_0(t_0) = 0; \qquad dE_0(t_0)/dt_0 \neq 0, \tag{1.6}$$

the stable and unstable manifold of the perturbed saddle of equation (1.1) admit a transversal homoclinic intersection for all $\varepsilon \in \left(0, K_0^{-1}|dE_0(t_0)/dt_0|\right)$ where $K_0 > 0$ is a constant independent of ω.

We emphasize that this proposition, though not entirely trivial to prove, is only a necessary component of auxiliary importance for the Poincare-Melnikov method. The gist of this method resides in the integral formula (1.5) for $E_0(t_0)$. We further observe that, though Poincare-Melnikov method offers a thorough handling of $E_0(t_0)$, it does not do much, in view of (1.4), in handling $E_n(t_0)$ for $n \geq 1$.

In this paper, we study the periodically perturbed Duffing equation

$$\frac{d^2x}{dt^2} = x - x^3 + \varepsilon x^2 \left(\cos 2\omega t + \cos 3\omega t\right)$$

where ε is the magnitude and ω is the frequency of a time periodic perturbation. We denote $y = dx/dt$ to rewrite this equation as

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon x^2 \left(\cos 2\omega t + \cos 3\omega t\right). \tag{1.7}$$

The main theorem of this paper is as follows.

Main Theorem *There exists a positive integer $\kappa > 0$ and a sufficiently large constant $\omega_0 > 0$, so that for all $\omega \in (\omega_0, +\infty)$ and all $\varepsilon \in (0, \omega^{-\kappa})$, the stable and unstable manifold of the saddle fixed point $(x, y) = (0, 0)$ of equation (1.7) admits a transversal homoclinic intersection.*

We note that the Poincare-Melnikov method and the conclusions of [1] and its predecessors can not be applied to equation (1.7) to prove this theorem. Our proof of the Main Theorem relies on the recent theory on $E_1(t_0)$ developed in [3] and that on $E_n(t_0)$ for all n developed in [18].

2. On the Proof of the Main Theorem

We start by trying to apply the Poincare-Melnikov method. For equation (1.7), the unperturbed homoclinic solution $(a(t), b(t))$ is

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \qquad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2} \tag{2.1}$$

and by (1.6),

$$E_0(t_0) = \int_{-\infty}^{+\infty} b(t) a^2(t) \left(\sin 2\omega(t + t_0) + \cos 3\omega(t + t_0)\right) dt. \tag{2.2}$$

Proposition 2.1. *We have, for equation (1.7),*

$$E_0(t_0, \omega) = A_2 \sin 2\omega t_0 + A_3 \sin 3\omega t_0$$

where

$$A_2 = \frac{4\sqrt{2}\pi\omega(4\omega^2 + 1)}{3(1 + e^{-2\pi\omega})} e^{-\pi\omega}, \qquad A_3 = \frac{2\sqrt{2}\pi\omega(9\omega^2 + 1)}{(1 + e^{-3\pi\omega})} e^{-3\pi\omega/2}.$$

Proof. This proposition follows from a routine application of the residue theorem. □

Proposition 1.1 is then applied to equation (1.7) to assert the existence of a transversal homoclinic intersection, but only for $0 < \varepsilon < K_0^{-1}\omega^4 e^{-\pi\omega}$. Unfortunately, for ω sufficiently large,

$$K_0^{-1}\omega^4 e^{-\pi\omega} << \omega^{-\kappa}.$$

Consequently, the Poincare-Melnikov method can not be used to prove the existence of transversal homoclinic intersections for the ε-interval acclaimed in the Main Theorem.

A. The High Order Melnikov Function $E_1(t_0)$: The strict restriction on ε in applying Proposition 1.1 to equation (1.7) is due to the exponentially small factors $e^{-\pi\omega}$ for A_2 and $e^{-3\pi\omega/2}$ for A_3 in Proposition 2.1. To enlarge the size of the interval of ε for transversal homoclinic intersection, we need to carry out a refined analysis on $E_n(t_0)$. An explicit integral formula for $E_1(t_0)$ is recently derived in [3], and it is as follows for equation (1.7). Let

$$P(x, t, \omega) = x^2 \left(\cos 2\omega t + \cos 3\omega t\right) \tag{2.3}$$

and denote

$$\mathcal{P}(t,t_0,\omega)=P(a(t),t+t_0,\omega),\quad \mathcal{P}_t(t,t_0,\omega)=\partial_t P(a(t),t+t_0,\omega). \tag{2.4}$$

Also denote

$$H(t)=3b(t)a(t)\int_0^t a^{-2}(\tau)d\tau+1. \tag{2.5}$$

We have, according to [3],

$$E_1(t_0,\omega)=\frac{\mathcal{P}(0,t_0,\omega)}{\sqrt{2}}E_0(t_0,\omega)-\int_{-\infty}^{+\infty}\int_0^{\tau_2} f(\tau_1,\tau_2,t_0,\omega)d\tau_1 d\tau_2 \tag{2.6}$$

where

$$f(\tau_1,\tau_2,t_0,\omega)=\frac{b(\tau_2)H(\tau_1)}{a(\tau_1)}\left[\mathcal{P}_t(\tau_2,t_0,\omega)\mathcal{P}(\tau_1,t_0,\omega)+\mathcal{P}_t(\tau_1,t_0,\omega)\mathcal{P}(\tau_2,t_0,\omega)\right]. \tag{2.7}$$

Theorem 1. *For equation (1.7), we have, under the assumption that ω is sufficiently large,*

$$E_1(t_0,\omega)=\sum_{m=1}^{6}B_m(\omega)\sin m\omega t_0$$

where $B_1(\omega)$ is such that

$$B_1(\omega)=-2\pi\omega e^{-\pi\omega/2}\left(\frac{31}{270}+O(\omega^{-1})\right). \tag{2.8}$$

We also have, for $2\le m\le 6$,

$$|B_m(\omega)|<e^{-2\pi\omega/3}. \tag{2.9}$$

Theorem 1 and Proposition 2.1 together implies that the C^1-norm of $\varepsilon E_1(t_0)$ is much larger than that of $E_0(t_0)$ assuming ε is in polynomial power of ω^{-1}. In another word, $E_0(t_0)$ fails to dominate $\varepsilon E_1(t_0)$ for equation (1.7).

B. High Order Melnikov Functions: For the proof of the Main Theorem, the conclusions presented in the last paragraphs induce additional uncertainty: in order to control the C^1 norm of the splitting distance $D(t_0,\varepsilon,\omega)$, we need to further calculate $E_2(t_0), E_3(t_0)$ and so on. For this calculation, we rely on a recent theory developed in [18].

Theorem 2. *For equation (1.7), we have*

$$E_n(t_0,\omega)=\sum_{k=1}^{6(n+1)}A_{k,n}(\omega)\sin k\omega t_0$$

for all $n\ge 1$. Furthermore, there exist positive constants κ_0 and ω_0, so that for all $\omega>\omega_0$ and all $1\le k\le 6(n+1)$,

$$|A_{k,n}(\omega)|<\omega^{\kappa_0(n+1)}e^{-\omega\pi/2}. \tag{2.10}$$

We are ready to prove the Main Theorem assuming Theorems 1 and 2.

Proof of the Main Theorem: Let κ_0 be as in the estimates on $A_{k,n}$ from Theorem 2. We let $\kappa = 2(\kappa_0+1)$ so the acclaimed interval of ε in the Main Theorem is $(0, \omega^{-2(\kappa_0+1)})$. By using Proposition 2.1 and Theorems 1 and 2, we have, for the splitting distance,

$$\begin{aligned} D(t_0) &= E_0(t_0) + \varepsilon E_1(t_0) + \sum_{n=2}^{\infty} \varepsilon^n E_n(t_0) \\ &= A_2(\omega)\sin 2\omega t_0 + A_3(\omega)\sin 3\omega t_0 + \varepsilon B_1(\omega)\sin \omega t_0 \\ &\quad + \sum_{m=2}^{6} \varepsilon B_m(\omega)\sin m\omega t_0 + \sum_{n=2}^{\infty} \varepsilon^n \left[\sum_{k=1}^{6(n+1)} A_{k,n}(\omega)\sin k\omega t_0 \right]. \end{aligned}$$

It then follows that

$$D(0) = D\left(\omega^{-1}\pi\right) = 0.$$

We move to calculate the derivative with respect to t_0. We have

$$\begin{aligned} dD/dt_0(t_0) &= 2\omega A_2(\omega)\cos 2\omega t_0 + 3\omega A_3(\omega)\cos 3\omega t_0 + \varepsilon\omega B_1(\omega)\cos \omega t_0 \\ &\quad + \varepsilon \sum_{m=2}^{6} m\omega B_m(\omega)\cos m\omega t_0 + \sum_{n=2}^{\infty} \varepsilon^n \left[\sum_{k=1}^{6(n+1)} k\omega A_{k,n}(\omega)\cos k\omega t_0 \right]. \end{aligned}$$

This is for us to have, at $t_0 = 0$,

$$\begin{aligned} \omega^{-1} dD/dt_0(0) &= 2A_2(\omega) + 3A_3(\omega) + \varepsilon B_1(\omega) + \varepsilon \sum_{m=2}^{6} m B_m(\omega) \\ &\quad + \sum_{n=2}^{\infty} \varepsilon^n \left[\sum_{k=1}^{6(n+1)} k A_{k,n}(\omega) \right]. \end{aligned}$$

We divide into two cases. The first case is for $\varepsilon \in (0, e^{-3\omega\pi/4})$, and the second case is for $\varepsilon \in [e^{-3\omega\pi/4}, \omega^{-2(\kappa_0+1)}]$. For $\varepsilon \in (0, e^{-3\omega\pi/4})$, we have

$$\left|\omega^{-1} dD/dt_0(0)\right| \geq A_2(\omega) - K e^{-5\omega\pi/4}$$

because by Theorems 1 and 2, B_m and $A_{k,n}$ are all bounded by $e^{-\omega\pi/2}$. It then follows that, by using Proposition 2.1 for A_2,

$$dD/dt_0(0) \neq 0.$$

For the second case, that is, for ε such that

$$e^{-3\omega\pi/4} \leq \varepsilon < \omega^{-2(\kappa_0+1)}, \tag{2.11}$$

we have, for $m, n \geq 2$,

$$|B_1^{-1}(\omega) B_m(\omega)| < e^{-\pi\omega/6}; \qquad |B_1^{-1}(\omega) A_{k,n}(\omega)| < \omega^{\kappa_0} \tag{2.12}$$

by Theorems 1 and 2. We re-write $D(t_0)$ as

$$\begin{aligned} D(t_0) &= \varepsilon \left(E_1(t_0,\omega) + \varepsilon^{-1} E_0(t_0,\omega) + \sum_{n=2} \varepsilon^{n-1} E_n(t_0,\omega) \right) \\ &= \varepsilon B_1(\omega) \left(\sin \omega t_0 + \varepsilon^{-1} B_1^{-1}(\omega) \left(A_2(\omega)\sin 2\omega t_0 + A_3(\omega)\sin 3\omega t_0 \right) + H(t_0) \right) \end{aligned}$$

where

$$H(t_0) = \sum_{m=2}^{6} B_1^{-1}(\omega)B_m(\omega)\sin m\omega t_0 + \sum_{n=2}^{\infty}\varepsilon^{n-1}\left[\sum_{k=1}^{6(n+1)} B_1^{-1}(\omega)A_{k,n}(\omega)\sin k\omega t_0\right].$$

Taking derivative with respect to t_0, we have

$$dD/dt_0(t_0) = \varepsilon B_1(\omega)\omega\left(\cos\omega t_0 + \varepsilon^{-1}B_1^{-1}\left(2A_2\cos 2\omega t_0 + 3A_3\cos 3\omega t_0\right) + H_1(t_0)\right)$$

where

$$H_1(t_0) = \sum_{m=2}^{6} B_1^{-1}(\omega)B_m(\omega)m\cos m\omega t_0 + \sum_{n=2}\varepsilon^{n-1}\left[\sum_{k=1}^{6(n+1)} B_1^{-1}(\omega)A_{k,n}(\omega)k\cos k\omega t_0\right].$$

We conclude that

$$|H_1(t_0)| < 30e^{-\pi\omega/6} + \sum_{n=2}^{\infty} 36(n+1)^2\omega^{-\kappa_0 n} << 1. \tag{2.13}$$

Evaluating $dD/dt_0(t_0)$ at $t_0 = 0$ and $t_0 = \omega^{-1}\pi$, we have

$$\begin{aligned} dD/dt_0(0) &= \varepsilon B_1(\omega)\omega\left(1 + \varepsilon^{-1}B_1^{-1}\left(2A_2 + 3A_3\right) + H_1(0)\right) \\ dD/dt_0(\omega^{-1}\pi) &= \varepsilon B_1(\omega)\omega\left(-1 + \varepsilon^{-1}B_1^{-1}\left(2A_2 - 3A_3\right) + H_1(\omega^{-1}\pi)\right). \end{aligned}$$

This is for us to have

$$dD/dt_0(0) - dD/dt_0(\omega^{-1}\pi) = \varepsilon B_1(\omega)\omega\left(2 + 6\varepsilon^{-1}B_1^{-1}A_3 + H_1(0) - H_1(\omega^{-1}\pi)\right). \tag{2.14}$$

By using (2.11) and (2.12), we have

$$\begin{aligned} \varepsilon^{-1}B_1^{-1}A_3 &\le e^{3\pi\omega/4}\frac{2\sqrt{2}\pi\omega(9\omega^2+1)}{(1+e^{-3\pi\omega})}e^{-3\pi\omega/2}2\pi\omega^{-1}e^{\pi\omega/2}\left(\frac{31}{270} + O(\omega^{-1})\right)^{-1} \\ &< e^{-\omega\pi/6} << 1. \end{aligned}$$

This estimate, together with (2.13), (2.14), implies that $dD/dt_0(t_0)$ can not be both zero at $t_0 = 0$ and $t_0 = \omega^{-1}\pi$. □

The rest of this paper is devoted to prove Theorem 1 and Theorem 2.

3. Proof of Theorem 1

In this section, we prove Theorem 1 by evaluating $E_1(t_0)$. Recall that, according to [3],

$$E_1(t_0,\omega) = \frac{\mathcal{P}(0,t_0,\omega)}{\sqrt{2}}E_0(t_0,\omega) - \int_{-\infty}^{+\infty}\int_0^{\tau_2} f(\tau_1,\tau_2,t_0,\omega)d\tau_1 d\tau_2 \tag{3.1}$$

where $E_0(t_0,\omega)$ is as in Proposition 2.1,

$$f(\tau_1,\tau_2,t_0,\omega) = \frac{b(\tau_2)H(\tau_1)}{a(\tau_1)}\left[\mathcal{P}_t(\tau_2,t_0,\omega)\mathcal{P}(\tau_1,t_0,\omega) + \mathcal{P}_t(\tau_1,t_0,\omega)\mathcal{P}(\tau_2,t_0,\omega)\right], \tag{3.2}$$

and $(a(t), b(t))$ are as in (2.1); $H(t)$ is as in (2.5); and $P(x,t,\omega)$, $\mathcal{P}(t,t_0,\omega)$ and $\mathcal{P}_t(t,t_0,\omega)$ are as in (2.3) and (2.4).

The evaluation of the double integral in (3.1) for $E_1(t_0)$ is long and hard. Fortunately, there has been a step by step evaluation procedure established in [3], which we can follow closely to prove Theorem 1.

3.1. **The integrals for $E_1(t_0,\omega)$.** We start calculating $\mathcal{P}_t(\tau_2,t_0)\mathcal{P}(\tau_1,t_0)$. Recall that

$$\mathcal{P}(t,t_0,\omega)=P(a(t),t+t_0,\omega),\quad \mathcal{P}_t(t,t_0,\omega)=\partial_t P(a(t),t+t_0,\omega)$$

where

$$P(x,t,\omega)=x^2\left(\cos 2\omega t+\cos 3\omega t\right).$$

Lemma 3.1. *We have*

$$\mathcal{P}_t(\tau_2,t_0)\mathcal{P}(\tau_1,t_0)=-\omega a^2(\tau_1)a^2(\tau_2)\left(g_0(\tau_1,\tau_2)+\sum_{n=1}^{6}\left(g_n(\tau_1,\tau_2)\cos n\omega t_0+f_n(\tau_1,\tau_2)\sin n\omega t_0\right)\right)$$

where

$$\begin{aligned}
g_0(\tau_1,\tau_2) &= \frac{3}{2}\sin 3\omega(\tau_2-\tau_1)+\sin 2\omega(\tau_2-\tau_1)\\
g_1(\tau_1,\tau_2) &= \frac{3}{2}\sin(3\omega\tau_2-2\omega\tau_1)-\sin(3\omega\tau_1-2\omega\tau_2)\\
g_4(\tau_1,\tau_2) &= \sin 2\omega(\tau_2+\tau_1)\\
g_5(\tau_1,\tau_2) &= \frac{3}{2}\sin(3\omega\tau_2+2\omega\tau_1)+\sin(2\omega\tau_2+3\omega\tau_1)\\
g_6(\tau_1,\tau_2) &= \frac{3}{2}\sin 3\omega(\tau_2+\tau_1)
\end{aligned}$$

and

$$\begin{aligned}
f_1(\tau_1,\tau_2) &= \frac{3}{2}\cos(3\omega\tau_2-2\omega\tau_1)-\cos(3\omega\tau_1-2\omega\tau_2)\\
f_4(\tau_1,\tau_2) &= \cos 2\omega(\tau_2+\tau_1)\\
f_5(\tau_1,\tau_2) &= \frac{3}{2}\cos(3\omega\tau_2+2\omega\tau_1)+\cos(2\omega\tau_2+3\omega\tau_1)\\
f_6(\tau_1,\tau_2) &= \frac{3}{2}\cos 3\omega(\tau_2+\tau_1).
\end{aligned}$$

We have, in addition,

$$g_2(\tau_1,\tau_2)=f_2(\tau_1,\tau_2)=g_3(\tau_1,\tau_2)=f_3(\tau_1,\tau_2)=0.$$

Proof. This lemma follows directly from an elementary calculation. □

In what follows, we let

$$\begin{aligned}
I_{n,c} &= \int_{-\infty}^{+\infty}\int_0^{\tau_2} a(\tau_1)H(\tau_1)b(\tau_2)a^2(\tau_2)\left[g_n(\tau_1,\tau_2)+g_n(\tau_2,\tau_1)\right]d\tau_1 d\tau_2\\
I_{n,s} &= \int_{-\infty}^{+\infty}\int_0^{\tau_2} a(\tau_1)H(\tau_1)b(\tau_2)a^2(\tau_2)\left[f_n(\tau_1,\tau_2)+f_n(\tau_2,\tau_1)\right]d\tau_1 d\tau_2.
\end{aligned}\tag{3.3}$$

We have, by (3.1) and (3.2),

$$E_1(t_0,\omega)=\frac{\mathcal{P}(0,t_0,\omega)}{\sqrt{2}}E_0(t_0,\omega)-\omega I_{0,c}-\omega\sum_{n=1}^{6}\left(I_{n,c}\cos n\omega t_0+I_{n,s}\sin n\omega t_0\right).\tag{3.4}$$

Our task is then to evaluate $I_{0,c}$ and $I_{n,c}, I_{n,s}$ for $n=1,4,5,6$.

Lemma 3.2. *(1) Assume $F(\tau_1,\tau_2)$ is even in the sense that $F(\tau_1,\tau_2)=F(-\tau_1,-\tau_2)$. Then,*

$$\int_0^{+\infty}\int_0^{\tau_2} F(\tau_1,\tau_2)d\tau_1 d\tau_2 = \int_0^{-\infty}\int_0^{\tau_2} F(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

(2) Assume $F(\tau_1,\tau_2)$ is odd in the sense that $F(\tau_1,\tau_2)=-F(-\tau_1,-\tau_2)$. Then,

$$\int_0^{+\infty}\int_0^{\tau_2} F(\tau_1,\tau_2)d\tau_1 d\tau_2 = -\int_0^{-\infty}\int_0^{\tau_2} F(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

Proof. By changing (τ_1,τ_2) to $(-\tau_1,-\tau_2)$. □

We have, as a direct corollary,

Corollary 3.1. *For $n=0$ to 6, $I_{n,c}=0$.*

Proof. By Lemma 3.1, $g_n(\tau_1,\tau_2)+g_n(\tau_2,\tau_1)$ are odd functions of τ_1,τ_2. Note that $H(t),a(t)$ are even, but $b(t)$ is odd in t. It then follows that the integral function for $I_{n,c}$,

$$\frac{b(\tau_2)H(\tau_1)}{a(\tau_1)}\left[g_n(\tau_1,\tau_2)+g_n(\tau_2,\tau_1)\right],$$

is an even function of τ_1,τ_2. By Lemma 3.2(1), we have $I_{n,c}=0$ for all n. □

In conclusion, we have

$$E_1(t_0,\omega)=\frac{\mathcal{P}(0,t_0,\omega)}{\sqrt{2}}E_0(t_0,\omega)-\omega\sum_{n=1}^{6} I_{n,s}\sin n\omega t_0. \tag{3.5}$$

We also have $I_{2,s}=I_{3.s}=0$ by Lemma 3.1. We need to further evaluate $I_{1,s},I_{4,s},I_{5,s},I_{6,s}$.

3.2. **Evaluation of $I_{1,s}$.** By (3.3) and Lemma 3.1, we have

$$I_{1,s}=\frac{1}{2}\int_{-\infty}^{+\infty}\int_0^{\tau_2} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[\cos(3\omega\tau_2-2\omega\tau_1)+\cos(3\omega\tau_1-2\omega\tau_2)\right]d\tau_1 d\tau_2.$$

We calculate $I_{1,s}$ in this subsection to prove

Proposition 3.1.

$$I_{1,s}=-2\pi\omega e^{-\pi\omega/2}\left(\frac{31}{270}+O(\omega^{-1})\right).$$

The proof of this proposition is highly technical. We divide into a few sub-steps.

3.2.1. *Preparations.* Let $G(t)$ be such that

$$G(t)=H(t)a(t)-\frac{3\pi i}{8}b(t)a^2(t) \tag{3.6}$$

where $a(t),b(t)$ are as in (1.5) and $H(t)$ is as in (2.5). We also denote

$$g(t)=G(t+i\pi/2),\quad f(t)=t^4 b(t+i\pi/2)a^2(t+i\pi/2). \tag{3.7}$$

Our next lemma is preparatory in nature.

Lemma 3.3. *The functions $g(t)$, $f(t)$ are analytic at $t=0$. In addition, we have (i) $g(t)$ is odd in t, $f(t)$ is even in t; (ii) $g(0)=0$, $g'(0)=\frac{\sqrt{2}}{5}i$; and (iii) $f(0)=-2\sqrt{2}i$, $f'(0)=0$.*

Proof. That $a(t+i\pi/2)$ is odd and $b(t+i\pi/2)$ is even in t follows from

$$a(t+i\pi/2) = -\frac{2\sqrt{2}i}{(e^t-e^{-t})}, \qquad b(t+i\pi/2) = \frac{2\sqrt{2}i(e^{-t}+e^t)}{(e^t-e^{-t})^2}.$$

We then have, first,

$$g(t) \;=\; \frac{2\sqrt{2}i(e^{-t}+e^t)}{(e^t-e^{-t})^2}\left(\frac{3(e^{2t}-e^{-2t}-4t)}{2(e^t-e^{-t})^2}\right) - \frac{2\sqrt{2}i}{(e^t-e^{-t})}.$$

This is odd in t. Expanding into power series at $t=0$, we obtain

$$g(t) \;=\; \frac{\sqrt{2}}{5}it + O(t^3).$$

Next, we have

$$f(t) = t^4\frac{2\sqrt{2}i(e^{-t}+e^t)}{(e^t-e^{-t})^2}\left(\frac{2\sqrt{2}i}{(e^t-e^{-t})}\right)^2.$$

This is even in t. Again, expanding into power series at $t=0$, we obtain

$$f(t) = -2\sqrt{2}i + O(t^2).$$

All items of this lemma now follow. □

We note that, for $a(z)$ and $b(z)$, the point $z=i\pi/2$ is a pole. This lemma acclaims that $G(z)$, however, is analytic at $z=i\pi/2$. It is going to facilitate greatly our evaluation of $I_{1,s}$.

3.2.2. *Reduction.* In this subsection, we decompose $I_{1,s}$ into a set of simpler integrals.

Lemma 3.4. *We have* $I_{1,s} = \frac{1}{2}\left(\mathcal{R}_1+\mathcal{R}_2+\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3\right)$ *where*

$$\begin{aligned}
\mathcal{R}_1 &= \int_0^{-\infty} e^{3i\omega s_2}\left(\int_{-\infty}^{+\infty} G(s_1)b(s_1+s_2)a^2(s_1+s_2)e^{i\omega s_1}ds_1\right)ds_2\\
\mathcal{R}_2 &= \int_0^{-\infty} e^{-2i\omega s_2}\left(\int_{-\infty}^{+\infty} G(s_1)b(s_1+s_2)a^2(s_1+s_2)e^{i\omega s_1}ds_1\right)ds_2\\
\mathcal{E}_1 &= \left(\int_{-\infty}^{+\infty} b(\tau)a^2(\tau)e^{3i\omega\tau}d\tau\right)\left(\int_0^{+\infty} H(\tau)a(\tau)e^{-2i\omega\tau}d\tau\right)\\
\mathcal{E}_2 &= \left(\int_{-\infty}^{+\infty} b(\tau)a^2(\tau)e^{-2i\omega\tau}d\tau\right)\left(\int_0^{+\infty} H(\tau)a(\tau)e^{3i\omega\tau}d\tau\right)\\
\mathcal{E}_3 &= -\frac{3\pi i}{8}\left(\int_{-\infty}^{+\infty} b(\tau)a^2(\tau)e^{3i\omega\tau}d\tau\right)\left(\int_{-\infty}^{+\infty} b(\tau)a^2(\tau)e^{-2i\omega\tau}d\tau\right).
\end{aligned} \tag{3.8}$$

Proof. We use Lemma 3.2(2) to write $I_{1,s}$ as

$$I_{1,s} = \int_0^{+\infty}\int_0^{\tau_2} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[\cos(3\omega\tau_2-2\omega\tau_1)+\cos(3\omega\tau_1-2\omega\tau_2)\right]d\tau_1 d\tau_2.$$

We divide the proof of this lemma into three steps.

Step 1: Initial Reduction. We write trigonometric functions in complex form to obtain

$$\begin{aligned}
I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_0^{\tau_2} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[e^{i(3\omega\tau_2-2\omega\tau_1)}+e^{-i(3\omega\tau_2-2\omega\tau_1)}\right]d\tau_1 d\tau_2\\
&+\frac{1}{2}\int_0^{+\infty}\int_0^{\tau_2} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[e^{i(3\omega\tau_1-2\omega\tau_2)}+e^{-i(3\omega\tau_1-2\omega\tau_2)}\right]d\tau_1 d\tau_2.
\end{aligned}$$

We then switch the order of integration from $d\tau_1 d\tau_2$ to $d\tau_2 d\tau_1$ to obtain

$$\begin{aligned} I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_{\tau_1}^{+\infty} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[e^{i(3\omega\tau_2-2\omega\tau_1)}+e^{-i(3\omega\tau_2-2\omega\tau_1)}\right]d\tau_2 d\tau_1 \\ &+\frac{1}{2}\int_0^{+\infty}\int_{\tau_1}^{+\infty} H(\tau_1)a(\tau_1)b(\tau_2)a^2(\tau_2)\left[e^{i(3\omega\tau_1-2\omega\tau_2)}+e^{-i(3\omega\tau_1-2\omega\tau_2)}\right]d\tau_2 d\tau_1. \end{aligned}$$

Let $s_1=\tau_1, s_2=\tau_2-\tau_1$. We have

$$\begin{aligned} I_{1,s} =& \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} H(s_1)a(s_1)b(s_1+s_2)a^2(s_1+s_2)\left[e^{i(3\omega s_2+\omega s_1)}+e^{-i(3\omega s_2+\omega s_1)}\right]ds_2 ds_1 \\ &+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} H(s_1)a(s_1)b(s_1+s_2)a^2(s_1+s_2)\left[e^{i(\omega s_1-2\omega s_2)}+e^{-i(\omega s_1-2\omega s_2)}\right]ds_2 ds_1 \\ =& \frac{1}{2}\left(A^+ + A^- + B^+ + B^-\right) \end{aligned}$$

where

$$\begin{aligned} A^{\pm} &= \int_0^{+\infty} H(s_1)a(s_1)e^{\pm i\omega s_1}\int_0^{+\infty} b(s_1+s_2)a^2(s_1+s_2)e^{\pm 3i\omega s_2}ds_2 ds_1 \\ B^{\pm} &= \int_0^{+\infty} H(s_1)a(s_1)e^{\pm i\omega s_1}\int_0^{+\infty} b(s_1+s_2)a^2(s_1+s_2)e^{\mp 2i\omega s_2}ds_2 ds_1. \end{aligned} \tag{3.9}$$

Step 2: Extending integral bounds. We have

$$\begin{aligned} A^+ &= \int_0^{+\infty} H(s_1)a(s_1)e^{i\omega s_1}\int_0^{+\infty} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1 \\ &= \int_0^{+\infty} H(s_1)a(s_1)e^{i\omega s_1}\int_{-\infty}^{+\infty} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1 \\ &\quad - \int_0^{+\infty} H(s_1)a(s_1)e^{i\omega s_1}\int_{-\infty}^{0} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1. \end{aligned}$$

Here, we first change the lower bound of the inner integral from 0 to $-\infty$, then nullify the effect of this replacement by subtracting the second integral. We do the same to the lower integral bound of the outer integration to obtain

$$A^+ = (I) + (II) + (III)$$

where

$$\begin{aligned} (I) &= \int_0^{+\infty} H(s_1)a(s_1)e^{i\omega s_1}\int_{-\infty}^{+\infty} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1 \\ (II) &= -\int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1}\int_{-\infty}^{0} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1 \\ (III) &= \int_{-\infty}^{0} H(s_1)a(s_1)e^{i\omega s_1}\int_{-\infty}^{0} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2}ds_2 ds_1. \end{aligned}$$

For (I), we let $\tau = s_1+s_2$ to obtain

$$(I) = \int_0^{+\infty} H(s_1)a(s_1)e^{-2i\omega s_1}\int_{-\infty}^{+\infty} b(\tau)a^2(\tau)e^{3i\omega\tau}d\tau ds_1 = \mathcal{E}_1$$

where $\mathcal{E}_1$ is as in (3.8). We also note that $(III) = -A^-$. This is for us to conclude that

$$A^+ + A^- = \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} \int_0^{-\infty} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 + \mathcal{E}_1. \tag{3.10}$$

In parallel, we have

$$B^+ + B^- = \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} \int_0^{-\infty} b(s_1+s_2)a^2(s_1+s_2)e^{-2i\omega s_2} ds_2 ds_1 + \mathcal{E}_2 \tag{3.11}$$

where $\mathcal{E}_2$ is as in (3.8). It then follows from (3.10) and (3.11) that

$$I_{1,s} = -\frac{\omega}{2}\left(\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \mathcal{E}_1 + \mathcal{E}_2\right)$$

where

$$\begin{aligned} \tilde{\mathcal{R}}_1 &= \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} \int_0^{-\infty} b(s_1+s_2)a^2(s_1+s_2)e^{3i\omega s_2} ds_2 ds_1 \\ \tilde{\mathcal{R}}_2 &= \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} \int_0^{-\infty} b(s_1+s_2)a^2(s_1+s_2)e^{-2i\omega s_2} ds_2 ds_1. \end{aligned} \tag{3.12}$$

Step 3: Final Reduction. Switching the order of integration from ds_2ds_1 to ds_1ds_2, we obtain

$$\begin{aligned} \tilde{\mathcal{R}} &= \int_0^{-\infty} e^{3i\omega s_2} \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} b(s_1+s_2)a^2(s_1+s_2) ds_1 ds_2 \\ \tilde{\mathcal{R}}_2 &= \int_0^{-\infty} e^{-2i\omega s_2} \int_{-\infty}^{+\infty} H(s_1)a(s_1)e^{i\omega s_1} b(s_1+s_2)a^2(s_1+s_2) ds_1 ds_2. \end{aligned}$$

Recall that, by definition,

$$H(t)a(t) = G(t) + \frac{3\pi i}{8} b(t)a^2(t).$$

This is for us to have

$$\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 = \mathcal{R}_1 + \mathcal{R}_2 + W_1 + W_2$$

where $\mathcal{R}_1, \mathcal{R}_2$ are as in (3.8), and

$$\begin{aligned} W_1 =& \frac{3\pi i}{8} \int_0^{-\infty} e^{3i\omega s_2} \left(\int_{-\infty}^{+\infty} b(s_1)a^2(s_1)b(s_1+s_2)a^2(s_1+s_2)e^{i\omega s_1} ds_1 \right) ds_2 \\ W_2 =& \frac{3\pi i}{8} \int_0^{-\infty} e^{-2i\omega s_2} \left(\int_{-\infty}^{+\infty} b(s_1)a^2(s_1)b(s_1+s_2)a^2(s_1+s_2)e^{i\omega s_1} ds_1 \right) ds_2. \end{aligned} \tag{3.13}$$

We work on $W_1 + W_2$. Letting $t_1 = s_1 + s_2$, we have

$$\begin{aligned} W_1 =& \frac{3\pi i}{8} \int_0^{-\infty} e^{2i\omega s_2} \left(\int_{-\infty}^{+\infty} b(t_1-s_2)a^2(t_1-s_2)b(t_1)a^2(t_1)e^{i\omega t_1} dt_1 \right) ds_2 \\ =& -\frac{3\pi i}{8} \int_0^{+\infty} e^{-2i\omega t_2} \left(\int_{-\infty}^{+\infty} b(t_1+t_2)a^2(t_1+t_2)b(t_1)a^2(t_1)e^{i\omega t_1} dt_1 \right) dt_2 \end{aligned}$$

where the second equality is obtained by letting $t_2 = -s_2$. It then follows that $W_1 + W_2 = \mathcal{E}_3$ where $\mathcal{E}_3$ is as in (3.8). □

3.2.3. *A generic setting.* In this subsection, we introduce a generic setting to facilitate the computation of $\mathcal{R}_1$ and $\mathcal{R}_2$. Let

$$\mathcal{K}_{g,f}(z,t) = \frac{e^{i\omega z} g(z) f(t+z)}{(t+z)^4} \tag{3.14}$$

where $t \neq 0$ is a real parameter and z is a complex variable. We assume

(A1) the functions $f(z), g(z)$ are independent of the forcing frequency ω,
(A2) $f(z), g(z)$ are real analytic, non-constant functions and $f(0) \neq 0$,
(A3) $|f^{(k)}(t)|,\ |g^{(k)}(t)| \leq C_0 e^{-c_0|t|}$ for all $0 \leq k < 4$ where both C_0 and c_0 are positive constants.

Letting $t \neq 0$ be fixed, we regard $\mathcal{K}_{g,f}(z,t)$ as a function of z. The function $\mathcal{K}_{g,f}(z,t)$ has a pole of order 4 on the real z axis at $z = -t$ (with the possible exception of countably many values of t so that $g(t) = 0$). Denote the residue of this pole as $R(t)$. Our purpose is to calculate

$$I_{g,f}^{(m)}(\omega) = \int_0^{-\infty} e^{im\omega t} R(t) dt. \tag{3.15}$$

Lemma 3.5. *We have*

$$\begin{aligned} I_{g,f}^{(m)}(\omega) = &- \frac{(i\omega)^2}{3!(m-1)} f(0)g(0) - \frac{i\omega}{2!(m-1)} \left(f(0)g'(0) + f'(0)g(0)\right) \\ &+ \frac{i\omega}{3!(m-1)^2} f(0)g'(0) + O(1). \end{aligned} \tag{3.16}$$

Proof. Denote

$$K(z) = e^{i\omega z} g(z) \tag{3.17}$$

to write $\mathcal{K}_{g,f}(z,t)$ as

$$\mathcal{K}_{g,f}(z,t) = \frac{K(z) f(t+z)}{(t+z)^4}. \tag{3.18}$$

We have

$$R(t) \quad = \quad \frac{1}{3!} \partial_{z^3} \left. \left(f(z) K(z-t)\right) \right|_{z=0}.$$

To compute $R(t)$, we start with Leibniz's formula (product rule)

$$\partial_{z^3}(h_1(z)h_2(z)) = \sum_{\alpha=0}^{3} \frac{3!}{\alpha!(3-\alpha)!} \partial_{z^\alpha} h_1(z) \cdot \partial_{z^{3-\alpha}} h_2(z). \tag{3.19}$$

We have, by using (3.19),

$$R(t) \quad = \quad \sum_{\alpha=0}^{3} \frac{1}{\alpha!(3-\alpha)!} f^{(\alpha)}(0) \partial_{\tau^{(3-\alpha)}} K(\tau)$$

where

$$\tau = -t.$$

We use $K(\tau) = e^{i\omega\tau} g(\tau)$ to obtain

$$K^{(3-\alpha)}(\tau) = e^{i\omega\tau} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}(3-\alpha)!}{\gamma!(3-\alpha-\gamma)!} g^{(\gamma)}(\tau).$$

This is then for us to have

$$\begin{aligned} R(t) &= \sum_{\alpha=0}^{3} \frac{1}{\alpha!(3-\alpha)!} f^{(\alpha)}(0) e^{i\omega\tau} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}(3-\alpha)!}{\gamma!(3-\alpha-\gamma)!} g^{(\gamma)}(\tau) \\ &= e^{i\omega\tau} \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0) g^{(\gamma)}(\tau) \\ &= e^{-i\omega t} \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0) g^{(\gamma)}(-t). \end{aligned}$$

This is for us to have

$$I_{g,f}^{(m)} = \sum_{\alpha=0}^{3} \sum_{\gamma=0}^{3-\alpha} \frac{(i\omega)^{3-\alpha-\gamma}}{\alpha!\gamma!(3-\alpha-\gamma)!} f^{(\alpha)}(0) \mathcal{I}_g^{\alpha,\gamma,(m)} \tag{3.20}$$

where

$$\mathcal{I}_g^{\alpha,\gamma,(m)} = \int_0^{-\infty} e^{i(m-1)\omega t} g^{(\gamma)}(-t) dt. \tag{3.21}$$

We have

$$\begin{aligned} \mathcal{I}_g^{\alpha,\gamma,(m)} &= -\int_0^{+\infty} e^{-i(m-1)\omega t} g^{(\gamma)}(t) dt = \frac{1}{i(m-1)\omega} \int_0^{+\infty} g^{(\gamma)}(t) de^{-i(m-1)\omega t} \\ &= \frac{-1}{i(m-1)\omega} g^{(\gamma)}(0) + \frac{1}{(i(m-1)\omega)^2} \int_0^{+\infty} g^{(\gamma+1)}(t) de^{-i(m-1)\omega t} \\ &= \frac{-1}{i(m-1)\omega} g^{(\gamma)}(0) - \frac{1}{(i(m-1)\omega)^2} g^{(\gamma+1)}(0) + O(\omega^{-3}). \end{aligned} \tag{3.22}$$

It then follows from (3.20) that

$$\begin{aligned} I_{g,f}^{(m)} = &- \frac{(i\omega)^2}{3!(m-1)} f(0)g(0) - \frac{i\omega}{2!(m-1)} \left(f(0)g'(0) + f'(0)g(0)\right) \\ &+ \frac{i\omega}{3!(m-1)^2} f(0)g'(0) + O(1). \end{aligned}$$

□

3.2.4. *Evaluating $\mathcal{R}_1$ and $\mathcal{R}_2$.* For a continuous curve ℓ in the complex z-plane, we let

$$I_\ell(t) = \int_\ell e^{i\omega z} G(z) b(t+z) a^2(t+z) dz$$

where $t \neq 0$ is a real parameter. Let

$$\begin{aligned} \ell_1(t) =& \{z = t_1 + is_1, \quad t_1 \in (-\infty, +\infty), \quad s_1 = 0\} \\ \ell_2(t) =& \{z = t_1 + is_1, \quad t_1 \in (-\infty, +\infty), \quad s_1 = i\rho\}. \end{aligned}$$

where $\rho = 3\pi/2 - \omega^{-1}$ for ℓ_2. First, we work on I_{ℓ_2}.

Lemma 3.6. *There exists a constant C independent of ω such that*

$$|I_{\ell_2}| < C\omega^4 e^{-3\pi\omega/2} e^{-|t|}.$$

Proof. Recall that $\rho = 3\pi/2 - \omega^{-1}$. First, by the asymptotic behavior of $G(t), b(t), a(t)$ on ℓ_2, we have

$$|G(t+i\rho)| \le Ce^{-|t|}, \quad |b(t+i\rho)a^2(t+i\rho)| \le C\omega^4 e^{-3|t|}. \tag{3.23}$$

We have a factor ω^4 in the second estimate because the order of the pole of the function $b(t)a^2(t)$ at $t = 3i\pi/2$ is four. Note that the distance from ℓ_2 to the pole located at $-t+i3\pi/2$ is $\ge \omega^{-1}$. We have

$$\begin{aligned} I_{\ell_2} &= e^{-3\pi\omega/2+1}\int_{-\infty}^{+\infty} e^{i\omega t_1} G(t_1+i\rho)b(t+t_1+i\rho)a^2(t+t_1+i\rho)dt_1 \\ &= e^{-3\pi\omega/2+1}e^{-i\omega t}\int_{-\infty}^{+\infty} e^{i\omega s_1} G(s_1-t+i\rho)b(s_1+i\rho)a^2(s_1+i\rho)ds_1. \end{aligned}$$

This implies

$$\begin{aligned} |I_{\ell_2}| &\le e^{-3\pi\omega/2+1}\int_{-\infty}^{+\infty} \left|G(s_1-t+i\rho)b(s_1+i\rho)a^2(s_1+i\rho)\right| ds_1 \\ &\le C\omega^4 e^{-3\pi\omega/2}\int_{-\infty}^{+\infty} e^{-|s_1-t|}e^{-3|s_1|}dt_1 \\ &\le C\omega^4 e^{-3\pi\omega/2}\int_{-\infty}^{+\infty} e^{-(|t|-|s_1|)}e^{-3|s_1|}dt_1 \\ &\le C\omega^4 e^{-3\pi\omega/2}e^{-|t|}. \end{aligned}$$

For the second inequality here, we use (3.23); for the third, we use $|s_1 - t| \ge |t| - |s_1|$. □

Lemma 3.7. *We have*

$$\mathcal{R}_1 + \mathcal{R}_2 = -2\pi\omega e^{-\pi\omega/2}\left(\frac{31}{270} + O(\omega^{-1})\right) + O(\omega^4 e^{-3\pi\omega/2}).$$

Proof. We have, by definition,

$$\mathcal{R}_1 = \int_0^{-\infty} e^{3i\omega t} I_{\ell_1} dt, \qquad \mathcal{R}_2 = \int_0^{-\infty} e^{-2i\omega t} I_{\ell_1} dt.$$

By the residue theorem,

$$I_{\ell_1} = I_{\ell_2} + 2\pi i\ Res(e^{i\omega z}G(z)b(t+z)a^2(t+z))\big|_{z=-t+\pi i/2} = I_{\ell_2} + 2\pi i e^{-\pi\omega/2}R(t)$$

where

$$R(t) = Res(e^{i\omega z}G(z)b(t+z)a^2(t+z))\big|_{z=-t+\pi i/2} = Res\left(\frac{e^{i\omega z_1}g(z_1)f(z_1+t)}{(t+z_1)^4}\right)\Bigg|_{z_1=-t}.$$

We note that z_1 in the second equality is such that $z_1 = z - \pi i/2$ and the functions f, g are as in (3.8). We then have by Lemma 3.6,

$$\mathcal{R}_1 = \int_0^{-\infty} e^{3i\omega t} I_{\ell_1}(t)dt = \int_0^{-\infty} e^{3i\omega t}\left(I_{\ell_2} + 2\pi i e^{-\pi\omega/2}R(t)\right)dt = 2\pi i e^{-\pi\omega/2} I_{g,f}^{(3)} + O(\omega^4 e^{-3\pi\omega/2}).$$

In parallel, we have

$$\mathcal{R}_2 = 2\pi i e^{-\pi\omega/2} I_{g,f}^{(-2)} + O(\omega^4 e^{-3\pi\omega/2}).$$

By using Lemmas 3.5 and 3.3,

$$I_{g,f}^{(m)}(\omega) = \left(-\frac{1}{2(m-1)} + \frac{1}{6(m-1)^2}\right)(i\omega)f(0)g'(0) + O(1).$$

This gives us

$$I_{g,f}^{(3)}(\omega) + I_{g,f}^{(-2)}(\omega) = \frac{31\omega i}{54\times 5} + O(1).$$

We finally conclude

$$\mathcal{R}_1 + \mathcal{R}_2 = 2\pi\omega e^{-\pi\omega/2}\left(-\frac{31}{270} + O(\omega^{-1})\right) + O(\omega^4 e^{-3\pi\omega/2}).$$

□

Proof of Proposition 3.1. Let

$$I_{m_1,m_2,n} = \int_{-\infty}^{+\infty} e^{in\omega t} a^{m_1}(t) b^{m_2}(t) dt.$$

Recall that $a(t), b(t)$ are meromorphic function in the complex t-plane and both functions take $(2n+1)\pi i/2$ for all n as poles. By a direct application of the residue theorem, we obtain

$$|I_{m_1,m_2,n}| \leq K\omega^{m_1+2m_2-1} e^{-n\omega\pi/2}. \tag{3.24}$$

We leave the details of this estimate to the reader. Applying (3.24) to $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, we have

$$|\mathcal{E}_1| \leq K\omega^3 e^{-3\pi\omega/2}, \quad |\mathcal{E}_2| \leq K\omega^3 e^{-\pi\omega}, \quad |\mathcal{E}_3| \leq K\omega^6 e^{-5\pi\omega/2}.$$

Proposition 3.1 then follows by combining these estimates with the conclusion of Lemma 3.7. □

3.3. **Proof of Theorem 1.** First, we prove

Lemma 3.8. $|I_{4,s}| < K\omega^3 e^{-\pi\omega}, \quad |I_{5,s}| < K\omega^3 e^{-5\pi\omega/4}, \quad |I_{6,s}| < K\omega^3 e^{-3\pi\omega/2}.$

Proof. We start with $I_{4,s}$. We follow the process of Sect. 3.2 closely to decompose $I_{4,s}$ into a collection of integrals to obtain

$$I_{4,s} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

where

$$\begin{aligned}
\mathcal{R}_1 =& -\int_{-\infty}^{0} e^{2i\omega s_2}\left(\int_{-\infty}^{+\infty} e^{4i\omega s_1} G(s_1) b(s_1+s_2) a^2(s_1+s_2) ds_1\right) ds_2 \\
\mathcal{R}_2 =& -\int_{-\infty}^{0} e^{-2i\omega s_2}\left(\int_{-\infty}^{+\infty} e^{-4i\omega s_1} G(s_1) b(s_1+s_2) a^2(s_1+s_2) ds_1\right) ds_2 \\
\mathcal{E}_1 =& \left(\int_{0}^{+\infty} a(\tau) H(\tau) e^{2i\omega\tau} d\tau\right)\left(\int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega\tau} d\tau\right) \\
\mathcal{E}_2 =& \left(\int_{0}^{+\infty} a(\tau) H(\tau) e^{-2i\omega\tau} d\tau\right)\left(\int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{-2i\omega\tau} d\tau\right) \\
\mathcal{E}_3 =& -\frac{3\pi i}{8}\left(\int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega\tau} d\tau\right)\left(\int_{-\infty}^{+\infty} b(\tau) a^2(\tau) e^{2i\omega\tau} d\tau\right).
\end{aligned}$$

In contrast to $\mathcal{R}_1$, $\mathcal{R}_2$ for $I_{1,s}$, the triangular part of the inner integrals for $\mathcal{R}_1, \mathcal{R}_2$ here is respectively $e^{4i\omega s_1}$ and $e^{-4i\omega s_1}$. Consequently, applying the residue theorem to the inner

integral would induce a factor $e^{-2\pi\omega}$ for $I_{4,s}$ instead of $e^{-\pi\omega/2}$ for $I_{1,s}$. It then follows directly that

$$|\mathcal{R}_1|,\ |\mathcal{R}_2| < K\omega^3 e^{-2\pi\omega}.$$

We further have, for $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$,

$$|\mathcal{E}_1|,\ |\mathcal{E}_2|,\ |\mathcal{E}_3| < K\omega^3 e^{-\pi\omega}.$$

In conclusion, we have

$$|I_{4,s}| < K\omega^3 e^{-\pi\omega}.$$

The proofs for $I_{5,s}$ and$I_{6,s}$ are similar. □

Proof of Theorem 1: By (3.5),

$$\begin{aligned} E_1(t_0,\omega) &= \frac{\mathcal{P}(0,t_0,\omega)}{\sqrt{2}} E_0(t_0,\omega) - \omega \sum_{n=1}^{6} I_{n,s} \sin n\omega.t_0 \\ &= \sqrt{2}\,(A_2 \sin 2\omega t_0 + A_3 \sin 3\omega t_0)\,(\cos 2\omega t_0 + \cos 3\omega t_0) \\ &\quad -\omega \sum_{n=1}^{6} I_{n,s} \sin n\omega.t_0 \\ &= \sum_{n=1}^{6} B_n(\omega) \sin n\omega t_0 \end{aligned}$$

where

$$\begin{aligned} B_1(\omega) &= \omega I_{1,s} + \frac{\sqrt{2}}{2}(A_3 - A_2); \quad B_2(\omega) = B_3(\omega) = 0; \quad B_4(\omega) = \omega I_{4,s} + \frac{\sqrt{2}}{2} A_2; \\ B_5(\omega) &= \omega I_{5,s} + \frac{\sqrt{2}}{2}(A_3 + A_2); \quad B_6(\omega) = \omega I_{6,s} + \frac{\sqrt{2}}{2} A_3. \end{aligned}$$

The acclaimed estimates on B_n then follow from Proposition 3.1 for $I_{1,s}$, Lemma 3.8 for $I_{4,s}, I_{5,s}$ and $I_{6,s}$, and Proposition 2.1 for A_2 and A_3. □

4. On High Order Melnikov Integrals

We rely on a recent theory of [18] on E_n to prove Theorem 2. The theory of [18], however, is on the second order equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon \cos \omega t \cdot y^2. \tag{4.1}$$

Note that the perturbation function of equation (4.1) is non-Hamiltonian and assumes only one forcing frequency. In contrast, the perturbation function of equation (1.7) is Hamiltonian and assumes two forcing frequencies. Though these differences are, by large, inconsequential for the analysis of [18] to be extended to cover equation (1.7), we can not quote the main result of [18] to prove Theorem 2. Here, an overhaul is called for to extend the scope of the theory of [18] to cover more equations including (1.7).

The theory of [18] has two main components. The first component is on the derivation of explicit integral formulas for all E_n for equation (4.1). The conclusion reached at the end of this derivation is that all E_n are sums of certain well structured multiple integrals, which we name as *high order Melnikov integrals*. The second component is on how to manipulate high order Melnikov integrals to acquire an upper bound estimates on E_n.

In the rest of this paper, we work to extend the scope of application of the theory of [18] by treating these two components as independent modules, each affords a setting that is flexible enough to be applied to situations arising from a variety of second order equations. In between the two modules, we elect to present the analysis on high order Melnikov integrals first in this section, leaving the setting on equations and derivations of integral formula for E_n to the next.

4.1. **Definition of High Order Melnikov Integrals.** To define a high order Melnikov integral, we start with a *structure tree*. A structure tree $\mathcal{T}$ is originated from a *root node*, from which some direct descendants are branched out. Each of these descendant nodes then serves as a root node for a sub-tree, from which the second generation descendants are branched out, and so on. We assume that only a maximum number of k_0 nodes are allowed to branch out from any given node. Eventually the branching out stops, and we obtain a structure tree.

Assume a given structure tree has p nodes in total. We index the tree nodes from the bottom level to the top level, and at a fixed level, from the right to the left, as $N_1, \cdots, N_p$. The root node of the entire tree is then indexed as N_p. All nodes N_j are of two types: the M-type and the W-type. To each node N_j, we assign an integral variable which we denote as t_j. We also assign to N_j a kernel function $f_j(t_j)$ and an interval of integration I_j. Assume N_j is branched out directly from $N_{j'}$. We let $I_j = (t_{j'}, +\infty)$ if N_j is an M-node but $I_j = (0, t_{j'})$ if it is a W-node. This way all I_j are well-defined except I_p for the root node N_p, for which we introduce a new variable t and let $I_p = (t, +\infty)$ if N_p is a M-node and $I_p = (0, t)$ if N_p is a W-node.

Notation: For a given structure tree $\mathcal{T}$, $C(j)$ is the set of all k so that N_k is a direct descendant of N_j, $T(j)$ is the set of all k so that N_k is inside of the subtree rooted at N_j (including j), and $P(j)$ is the set of all k such that $j \in T(k)$ (excluding j). The set of N_k, $k \in P(j)$ is the *ancestry line* of N_j.

For a given tree-node $N_j \in \mathcal{T}$, we denote the sub-tree rooted at N_j as $\mathcal{T}_j$, and assume N_j is branched out from $N_{j'}$. We define a formal multiple integral for $\mathcal{T}_j$, which we denote as $\mathcal{T}_j(t_{j'})$, by inductively letting

$$\mathcal{T}_j(t_{j'}) = \int_{I_j} f_j(t_j) \prod_{k \in C(j)} \mathcal{T}_k(t_j) dt_j. \tag{4.2}$$

In particular, the integral for $\mathcal{T}$ is a multiple integral of multiplicity p, which can also be written as,

$$\mathcal{T}(t) = \int_{I_p} f_p(t_p) \left(\cdots \left(\int_{I_j} f_j(t_j) \left(\cdots \left(\int_{I_1} f_1(t_1) dt_1 \right) \cdots \right) dt_j \right) \cdots \right) dt_p. \tag{4.3}$$

We now present assumptions on the kernel functions $f_j(t)$. In what follows, $z = t + is$ is a complex variable, and $\rho > 0$, $K_0 > 1$ and κ_0 are constants independent of ω, all three are fixed throughout.

Definition 4.1. *A function $d(z)$ is admissible if there exist a constant r_d independent of ω so that*

$$K_0^{-1} e^{-r_d|t|} \leq |d(t + is)| \leq K_0 e^{-r_d|t|} \tag{4.4}$$

for all $|t| > 1$ and $|s| < 2\rho$. For an admissible function $d(z)$, r_d in (4.4) is the declining rate for $d(z)$.

Let $\mathcal{K}_d$ be a finite set of admissible functions. We assume for $d(z) \in \mathcal{K}_d$,

(A1) $d(z)$ is analytic on $|Im(z)| \leq \rho$ minus the negative half of the real axis and $z = \pm i\rho$. In addition, $d(z)$ is bounded around $z = 0$, and $z = \pm i\rho$ are poles of order $\leq \kappa_0$.

(A2) $d(-z) = -d(z)$.

We also let $\mathcal{K}_g$ be a finite set of characteristic cosine functions in the form of $\cos n\omega z$. Let

$$\mathcal{K} = \{f(t, t_0) = g^{n_0}(t + t_0)d(t) : \quad g(z) \in \mathcal{K}_g;\ d(z) = \mathcal{K}_d;\ n_0 = 0 \text{ or } 1\} \tag{4.5}$$

be the set of kernel functions. We also call $g^{n_0}(t + t_0)$ a *trigonometric part*, and $d(t)$ the *dynamic part* of the kernel function $f(t, t_0)$.

Definition 4.2. *Let $\mathcal{T}$ be a structure tree. We assume there are n_j-many M-nodes branching out of N_j and all $f_j(t_j, t_0) = g^{n_0}(t_j + t_0)d_j(t_j)$ are from $\mathcal{K}$ in (4.5). We assume, in addition,*

(i) *if N_j is an M-node, then $r_j + 3n_j \geq 3$;*
(ii) *if N_j is a W-node, then $r_j + 3n_j \geq 1$*

where r_j is the declining rate for $d_j(t_j)$. **The high order Melnikov integral** *$\mathcal{T}(t)$ for $\mathcal{T}$ is then defined by using (4.3).*

Remark: Our assumption on $f \in \mathcal{K}$ is conceptually rather straight forward: f is a product of two functions, one we name as the trigonometric kernel and the other we name as the dynamic kernel. The trigonometric kernel is either missing ($n_0 = 0$) or it is a simple characteristic cosine function ($n_0 = 1$). The dynamic kernels are odd functions, for all of which $z = 0$ is a bounded branch point and $z = \pm i\rho$ are poles. Properties of dynamic kernels, including Definition 4.2(i)(ii) on the declining rates, as we will see, are imposed on us by the perturbed equation in a small neighborhood of an unperturbed homoclinic solution $(a(t), b(t))$. The precise setting on these equations and the ways the dynamic kernels are derived, however, are not yet relevant here. They are left entirely to the next section. Note that it follows directly from Definition 4.2(i)(ii) that there exists a constant K independent of ω so that $|\mathcal{T}_j(t)| < K^p e^{-3t}$ if N_j is an M-node, and $|\mathcal{T}_j(t)| < K^p$ if N_j is a W-node.

Notation: In the rest of this section we let the structure tree $\mathcal{T}$ be fixed throughout. We let m_0 be such that $\cos m_0\omega t$ is the cosine function of the largest frequency in $\mathcal{K}_g$. We also let κ_0 be the highest order of all $d(z) \in \mathcal{K}_d$ at $\pm i\rho$. The letter K is reserved for generic constants, the precise values of which are allowed to vary from line to line.

We define the dual of $\mathcal{T}(t)$ by changing $+\infty$ in intervals of integration for all M-nodes in $\mathcal{T}$ to $-\infty$. To distinguish the two, we use $\mathcal{T}^s(t)$ for $\mathcal{T}(t)$, and denote its dual as $\mathcal{T}^u(t)$. The main proposition of this section is as follows.

Proposition 4.1. *Let $\mathcal{T}^s(t)$ be a high order Melnikov integral defined by using a structure tree $\mathcal{T}$, and $\mathcal{T}^u(t)$ its dual, we have*

$$\mathcal{T}^s(0) - \mathcal{T}^u(0) = \sum_{j=1}^{m_0 p} A_j \sin j\omega t_0 \tag{4.6}$$

where A_j for all j in range is such that

$$|A_j| < \omega^{5\kappa_0 p} e^{-\rho\omega}. \tag{4.7}$$

assuming ω is sufficiently large.

We will define, in precise terms, the set of equations to be covered by this overhaul version of [18] in Section 5. This set covers both equations (1.7) and (4.1). We will also prove, in the next section, that for this set of equations, E_n are sums of a number of high order Melnikov integral of order $\leq K_1(n+2)$ and the number of integrals for E_n is $< K_2^{(n+2)}$. Theorem 2 will then follow from Proposition 4.1.

4.2. **Extended, Pure and Complex Integrals.** From this point on, structure tree is the adopted convention for us to represent multiple integrals. The traditional convention of using integral sign with implicitly or explicitly defined domain and function is still in use in occasions, but the defining structure tree is the main tool for us to represent and to manipulate multiple integrals. We start by applying

$$\int_0^t = \int_0^{+\infty} - \int_t^{+\infty} \tag{4.8}$$

to all W-node to make upper integral bounds uniformly $+\infty$.

Definition 4.3. *Let $\mathcal{T}$ be the structure tree for a high order Melnikov integral $\mathcal{T}(0)$. We obtain an extended Melnikov integral $\tilde{\mathcal{T}}(0)$ from $\mathcal{T}(0)$ by adopting the following changes to $\mathcal{T}$. We change the label of all W-node in $\mathcal{T}$ to either W_1 or W_2; and we change the interval of integration to $[0,+\infty)$ for all newly assigned W_1-node, but to $[t_{j'},+\infty)$ for all newly assigned W_2-node.*

Lemma 4.1. *Let $\mathcal{E}(\mathcal{T}(0))$ be the collection of all extended Melnikov integrals $\tilde{\mathcal{T}}(0)$ induced by using $\mathcal{T}(0)$. Then,*

(a) *the number of extended integrals in $\mathcal{E}(\mathcal{T}(0))$ equals to $2^{\hat{p}}$, where $\hat{p}$ is the total number of W-node in $\mathcal{T}(0)$; and*
(b) *we have*

$$\mathcal{T}(0) = \sum_{\tilde{\mathcal{T}}(0)\in\mathcal{E}(\mathcal{T}(0))} (-1)^{w(\tilde{\mathcal{T}})}\tilde{\mathcal{T}}(0)$$

where $w(\tilde{\mathcal{T}})$ is the total number of W_2-node in $\tilde{\mathcal{T}}(0)$.

Proof. Item (a) holds because changing every W to either W_1 or W_2 is a process of binary splitting. Item (b) is straight forward from (4.8). □

Definition 4.4. *Let $\tilde{\mathcal{T}}(0)$ be an extended integral induced by using $\mathcal{T}(0)$. We define* **pure blocks** *of $\tilde{\mathcal{T}}(0)$ as follows:*

(a) *the remainder tree obtained by dropping all W_1-subtree from $\tilde{\mathcal{T}}(0)$ is a pure block;*
(b) *a W_1-subtree is a pure block if it contains no smaller subtree of W_1-type; and*
(c) *if a W_1-subtree contains other W_1-subtree inside, then the remainder tree obtained by deleting all inside W_1-subtree is a pure block.*

We also call a pure block a **pure integral**.

Lemma 4.2. *An extended integral $\tilde{\mathcal{T}}(0)$ with m W_1-node has $m+1$ pure blocks, which we denote as $\mathcal{B}_1(0),\cdots,\mathcal{B}_{m+1}(0)$; and we have*

$$\tilde{\mathcal{T}}(0) = \mathcal{B}_1(0)\mathcal{B}_2(0)\cdots\mathcal{B}_{m+1}(0).$$

Proof. This is because the interval of integration for a W_1-node is $[0,+\infty)$, causing the sub-tree rooted at it to factor out. □

We now focus our attention on a given pure block $\mathcal{B}_j(0)$. To avoid unnecessarily complicated indexing and sub-indexing, we re-write $\mathcal{B}_j(0)$ as $\mathcal{T}(0)$. In the rest of this subsection, we assume $\mathcal{T}$ is a pure integral, the order of which we denote as p.

Recall that, for a given node N_j in $\mathcal{T}$, the kernel function is in the form of

$$f_j(t_j, t_0) = \cos^{n_0} n_j\omega(t_j + t_0)d_j(t_j)$$

where $n_0 = n_0(j)$ is either 0 or 1.

Definition 4.5. *For a pure integral $\mathcal{T}$ of order p, let*

$$\mathbf{q} = (q_1, \cdots, q_p), \quad q_j = n_j n_0(j) \text{ or } q_j = -n_j n_0(j).$$

We define a **complex pure integral** *for a given $\mathbf{q}$, which we denote as $\mathcal{T}_{\mathbf{q}}$, by changing $\cos^{n_0(j)} n_j\omega(t_j + t_0)$ in $f_j(t_j, t_0)$ to $e^{i\omega q_j t_j}$ for all N_j in $\mathcal{T}$.*

Let

$$\mathcal{S} = \{\mathbf{q} = (q_1, \cdots, q_p); \quad q_j = n_j n_0(j) \text{ or } -n_j n_0(j)\}.$$

The set $\mathcal{S}$ is the collection of all possible $\mathbf{q}$ vectors.

Lemma 4.3. *We have*

$$\mathcal{T}(0) = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q}\in\mathcal{S}} e^{i\omega Q_{\mathbf{q}} t_0} \mathcal{T}_{\mathbf{q}}(0)$$

where $\hat{p} = n_0(1) + \cdots + n_0(p)$, $Q_{\mathbf{q}} = q_1 + \cdots + q_p$.

Proof. We substitute by using $\cos n_j\omega(t_j + t_0) = \frac{1}{2}\left(e^{i\omega n_j(t_j+t_0)} + e^{-i\omega n_j(t_j+t_0)}\right)$. □

Let $\mathcal{T}^s_{\mathbf{q}}(0)$ be a complex pure integral for $\mathcal{T}^s(0)$ and $\mathcal{T}^u_{\mathbf{q}}(0)$ be its dual. We write $\mathcal{T}^s_{\mathbf{q}}(0)$ as a multiple integral defined on a region $\mathcal{R} \subset (0, +\infty)^p$ as

$$\mathcal{T}^s_{\mathbf{q}}(0) = \int_{\mathcal{R}} e^{i\omega\mathbf{q}\cdot\mathbf{t}} F(\mathbf{t})d\mathbf{t} \tag{4.9}$$

where $\mathbf{t} = (t_1, \cdots, t_p)$ and $F(\mathbf{t}) = \prod_{j=1}^p d_j(t_j)$. By definition,

$$\mathcal{T}^u_{\mathbf{q}}(0) = \int_{-\mathcal{R}} e^{i\omega\mathbf{q}\cdot\mathbf{t}} F(\mathbf{t})d\mathbf{t}. \tag{4.10}$$

We let

$$\mathbb{D}_{\mathbf{q}}(0) := e^{i\omega Q_{\mathbf{q}} t_0}\left(\mathcal{T}^s_{\mathbf{q}}(0) - \mathcal{T}^u_{\mathbf{q}}(0)\right).$$

Proposition 4.2. *By definition,*

$$\mathbb{D}_{\mathbf{q}}(0) + \mathbb{D}_{-\mathbf{q}}(0) = -4\sin(\omega Q_{\mathbf{q}} t_0)\int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}. \tag{4.11}$$

Recall that for $\mathbf{q} = (q_1, \cdots, q_p)$, $Q_{\mathbf{q}} = q_1 + \cdots + q_p$.

Proof. We have, by $d(t) = -d(-t)$, that

$$\begin{aligned} \int_{\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} &= \int_{-\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}, \\ \int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} &= -\int_{-\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}. \end{aligned} \tag{4.12}$$

It then follows that

$$\begin{aligned}
\mathbb{D}_{\mathbf{q}}(0)+\mathbb{D}_{-\mathbf{q}}(0) &= e^{i\omega Q_{\mathbf{q}}t_0}\left(\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^u_{\mathbf{q}}(0)\right)+e^{-i\omega Q_{\mathbf{q}}t_0}\left(\mathcal{T}^s_{-\mathbf{q}}(0)-\mathcal{T}^u_{-\mathbf{q}}(0)\right)\\
&= \left(\cos\left(\omega Q_{\mathbf{q}}t_0\right)+i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\right)\left(\int_{\mathcal{R}}\cos\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}+i\int_{\mathcal{R}}\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}\right)\\
&\quad-\left(\cos\left(\omega Q_{\mathbf{q}}t_0\right)+i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\right)\left(\int_{-\mathcal{R}}\cos\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}+i\int_{-\mathcal{R}}\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}\right)\\
&\quad+\left(\cos\left(\omega Q_{\mathbf{q}}t_0\right)-i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\right)\left(\int_{\mathcal{R}}\cos\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}-i\int_{\mathcal{R}}\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}\right)\\
&\quad-\left(\cos\left(\omega Q_{\mathbf{q}}t_0\right)-i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\right)\left(\int_{-\mathcal{R}}\cos\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}-i\int_{-\mathcal{R}}\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}\right)\\
&= -4\sin\left(\omega Q_{\mathbf{q}}t_0\right)\int_{\mathcal{R}}\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)F(\mathbf{t})d\mathbf{t}.
\end{aligned}$$

□

We now use

$$\sin\left(\omega\mathbf{q}\cdot\mathbf{t}\right)=\frac{1}{2i}\left(e^{i\omega\mathbf{q}\cdot\mathbf{t}}-e^{-i\omega\mathbf{q}\cdot\mathbf{t}}\right)$$

to rewrite (4.11) as

$$\mathbb{D}_{\mathbf{q}}(0)+\mathbb{D}_{-\mathbf{q}}(0)=2i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\left(\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)\right), \tag{4.13}$$

and we have from Lemma 4.3 and (4.13),

$$\mathcal{T}^s(0)-\mathcal{T}^u(0)=\frac{1}{2^{\hat{p}}}\sum_{\mathbf{q}\in\mathcal{S}^+}2i\sin\left(\omega Q_{\mathbf{q}}t_0\right)\left(\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)\right) \tag{4.14}$$

where $\mathcal{S}^+$ is the set of all $\mathbf{q}\in\mathcal{S}$ so that $Q_{\mathbf{q}}>0$. This proves that $\mathcal{T}^s(0)-\mathcal{T}^u(0)$ is a sine polynomial.

4.3. **An Illustrative Example of Complex Pure Integral.** Before presenting a general theory on complex pure integrals, we would like to use a simple example to illustrate our strategy on how to move to the complex plane to extract an exponentially small factor $e^{-\omega\rho}$ out of $\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)$. Let $\mathbf{q}=(2,3)$, and

$$\begin{aligned}
\mathcal{T}^s_{\mathbf{q}}(0) &= \int_0^{+\infty}e^{3i\omega t_2}d_2(t_2)\left(\int_{t_2}^{+\infty}e^{2i\omega t_1}d_1(t_1)dt_1\right)dt_2,\\
\mathcal{T}^s_{-\mathbf{q}}(0) &= \int_0^{+\infty}e^{-3i\omega t_2}d_2(t_2)\left(\int_{t_2}^{+\infty}e^{-2i\omega t_1}d_1(t_1)dt_1\right)dt_2.
\end{aligned}$$

Our intention is to motivate a decomposition scheme to extract an exponentially small factor out of $\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)$. As the first step, we introduce

$$\tau_1=t_1-t_2,\quad \tau_2=t_2$$

to write $\mathcal{T}^s_{\mathbf{q}}(0)$ as

$$\mathcal{T}^s_{\mathbf{q}}(0)=\int_0^{+\infty}e^{5i\omega\tau_2}d_2(\tau_2)\left(\int_0^{+\infty}e^{2i\omega\tau_1}d_1(\tau_1+\tau_2)d\tau_1\right)d\tau_2.$$

We switch the order of integration to rewrite it as

$$\mathcal{T}^s_{\mathbf{q}}(0)=\int_0^{+\infty}e^{2i\omega\tau_1}\left(\int_0^{+\infty}e^{5i\omega\tau_2}d_2(\tau_2)d_1(\tau_1+\tau_2)d\tau_2\right)d\tau_1.$$

Regarding τ_1 as a parameter and treating τ_2 as a complex variable $z = t_2 + is_2$, we focus on the inner integral,

$$I_{\ell_0^+} := \int_0^{+\infty} e^{5i\omega z} d_2(\tau_2) d_1(\tau_1 + z) dz.$$

Let

$$\ell_\omega^+ = \left\{ z = t_2 + is_2, \quad t_2 \in (0, +\infty), \quad s_2 = \rho - \omega^{-1} \right\};$$
$$\ell_v^+ = \left\{ z = t_2 + is_2, \quad t_2 = 0, \quad s_2 \in (0, \rho - \omega^{-1}] \right\}.$$

We have, by Cauchy integral theorem,

$$\begin{aligned} I_{\ell_0^+} &= \int_{\ell_v^+} e^{5i\omega z} d_2(z) d_1(\tau_1 + z) dz + \int_{\ell_\omega^+} e^{5i\omega z} d_2(z) d_1(\tau_1 + z) dz \\ &= i \int_0^{\rho - \omega^{-1}} e^{-5\omega s_2} d_2(is_2) d_1(is_2 + \tau_1) ds_2 \\ &\quad + e^{-5\omega\rho + 5} \int_0^{+\infty} e^{5i\omega t_2} d_2(t_2 + i(\rho - \omega^{-1})) d_1(\tau_1 + t_2 + i(\rho - \omega^{-1})) dt_2 \end{aligned}$$

This is for us to have

$$\mathcal{T}_{\mathbf{q}}^s(0) = I_v + I_\omega$$

where

$$\begin{aligned} I_v =& i \int_0^{+\infty} e^{2i\omega\tau_1} \int_0^{\rho - \omega^{-1}} e^{-5\omega s_2} d_2(is_2) d_1(is_2 + \tau_1) ds_2 d\tau_1 \\ I_\omega =& e^{-5\omega\rho + 5} \int_0^{+\infty} e^{2i\omega\tau_1} \int_0^{+\infty} e^{5i\omega t_2} d_2(t_2 + i(\rho - \omega^{-1})) d_1(\tau_1 + t_2 + i(\rho - \omega^{-1})) dt_2 d\tau_1. \end{aligned}$$

Note that we already have an exponentially small factor in front of I_ω. This is to imply

$$|I_\omega^s(0)| \leq \omega^{3\kappa_0} e^{-\omega\rho}.$$

Here by changing the integral path from the real axis to one that is ω^{-1}-distance away from the singular set of the integral function in complex plane, we magnify the dynamic part of the kernel function by a factor $(K\omega)^{2\kappa_0}$, but gain from the trigonometric part of the kernel function an exponentially small factor $e^{-\omega\rho}$.

Switching the order of integration for I_v, we write

$$I_v = i \int_0^{\rho - \omega^{-1}} e^{-5\omega s_2} d_2(is_2) \left(\int_0^{+\infty} e^{2i\omega\tau_1} d_1(is_2 + \tau_1) d\tau_1 \right) ds_2.$$

Regarding τ_1 as a complex variable $z_1 = t_1 + is_1$, we move to complex plane to decompose the inner integral once more. Note that, because the argument for $d_1(z)$ is $is_2 + is_1 + t_1$, we are only allowed to move in the imaginary direction so that

$$s_2 + s_1 \leq \rho - \omega^{-1}.$$

We have

$$\begin{aligned} I_v = &- \int_0^{\rho - \omega^{-1}} e^{-5\omega s_2} d_2(is_2) \left(\int_0^{\rho - s_2 - \omega^{-1}} e^{-2\omega s_1} d_1(is_2 + is_1) ds_1 \right) ds_2 \\ &+ ie^{-2\omega\rho + 2} \int_0^{\rho - \omega^{-1}} e^{-3\omega s_2} d_2(is_2) \left(\int_0^{+\infty} e^{2i\omega t_1} d_1(t_1 + i(\rho - \omega^{-1})) dt_1 \right) ds_2. \end{aligned}$$

The second integral is again with an exponentially small factor, but the first integral is not.

At this point, we recall that the quantity we need to estimate is not $\mathcal{T}^s_{\mathbf{q}}(0)$ but $\mathcal{T}^s_{\mathbf{q}}(0) - \mathcal{T}^s_{-\mathbf{q}}(0)$. We now decompose $\mathcal{T}^s_{-\mathbf{q}}(0)$, moving downward, instead of upward, in the imaginary direction. We readily conclude that $\mathcal{T}^s_{-\mathbf{q}}(0)$ is decomposed again into three integrals, two of them are with exponentially small factors but the third is without. However, using the symmetry induced by $d(z) = -d(-z)$, we conclude that the third integral is identical to its correspondence from $\mathcal{T}^s_{\mathbf{q}}(0)$. They cancel each other out in $\mathcal{T}^s_{\mathbf{q}}(0) - \mathcal{T}^s_{-\mathbf{q}}(0)$.

4.4. **Decomposition of Pure Integrals.** We want to replicate the decomposition process for a particular double integral presented in the last subsection to all complex pure integrals. This would necessarily lead us to an inductive process in which changes of variables are employed to transform various multiple integrals. A formal presentation of this induction for complex pure integrals of equation (4.1) is presented in details in Section 5 of [18]. To adopt this presentation for complex pure integrals as defined in this section, we need to make the following adjustment: (i) the pole for $d(z) \in \mathcal{K}_d$ for equation (4.1) are at $\pm i\pi/2$, so we need to replace $\pi/2$ by using ρ; (ii) the explicit formula of the dynamic kernels for equation (4.1) are replaced by the more generically defined $d(z) \in \mathcal{K}_d$. The proofs are virtually identical otherwise.

To save repetitive presentation of tedious technical details, here we present the intended induction in expository style, referring readers who is interested in verifying all technical details to Section 5 of [18]. We will use structure trees to define and redefine integrals involved in this process.

Definition 4.6. *Let $\mathcal{T}_{\mathbf{q}}(0)$ be a complex pure integral for a given $\mathcal{T}(0)$ where $\mathbf{q} = (q_1, \cdots, q_p)$ and $\mathcal{T}_j$ be the subtree of $\mathcal{T}_{\mathbf{q}}(0)$ rooted at N_j.*

(a) *We call p_j, the number of tree nodes in the subtree rooted at N_j, the* **order** *of the integral $\mathcal{T}_j$. In particular, the order of $\mathcal{T}_{\mathbf{q}}$ is p.*
(b) *We call*
$$Q_j = \sum_{j' \in T(j)} q_{j'}$$
the **total index** *of the subtree $\mathcal{T}_j$. In particular, the total index of $\mathcal{T}_{\mathbf{q}}$ is Q_p, which we sometimes also denote as $Q_{\mathbf{q}}$.*
(c) *We use σ_j to denote the* **sign** *of Q_j. This is to say that if $Q_j > 0$ then $\sigma_j = 1$, if $Q_j < 0$ then $\sigma_j = -1$, and if $Q_j = 0$ then $\sigma_j = 0$.*
(d) *We call $\mathcal{T}_j$ a* **zero subtree** *if $Q_j = 0$.*

4.4.1. *Unification of all Integral Bounds.* We employ a change of coordinates to transform the integral domain of $\mathcal{T}_{\mathbf{q}}(0)$ to $(0, +\infty)^p$. Let

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'} \tag{4.15}$$

for all j. We change the integral variables from $(t_1, \cdots, t_p)$ to $(\tau_1, \cdots, \tau_p)$. Recall that $P(j)$ is the index set for the ancestry line of N_j (excluding j). Corresponding to this change of variables, we modify $\mathcal{T}_{\mathbf{q}}(0)$ as follows to obtain a new structure tree $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ in τ variables. For all N_j, $j \le p$, (i) the integral variable for N_j is changed from t_j to τ_j; (ii) the interval of integration is changed to $(0, +\infty)$ for τ_j; and (iii) the new kernel function is

$$\hat{f}_j(\tau) = \hat{g}_j^{n_0(j)}(\tau_j)\hat{d}_j(\tau)$$

where

$$\hat{g}_j(\tau_j) = e^{i\omega Q_j \tau_j}; \qquad \hat{d}_j(\tau) = d_j\left(\tau_j + \sum_{j' \in P(j)} \tau_{j'}\right).$$

The integral defined by $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ is a multiple integral defined on $(0, +\infty)^p$ in the space of $\tau = (\tau_1, \cdots, \tau_p)$. We have

$$\hat{\mathcal{T}}_{\mathbf{q}}(0) = \int_{(0,+\infty)^p} \prod_{j=1}^{p} \hat{f}_j(\tau) d\tau \tag{4.16}$$

where $d\tau = d\tau_1 \cdots d\tau_p$.

Proposition 4.3. *We have $\mathcal{T}_p(0) = \hat{\mathcal{T}}_p(0)$.*

Proof. The determinant of the Jacobian of the coordinate change from t to τ defined by using

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'}$$

is 1. The integral interval for τ_j is $(0, +\infty)$ because, by definition,

$$\tau_j = t_j - t_{j'}$$

where j' is such that $j \in C(j')$. For the trigonometric part of the kernel function, we observe that

$$\sum_{j=1}^{p} q_j t_j = \sum_{j=1}^{p} q_j \left(\tau_j + \sum_{j' \in P(j)} \tau_{j'}\right) = \sum_{j=1}^{p} Q_j \tau_j \tag{4.17}$$

where $Q_j = \sum_{j' \in T(j)} q_{j'}$. In (4.17), the first equality is by definition and the second equality follows from the fact that, as far as variable τ_j is concerned, the contribution from $t_{j'}$ to $\sum_{j=1}^{p} q_j t_j$ is $q_{j'} \tau_j$ if $j' \in T(j)$, but none if $j' \notin T(j)$. We then have

$$\prod_{j=1}^{p} e^{i\omega q_j t_j} = \prod_{j=1}^{p} e^{i\omega q_j (\tau_j + \sum_{j' \in P(j)} \tau_{j'})} = \prod_{j=1}^{p} e^{i\omega Q_j \tau_j}$$

for the trigonometric part of the kernel function. □

4.4.2. *Inductive Decomposition.* We are going to introduce complex variables $z_j = \tau_j + is_j$, starting with $z_p = \tau_p + is_p$, one j at a time, to transform a pure integral that was originally defined on the positive half of the real axis $\tau_j \in [0, +\infty)$, to two integrals; one is defined on an interval of the purely imaginary z_j-axis, and the other is defined on a half line in the complex plane that is parallel to the real axis. By a careful design of integral paths in the complex plane, we move the argument of the integral functions towards the complex singularity of the function $d_j(z)$ (but keep it at least an ω^{-1}-distance away), in the hope of trading a worsening factor of polynomial power in ω from the dynamical part of the kernel function for an exponentially small factor $e^{-\omega\rho}$ from the trigonometric part of the kernel function.

In between the two integrals, the one that is defined on the purely imaginary axis is passed to the next step, and the other is put aside, which we will collect at the end. For the integral that is passed forward, the variables of integration are turned from τ_j to s_j one at a time. At the end, the integral passed forward is entirely in s-variable.

We start with a pure integral $\mathcal{T}_{\mathbf{q}}(0)$ in t variables to obtain the corresponding integral $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ in τ variables. What is outlined in the above for $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ is an inductive process of p steps. We use $k = 0, \cdots, p-1$ as the inductive index.

The Initial Step of Induction: We define two new integrals, $\mathbb{T}(0, v)$ and $\mathbb{T}(0, \omega)$, by using $\tilde{\mathcal{T}}$. The integral $\mathbb{T}(0, v)$ is defined by modifying $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ as follows: (i) in all kernel functions, change τ_p to is_p; and (ii) for the tree node N_p, change the variable of integration from τ_p to s_p; change the interval of integration from $[0, +\infty)$ to $[0, \sigma_p(\rho - \omega^{-1})]$.

The integral $\mathbb{T}(0, \omega)$ is defined by replacing, in the dynamic part of all kernel functions of $\hat{\mathcal{T}}_{\mathbf{q}}(0)$, the variable τ_p by using $\tau_p + i\sigma_p(\rho - \omega^{-1})$.

Structure trees for $\mathbb{T}(k, v)$ and $\mathbb{T}(k, \omega)$: At k-th step, starting from $k = 0$, we denote the integral that is going to be passed forward as $\mathbb{T}(k, v)$, and the integral that is going to be put aside as $\mathbb{T}(k, \omega)$. We again modify the memories of all tree nodes in $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ to define $\mathbb{T}(k, v)$ and $\mathbb{T}(k, \omega)$.

The Structure Tree of $\mathbb{T}(k, v)$: We divide all tree nodes into two groups. Group (a) is the top part of the tree. That is, all N_j such that $p - k \leq j \leq p$. Group (b) is the rest. At step k, integral variables for tree nodes in Group (a) have been converted from τ to s. For N_j, $p - k \leq j \leq p$, we set the variable of integration as s_j, the interval of integration as

$$I_j = \left[0, \sigma_j(\rho - \omega^{-1}) - \sum_{j' \in P(j)} s_{j'}\right],$$

the trigonometric part of the kernel function as $e^{-\omega Q_j s_j}$, and the dynamic part of the kernel function is $d_j(iS_j)$ where

$$S_j = s_j + \sum_{j' \in P(j)} s_{j'} \tag{4.18}$$

is obtained by adding to s_j all variables of integration along the ancestry line of N_j. The upper bound imposed on I_j ensures that S_j are in between $[-\rho + \omega^{-1}, \rho - \omega^{-1}]$ so the complex singularities of $d_j(z)$ do not interfere with our intended use of the Cauchy integral theorem.

At this step of the induction, the process of converting τ_j to s_j has not yet reached tree nodes in Group (b). So for N_j, $1 \leq j < p - k$, we set the variable of integration as τ_j, the interval of integration as $(0, +\infty)$, and the trigonometric part of the kernel function remains $e^{i\omega Q_j \tau_j}$. For the dynamic part of the kernel function, we recall that it was $d_j(t_j)$ where

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'}.$$

We change it to $\hat{d}_j(\tau, s) = d_j(w_j)$ where w_j is obtained by changing $\tau_{j'}$ in t_j to $is_{j'}$ if $N_{j'}$ is in Group (a). This is to say we have

$$w_j = \tau_j + \sum_{j' \in P(j),\ j' < p-k} \tau_{j'} + i \sum_{j' \in P(j),\ j' \geq p-k} s_{j'}. \tag{4.19}$$

This finishes the definition of $\mathbb{T}(k, v)$.

The Structure Tree of $\mathbb{T}(k, \omega)$: To define $\mathbb{T}(k, \omega)$, we divide all tree nodes into three groups and denote them as Group (a), (b) and (c) respectively. Group (a) is again at the top, but it differs from Group (a) in $\mathbb{T}(k, v)$ by one node: N_{p-k} is excluded. The rest of the tree nodes are divided into Group (b), containing the remaining tree nodes that reside

outside of the subtree rooted at N_{p-k}, and Group (c), containing all that reside inside the subtree rooted at N_{p-k}.

A tree node N_j in Group (a) for $\mathbb{T}(k,\omega)$ is identical to the corresponding N_j in $\mathbb{T}(k,v)$ as far as it is not in the ancestry line of N_{p-k}. For N_j that is on the ancestry line of N_{p-k}, all again remain the same as in $\mathbb{T}(k,v)$ except the trigonometric part of the kernel function, which we change from $e^{-\omega Q_j s_j}$ to

$$e^{-\omega(Q_j-Q_{p-k})s_j}$$

for $\mathbb{T}(k,\omega)$.

Tree nodes in Group (b) are identical to their correspondences in $\mathbb{T}(k,v)$. Finally, for N_j in Group (c), that is, a tree node in the subtree rooted at N_{p-k} (including N_{p-k}), all are the same as in Group (b) except the dynamic part of the kernel function is defined as $d_j(w_j)$ by using

$$w_j = \tau_j + \sum_{j'\in P(j)\cap T(p-k)} \tau_{j'} + i\sigma_{p-k}(\rho-\omega^{-1}). \tag{4.20}$$

Comparing this to w_j in Group (b), all s-variables are dropped but we add to it a constant shift $i\sigma_{p-k}(\rho-\omega^{-1})$.

Pushing the Induction Forward: For the intended induction, first we initiate the inductive process to prove

$$\mathcal{T}_{\mathbf{q}}(0) = i\mathbb{T}(0,v) + e^{-\sigma_p Q_p \omega(\rho-\omega^{-1})}\mathbb{T}(0,\omega).$$

We then prove

$$\mathbb{T}(k,v) = i\mathbb{T}(k+1,v) + e^{-\sigma_{p-k-1}Q_{p-k-1}\omega(\rho-\omega^{-1})}\mathbb{T}(k+1,\omega). \tag{4.21}$$

to move the induction forward from step k of this induction to step $k+1$. The result of this induction, obtained by a recursively using (4.21), is as follows

Proposition 4.4. *Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral. Then*

$$\mathcal{T}_{\mathbf{q}}(0) = i^p\mathbb{T}(p-1,v) + \sum_{k=0}^{p-1} i^k e^{-\sigma_{p-k}Q_{p-k}\omega(\rho-\omega^{-1})}\mathbb{T}(k,\omega). \tag{4.22}$$

Proof. We refer to Section 5 of [18] for a formal presentation of this induction with all detailed technical justifications. With the adjustments of replacing $\pi/2$ by using ρ, and the dynamic kernels of equation (4.1) by using the more generically defined $d(z)\in\mathcal{K}_d$, that presentation would go through seamlessly for complex pure integrals as defined in this section. The details are actually made easier because we can directly use the assumptions made on $d(z)$, including Definition 4.1(i)(ii) instead of proving them for a given equation. □

4.5. **Cancellations Based on Symmetry.** Let $\mathcal{T}_{\mathbf{q}}(0)$ be a complex pure integral of order p. We assume $Q_{\mathbf{q}}\neq 0$ in this subsection because complex pure integrals of $Q_{\mathbf{q}}=0$ do not contribute to $\mathcal{T}^s(0)-\mathcal{T}^u(0)$ by (4.14). There are three kind of terms on the right hand side of (4.22). The first is $\mathbb{T}(p-1,v)$ that is completely in s-variables. This term is canceled by $\mathbb{T}(p-1,v)$ for $\mathcal{T}_{-\mathbf{q}}$ because $d(z)=-d(-z)$ for all $d(z)\in\mathcal{K}_d$. The second kind is induced by subtrees $\mathcal{T}_j$ so that $Q_j\neq 0$. For these terms, we have

Proposition 4.5. *There exists a constant $K>1$ so that we have, under the assumption that $Q_p\neq 0$ and $Q_{p-k}\neq 0$,*

$$|e^{-\sigma_{p-k}Q_{p-k}\omega(\pi/2-\omega^{-1})}\mathbb{T}_{\mathbf{q}}(k,\omega)| \le \omega^{3\kappa_0 p}e^{-\omega\rho}.$$

Proof. We refer the readers to Section 8 of [18] for a proof of this Proposition. With the adjustment of replacing $\pi/2$ by using ρ, and the explicit dynamic kernels of equation (4.1) by using the more generically defined $d(z) \in \mathcal{K}_d$, the proof presented in Section 8 of [18] goes through seamlessly for complex pure integrals as defined in this section. □

What is problematic for us here is the terms of third kind, that is, $\mathbb{T}(k,\omega)$ from $\mathcal{T}_k$ of $Q_k = 0$ (zero sub-tree). In this case, there exists no exponentially small factor in front of $\mathbb{T}(k,\omega)$ on the right-side of (4.22). These terms are *not* exponentially small in magnitude, and they call for cancellations. However, in this case, $\mathbb{T}_{\mathbf{q}}(k,\omega)$ and $\mathbb{T}_{-\mathbf{q}}(k,\omega)$ do *not* cancel each other out by symmetry in $\mathbb{T}_{\mathbf{q}}(k,\omega) - \mathbb{T}_{-\mathbf{q}}(k,\omega)$.

Fortunately, we need not to prove that $\mathbb{T}_{\mathbf{q}}(k,\omega) - \mathbb{T}_{-\mathbf{q}}(k,\omega)$ is exponentially small in size for a fixed $\mathbf{q}$. Recall that by (4.14),

$$\mathcal{T}^s(0) - \mathcal{T}^u(0) = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q}\in\mathcal{S}^+} 2i \sin \omega Q_{\mathbf{q}} t_0 \left(\mathcal{T}^s_{\mathbf{q}} - \mathcal{T}^s_{-\mathbf{q}}\right).$$

To make the terms induced by zero-subtrees – at least the non-exponentially small part of these terms – go away in $\mathcal{T}^s(0) - \mathcal{T}^u(0)$, we need to enlist the help of the terms

$$\sin \omega Q_{\mathbf{q}} t_0 \left(\mathcal{T}^s_{\mathbf{q}} - \mathcal{T}^s_{-\mathbf{q}}\right)$$

for other $\mathbf{q} \in \mathcal{S}^+$ with the same $Q_{\mathbf{q}}$.

This goes as follows. For a given $\mathbf{q} \in \mathcal{S}^+$, we define *basic* $\mathbf{q}$*-block* as follows. First, we count $\mathbf{q}_R$, the part of $\mathbf{q}$ that defines the remainder tree obtained by deleting all zero subtrees in $\mathcal{T}_{\mathbf{q}}(0)$, as one basic $\mathbf{q}$-block. We also count the part of $\mathbf{q}$ that defines a zero subtree $\mathcal{T}_k$ as a basic $\mathbf{q}$-block if $\mathcal{T}_k$ contains no other zero subtree. Finally, if a zero subtree $\mathcal{T}_k$ contains other zero subtree, we take the part of $\mathbf{q}$ that defines the remainder tree of $\mathcal{T}_k$ obtained by deleting all zero subtrees in $\mathcal{T}_k$ as a basic $\mathbf{q}$-block.

Lemma 4.4. *Under the assumption that $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees, we have in total, counting $\mathbf{q}_R$, $n+1$ basic $\mathbf{q}$-block which we denote as*

$$\mathbf{q}_R,\ \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n.$$

These are, by definition, mutually disjoint subsets of $\mathbf{q}$ and we have

$$\mathbf{q} = \mathbf{q}_R \cup \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_n.$$

Proof. Let q_j be the component of $\mathbf{q}$ for N_j. Starting from N_j, we go backward along the ancestry line of N_j. If there is no root node of zero subtree on this ancestry line, then q_j is a component of $\mathbf{q}_R$. Otherwise, q_j is in the basic $\mathbf{q}$-block defined by the first root node of a zero subtree encountered on the way. □

For a given $\mathbf{q} \in \mathcal{S}^+_{\mathbf{q}}$, let

$$\mathbb{S}_{\mathbf{q}} = \{\hat{\mathbf{q}} = \mathbf{q}_R \cup \pm\mathbf{q}_1 \cup \cdots \cup \pm\mathbf{q}_n\} \tag{4.23}$$

where $\pm$ is either $+$ or $-$ if the basic $\mathbf{q}$-block it marks is not a zero vector. We set $\pm = 0$ if the basic $\mathbf{q}$-block it marks is a zero vector.

Lemma 4.5. *We have (i) for all $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$, $Q_{\hat{\mathbf{q}}} = Q_{\mathbf{q}}$; and (ii) if $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$, then $\mathbb{S}_{\hat{\mathbf{q}}} = \mathbb{S}_{\mathbf{q}}$.*

Proof. Item (i) holds because the total index of all basic blocks are zero except that of $\mathbf{q}_R$, the total index of which is the total index of the entire $\mathbf{q}$. Item (ii) follows from (4.23). □

Let

$$D_{\mathbb{S}_{\mathbf{q}}} = \sum_{\hat{q}\in\mathbb{S}_{\mathbf{q}}} (T_{\hat{\mathbf{q}}}(0) - T_{-\hat{\mathbf{q}}}(0)).$$

We have

Proposition 4.6. *There exists a constant $K > 0$ so that*

$$\left|\mathbb{D}_{\mathbb{S}_{\mathbf{q}}}\right| \le \omega^{4\kappa_0 p} e^{-\omega\rho}.$$

Proof. We refer the reader to Section 7 of [18] for the proof of this proposition. The intended cancellation works because, by (4.20), the zero sub-tree rooted at N_k in $\mathbb{T}(k,\omega)$ is independent of variable s and can be factored out. We also note that this proof relies on Proposition 4.5 to control all $\mathbb{T}(k,\omega)$ from non-zero sub-tree $\mathcal{T}_k$. □

4.6. **Proof of Proposition 4.1.** In this subsection, we pull the various things we have done so far into one place to prove Proposition 4.1. Let $\mathcal{T}$ be the structure tree for a high order Melnikov integral. We start with Lemma 4.1(c), that is,

$$D := \mathcal{T}^s(0) - \mathcal{T}^u(0) = \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \left(\tilde{\mathcal{T}}^s - \tilde{\mathcal{T}}^u\right) \tag{4.24}$$

where $\tilde{\mathcal{T}}$ is an extended integral and $\mathcal{E}(\tilde{\mathcal{T}})$ is the set of all extended integrals from $\mathcal{T}$. By Lemma 4.2, we have

$$\tilde{\mathcal{T}} = \mathcal{B}_1 \cdots \mathcal{B}_{m+1}.$$

where $\mathcal{B}_1, \cdots, \mathcal{B}_{m+1}$ are pure blocks for $\tilde{\mathcal{T}}$. We continue to write

$$D = \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \left(\mathcal{B}_1^s \cdots \mathcal{B}_{m+1}^s - \mathcal{B}_1^u \cdots \mathcal{B}_{m+1}^u\right).$$

For $0 \le k \le m+1$, let

$$\mathbb{B}_k := [\mathcal{B}_1^s \cdots \mathcal{B}_k^s] \cdot \left[\mathcal{B}_{k+1}^u \cdot \mathcal{B}_{m+1}^u\right].$$

We have

$$\tilde{\mathcal{T}}^s - \tilde{\mathcal{T}}^u = \mathbb{B}_{m+1} - \mathbb{B}_0 = \sum_{k=0}^{m} (\mathbb{B}_{k+1} - \mathbb{B}_k) = \sum_{k=0}^{m} [\mathcal{B}_1^s \cdots \mathcal{B}_k^s] \left(\mathcal{B}_{k+1}^s - \mathcal{B}_{k+1}^u\right) \left[\mathcal{B}_{k+2}^u \mathcal{B}_{m+1}^u\right].$$

This is to imply

$$D = \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \sum_{k=0}^{m} [\mathcal{B}_1^s \cdots \mathcal{B}_k^s] \left(\mathcal{B}_{k+1}^s - \mathcal{B}_{k+1}^u\right) \left[\mathcal{B}_{k+2}^u \cdot \mathcal{B}_{m+1}^u\right]. \tag{4.25}$$

To prove the acclaimed estimates on A_j in Proposition 4.1, we focus on extracting an exponentially small factor $e^{-\omega\rho}$ out of $\mathcal{B}^s - \mathcal{B}^u$ for a pure block $\mathcal{B}$. We also call a pure block a *pure integral*.

Let $\mathcal{B}$ be a pure integral of order p for stable solutions. We re-index all tree nodes of $\mathcal{B}$ from bottom level to top level, and from right to left, as $N_1, \cdots N_p$. Recall that the trigonometric part of the kernel function for N_j is $g_j(t_j) = \cos^{n_0(j)} n_j\omega(t_j + t_0)$ where $n_0(j)$ is either 0 or 1. If $n_0(j) = 0$ for all j, then $\mathcal{B}^s - \mathcal{B}^u = 0$ by symmetry.

Assume there exists at least one $n_0(j) \neq 0$. Let

$$\mathcal{S} = \{\mathbf{q} = (q_p, \cdots, q_1), \quad q_j \in \{-n_0(j), n_0(j)\}\}.$$

We have

$$\mathcal{B}^s - \mathcal{B}^u = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q}\in\mathcal{S}^+} 2i \sin \omega Q_{\mathbf{q}} t_0 \left(\mathcal{B}^s_{\mathbf{q}} - \mathcal{B}^s_{-\mathbf{q}}\right). \tag{4.26}$$

This is (4.14). Observe that only $\mathcal{B}_{\mathbf{q}}$ satisfying

$$Q_{\mathbf{q}} \neq 0 \tag{4.27}$$

contribute to $\mathcal{B}^s - \mathcal{B}^u$.

For a pure integral $\mathcal{B}_{\mathbf{q}}$ satisfying (4.27), we denote

$$\mathbf{q} = \mathbf{q}_R \cup \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_m.$$

where $\mathbf{q}_j$ are basic block defined by zero-subtrees. For a given $\mathbf{q} \in \mathcal{S}^+$, let

$$\mathbb{S}_{\mathbf{q}} = \{\hat{\mathbf{q}} \in \mathcal{S}^+, \quad \hat{\mathbf{q}} = \mathbf{q}_R \cup \pm\mathbf{q}_1 \cup \cdots \cup \pm\mathbf{q}_m\}$$

where $\pm$ is either $+$ or $-$. By definition, we have $Q_{\hat{\mathbf{q}}} = Q_{\mathbf{q}}$ for all $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$. It then follows that $\mathcal{S}^+$ is a union of mutually disjoint $\mathbb{S}_{\mathbf{q}}$. Regarding every $\mathbb{S}_{\mathbf{q}}$ as one element, we obtain a quotient set, which we denote as $\mathcal{S}^+/\sim$. This allows us to re-write (4.26) as

$$\mathcal{B}^s - \mathcal{B}^u = \frac{1}{2^{\hat{p}}} \sum_{\mathcal{S}^+/\sim} 2i \sin \omega Q_{\mathbf{q}} t_0 \cdot D_{\mathbb{S}_{\mathbf{q}}} \tag{4.28}$$

where

$$D_{\mathbb{S}_{\mathbf{q}}} = \sum_{\hat{q}\in\mathbb{S}_{\mathbf{q}}} (\mathcal{B}^s_{\hat{\mathbf{q}}} - \mathcal{B}^s_{-\hat{\mathbf{q}}}). \tag{4.29}$$

We have, by Proposition 4.6,

$$\left|D_{\mathbb{S}_{\mathbf{q}}}\right| < \omega^{4\kappa_0 p} e^{-\omega\rho}. \tag{4.30}$$

Putting (4.25), (4.28), (4.29) together, we have

$$D = \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \sum_{k=0}^{m} [\mathcal{B}^s_1 \cdots \mathcal{B}^s_k] \cdot \left[\mathcal{B}^u_{k+2} \cdot \mathcal{B}^u_{m+1}\right] \left(\frac{1}{2^{\hat{p}}} \sum_{\mathcal{S}^+/\sim} 2i \sin \omega Q_{\mathbf{q}} t_0 \cdot D_{\mathbb{S}_{\mathbf{q}}}\right)$$

where the bracketed quantity is from $\mathcal{B}_{k+1}$. We now apply the conclusion of Proposition 4.6 to obtain

$$|A_j| \quad \leq \quad K^p \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} \sum_{k=0}^{m} \left(\sum_{\mathcal{S}^+/\sim} \left|D_{\mathbb{S}_{\mathbf{q}}}\right|\right)_{(k+1)} \leq \omega^{5\kappa_0 p} e^{-\omega\rho}.$$

where for the first inequality, we used $\left|\mathcal{B}^s_1 \cdots \mathcal{B}^s_k \cdot \mathcal{B}^u_{k+2} \cdot \mathcal{B}^u_{m+1}\right| < K^p$. For the second inequality, we used (4.30). □

5. Derivation of E_n

In this section, we derive integral formula for E_n for a set of periodically perturbed second order equations. The equations we intend to study, in its most general form, would be

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = f(x) + \varepsilon P\left(x, y, \omega t, \varepsilon\right) \tag{5.1}$$

where $P(x, y, t, \varepsilon)$ is a periodic function of period 2π in t. Under the assumption that the unperturbed equation ($\varepsilon = 0$) has a saddle fixed point, and this fixed point has a homoclinic

solution which we denote as $(x, y) = (a(t), b(t))$, we can derive in ease an explicit integral formula for $E_0(t_0)$.

What we intend to do, however, goes way beyond $E_0(t_0)$. Not only we need to derive integral formulas for E_n, but also we need to be able to evaluate $E_n(t_0)$ for all $n \geq 1$. This is a much harder task, and it is unlikely we would be able to develop a complete theory on E_n that can be applied to all equations in the form of (5.1). A reasonable retreat would be to assume that $f(x)$ and all Fourier coefficients of $P(x, y, t, \varepsilon)$ are polynomials. This would exclude periodically perturbed pendulums, but pendulums are already excluded in view of the theory of Section 4 – the complex singularities of the homolcinic solutions for pendulums are not poles.

Let us weigh our options further for this particular paper. For the purpose of proving Theorem 2 for (1.7), no general setting on equations is required. It would be sufficient for us to derive E_n by a simple replication of part I of [18]. This is to say that, if we are only interested in proving Theorem 2 for (1.7), we can save much of repetitive technical writings by first offering a brief summary, then referring the details to part I of [18] . We certainly would like to do better, that is, to extend the scope of the current exposition to cover more equations. However, we shall not sacrifice too much of a well-motivated, clean presentation to pursue generality for the sake of generality.

Equations of Study: To strike a balance in between the two sides of the considerations, we elected to study, in this section, the second order equations in the form of

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon P(x, y, \omega t) \tag{5.2}$$

where $P(x, y, t)$ is a Fourier cosine polynomial in the form of

$$P(x, y, t) = \sum_{j=1}^{k_1} P_j(x, y) \cos jt. \tag{5.3}$$

We assume all $P_j(x, y)$ are polynomials in x, y satisfying $P_j(0, 0) = \partial_x P_j(0, 0) = \partial_y P_j(0, 0) = 0$ and

$$P_j(x, y) = P_j(x, -y). \tag{5.4}$$

Statement of Result: Let $D(t_0, \omega, \varepsilon)$ be the splitting distance for equation (5.2) and $E_n(t_0, \omega)$ are such that

$$D(t_0, \omega, \varepsilon) = E_0(t_0, \varepsilon) + \varepsilon E_1(t_0, \omega) + \cdots + \varepsilon^n E_n(t_0, \omega) + \cdots .$$

The main result of this section is as follows.

Proposition 5.1. *For equation (5.1), we let*

$$\mathcal{K}_g = \{\cos j\omega t : 1 \leq j \leq k_1\}$$

where k_1 is as in (5.3). We can also properly define a finite set $\mathcal{K}_d$ of admissible functions, so that for every $n \geq 0$, there exists a set $\mathcal{E}_n$ of high order Melnikov integrals so that

$$E_n(t_0, \omega) = \sum_{\mathcal{T} \in \mathcal{E}_n} c_{\mathcal{T}} (\mathcal{T}^s(0) - \mathcal{T}^u(0))$$

where $c_{\mathcal{T}}$ are constants. Also, there exists a k_0 so that for all $\mathcal{T} \in \cup_{n \geq 1} \mathcal{E}_n$, only a maximum number of k_0 nodes are allowed to branch out of every node in $\mathcal{T}$.

We further have

Proposition 5.2. *Let $\mathcal{E}_n$ be as in Proposition 5.1. We have, in addition,*

(1) *(Order of Integrals) The order p of all $\mathcal{T}$ in $\mathcal{E}_n$ is such that $n \le p \le 4(n+1)$.*
(2) *(Number of Integrals) The total number of Melnikov integral in $\mathcal{E}_n$ is $< K^{n+1}$.*
(3) *(Coefficients) We have $|c_{\mathcal{T}}| < K^{n+1}$ for all $\mathcal{T} \in \mathcal{E}_n$.*

where K is a constant independent of ω.

Proof of Theorem 2: The conclusions of Theorem 2 Follows directly from Proposition 4.1, Proposition 5.1 and Proposition 5.2. □

The rest of this section is devoted to the proof of Propositions 5.1 and 5.2. Our presentation is exclusively on $f(x) = x - x^3$. The correspondences for $f(x) = x - x^2$ can be easily replicated.

5.1. **Integral Equation for Stable Solutions.** Let $(a(t), b(t))$ be as in (2.1). We use a, a', b, b' for $a(t), a'(t), b(t), b'(t)$ respectively and we note that

$$a' = b, \qquad b' = a - a^3, \qquad b^2 = a^2 - a^4/2$$

where the last equality reflects the fact that (a, b) is a zero energy solution of the unperturbed equation (5.2). The contents of this subsection is in complete parallel to that of Sects. 2.1-2.3 of [18], and all claims share the same proofs of their correspondences in [18]: we of course need to replace the perturbation function $y^2 \cos \omega t$ in equation (4.1) by using the new perturbation function (5.3) throughout.

We start by solving the equations

$$\frac{d\xi}{dt} = \eta; \qquad \frac{d\eta}{dt} = (1 - 3a^2)\xi. \tag{5.5}$$

This is the equation of first variations for the unperturbed equation around $\ell(t) = (a, b)$. In what follows,

$$h(t) = 3a^2(t) \int_0^t a^{-2}(\tau) d\tau = \frac{3(e^{2t} - e^{-2t} + 4t)}{2(e^t + e^{-t})^2}; \tag{5.6}$$

$$H(t) = \frac{1}{a(t)} \left[b(t)h(t) + a(t) \right]; \qquad \tilde{H}(t) = \frac{1}{a(t)} \left[b'(t)h(t) + 2b(t) \right]. \tag{5.7}$$

Observe that $a(t), H(t)$ are even functions, but $b(t), h(t)$ and $\tilde{H}(t)$ are odd functions in t. In addition, $h(t), H(t)$ and $\tilde{H}(t)$ are all uniformly bounded for all $t \in (-\infty, +\infty)$. We also denote $h(t), H(t), \tilde{H}(t)$ respectively as $h, H, \tilde{H}$.

Lemma 5.1. *Let m, w be such that*

$$\xi = \frac{1}{a} (bw - aHm); \qquad \eta = \frac{1}{a} \left(b'w - a\tilde{H}m \right). \tag{5.8}$$

The equation of first variations (5.5) is transformed in new variables (m, w) to

$$\frac{dm}{dt} = -\frac{b}{a} m; \qquad \frac{dw}{dt} = \frac{b}{a} w. \tag{5.9}$$

Proof. The derivation to obtain m, w is a little tricky, but to verify the claim of this Lemma is straight forward. See also the proof of Lemma 2.2 of [18]. □

Let t_0 be a given initial time, and $(\hat{x}(t), \hat{y}(t))$ be the stable solution of the perturbed equation satisfying $(\hat{x}(t_0), \hat{y}(t_0)) = (x_0, y_0)$. Let $(x(t), y(t)) = (\hat{x}(t+t_0), \hat{y}(t+t_0))$. Then $(x(t), y(t))$ is well-defined on $t \in [0, +\infty)$ satisfying

$$\frac{dx}{dt} = y; \qquad \frac{dy}{dt} = x - x^3 + \varepsilon \left(\sum_{j=1}^{k_1} P_j(x,y) \cos j\omega(t+t_0) \right)$$

and $(x(0), y(0)) = (x_0, y_0)$. Let

$$X = x - a(t); \qquad Y = y - b(t). \tag{5.10}$$

We have

$$\frac{dX}{dt} = Y; \qquad \frac{dY}{dt} = (1-3a^2)X + Q(t,X) + \varepsilon \left(\sum_{j=1}^{k_1} P_j(X+a, Y+b) \cos j\omega(t+t_0) \right) \tag{5.11}$$

where

$$Q(t,X) = -3aX^2 - X^3. \tag{5.12}$$

Lemma 5.2. *Let M, W be such that*

$$X = \frac{1}{a}\left(bW - aHM\right); \qquad Y = \frac{1}{a}\left(b'W - a\tilde{H}M\right) \tag{5.13}$$

where h and $H, \tilde{H}$ are as in (5.6) and (5.7). Equation (5.11) is transformed in new variables (M, W) to

$$\begin{aligned} \frac{dM}{dt} =& -\frac{b}{a}M - \frac{b}{a}Q(t,X) - \varepsilon\frac{b}{a}\left(\sum_{j=1}^{k_1} P_j(X+a, Y+b) \cos j\omega(t+t_0) \right); \\ \frac{dW}{dt} =& \frac{b}{a}W - HQ(t,X) - \varepsilon H \left(\sum_{j=1}^{k_1} P_j(X+a, Y+b) \cos j\omega(t+t_0) \right). \end{aligned} \tag{5.14}$$

Proof. Straight forward derivation. See also the proof of Lemma 2.3 in [18]. □

We introduce one more change of variables by letting

$$\mathbb{M} = \frac{a}{\varepsilon}M; \qquad \mathbb{W} = \frac{\sqrt{(2-a^2)}}{\varepsilon b}W \tag{5.15}$$

to remove the linear part of the equation (5.14). We have

(5.16)

$$\begin{aligned} \frac{d\mathbb{M}}{dt} =& \varepsilon b\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - b\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b) \cos j\omega(t+t_0) \right); \\ \frac{d\mathbb{W}}{dt} =& \frac{2b}{a^2\sqrt{(2-a^2)}}\left[\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b) \cos j\omega(t+t_0) \right) \right]. \end{aligned}$$

where

$$\mathbb{X} = \frac{1}{a}\left[\frac{b^2}{\sqrt{(2-a^2)}}\mathbb{W} - H\mathbb{M} \right], \qquad \mathbb{Y} = \frac{1}{a}\left[\frac{bb'}{\sqrt{(2-a^2)}}\mathbb{W} - \tilde{H}\mathbb{M} \right]. \tag{5.17}$$

Lemma 5.3. *Let* $(\mathbb{M}(t), \mathbb{W}(t))$ *be the stable solution. We have*

(5.18)

$$\mathbb{M}(t) = -\varepsilon \int_t^{+\infty} b(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3)d\tau + \int_t^{+\infty} b\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b)\cos j\omega(t+t_0)\right)d\tau;$$

$$\mathbb{W}(t) = \int_0^t \frac{2b}{a^2\sqrt{(2-a^2)}}\left[\varepsilon H(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3) - H\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b)\cos j\omega(t+t_0)\right)\right]d\tau.$$

Proof. The proof of this lemma has been presented in both [3] and [18]. See, for instance, the proof of lemma 2.4 in [18]. □

5.2. **Proof of Proposition 5.1.** Let us write $\mathbb{M}(t) = \mathbb{M}(t, t_0, \omega, \varepsilon)$, $\mathbb{W}(t) = \mathbb{W}(t, t_0, \omega, \varepsilon)$ formally as power series of ε. This is to say,

$$\mathbb{M}(t) = \sum_{n=0}^{+\infty} \varepsilon^n \mathbb{M}_n(t, t_0, \omega); \qquad \mathbb{W}(t) = \sum_{n=0}^{+\infty} \varepsilon^n \mathbb{W}_n(t, t_0, \omega).$$

We can use (5.18) to determine the functions $\mathbb{M}_n = \mathbb{M}_n(t, t_0, \omega), \mathbb{W}_n = \mathbb{W}_n(t, t_0, \omega)$ recursively for all n: First we have, by letting $\varepsilon = 0$ on both side of (5.18),

$$\begin{aligned} \mathbb{M}_0(t, t_0, \omega) &= \int_t^{+\infty} b\left(\sum_{j=1}^{k_1} P_j(a, b)\cos j\omega(\tau + t_0)\right)d\tau; \\ \mathbb{W}_0(t, t_0, \omega) &= -\int_0^t H\left(\sum_{j=1}^{k_1} P_j(a, b)\cos j\omega(\tau + t_0)\right)d\tau. \end{aligned} \tag{5.19}$$

Assume that we have obtained $\mathbb{M}_k = \mathbb{M}_k(t, t_0, \omega), \mathbb{W}_k = \mathbb{W}_k(t, t_0, \omega)$ for all $k \le n$. We can solve for $\mathbb{M}_{n+1} = \mathbb{M}_{n+1}(t, t_0, \omega)$, $\mathbb{W}_{n+1} = \mathbb{W}_{n+1}(t, t_0, \omega)$ in terms of $\mathbb{M}_k = \mathbb{M}_k(t, t_0, \omega), \mathbb{W}_k = \mathbb{W}_k(t, t_0, \omega)$, $k \le n$ by using (5.18). This is because on the left hand side the only term of order ε^{n+1} are $\mathbb{M}_{n+1}$ and $\mathbb{W}_{n+1}$, but on the right hand side the terms of order ε^{n+1} only include $\mathbb{M}_k(t, t_0, \omega), \mathbb{W}_k(t, t_0, \omega)$, $k \le n$. All terms involving $\mathbb{M}_k, \mathbb{W}_k$, $k \ge n+1$ on the right hand side would be at least of order ε^{n+2}.

Proof of Proposition 5.1. We have, by (5.6) and (5.7) for $h, H, \tilde{H}$,

$$\begin{aligned} \mathbb{X} &= \left[ba^{-1}\frac{b}{\sqrt{(2-a^2)}}\mathbb{W} - \left[ba^{-1}h + 1\right]a^{-1}\mathbb{M}\right], \\ \mathbb{Y} &= \left[\frac{b(1-a^2)}{\sqrt{(2-a^2)}}\mathbb{W} - \left[(1-a^2)h(t) + 2ba^{-1}\right]a^{-1}\mathbb{M}\right]. \end{aligned}$$

By the assumption that $P_j(x, y)$ are even in y, we can rewrite $P_j(x, y)$ as function in y^2 instead of in y. This is to say that, in substitute y by using $\varepsilon\mathbb{Y} + b$, we actually substitute y^2 by using $(\varepsilon\mathbb{Y} + b)^2$. We have

$$\begin{aligned} (\varepsilon\mathbb{Y} + b)^2 = \varepsilon^2 &\left[\frac{b(1-a^2)}{\sqrt{(2-a^2)}}\mathbb{W} - \left[(1-a^2)h(t) + 2ba^{-1}\right]a^{-1}\mathbb{M}\right]^2 + b^2 \\ &+ 2\varepsilon b\left[\frac{b(1-a^2)}{\sqrt{(2-a^2)}}\mathbb{W} - \left[(1-a^2)h(t) + 2ba^{-1}\right]a^{-1}\mathbb{M}\right]. \end{aligned}$$

Denote

$$A = A(t) = \frac{1}{2}\sqrt{(2 - a^2(t))}. \tag{5.20}$$

The function $A(z)$ is even and it takes $z = 0$ as a branch point. In addition, $|b(z)A^{-1}(z)|$ is uniformly bounded by a constant K around $z = 0$.

Lemma 5.4. *The right hand side of equation (5.18) for $\mathbb{M}(t)$ and $\mathbb{W}(t)$ are linear combinations of finite collections of integrals, the integral functions of which are all in the form of*

$$f(t) = \cos^{n_0} n_3\omega(t + t_0) \cdot b^{m_1} h^{m_2} a^{n_1} A^{-n_2} \mathbb{M}^{n_4} \mathbb{W}^{n_5}. \tag{5.21}$$

where n_1 is allowed to be negative but $n_2, n_4, n_5, m_1, m_2 \geq 0$. In addition, there exists a constant $k_2 > 0$ so that

(i) *n_0 is either 0 or 1;*
(ii) *$m_1 \geq n_2$;*
(iii) *$m_1 + m_2$ is odd;*
(iv) *$|m_1 + m_2 + |n_1| + n_2 + m_4 + n_5| \leq k_2$;*
(v) *$m_1 + n_1 + 3n_4 \geq 3$ for $\mathbb{M}(t)$;*
(vi) *$m_1 + n_1 + 3n_4 \geq 1$ for $\mathbb{W}(t)$.*

Proof. The formula in (5.21) for $f(t)$ follows from the assumption that all $P_j(x, y)$ are polynomials. For item (i), all $f(t)$ of $n_0 = 0$ are from the first integral on the right hand of (5.18) and that of $n_0 = 1$ are from the second. For item (ii), we observe that A^{-1} is always multiplied by b in $\mathbb{X}, \mathbb{Y}$. Item (iii) follows from the use of the formula $(\varepsilon\mathbb{Y} + b)^2$ in $P_j(x, y)$. Item (iv) is a trivial upper bound. The constant k_2 here is determined by the highest degree of polynomials $P_j(x, y)$. Item (v) and (vi) follows by the observations that (1) all a^{-1} in $\mathbb{X}$ and $\mathbb{Y}$ are accompanied by a copy of b or $\mathbb{M}$, and (2) replacing $\mathbb{W}$ by 1, b by a, and $\mathbb{M}$ by a^3, an a^2 can be factored out of $\mathbb{X}$ and $\mathbb{Y}$. The difference in between the inequalities of (v) and (vi) is caused by an a^2 in the denominator of the integral function for $\mathbb{W}$ on the right side of equation (5.18). □

So far, we have derived the integral formula for stable solutions. We can compute unstable solutions the same way. To distinguish the two, let us denote a stable solution as $(\mathbb{M}^s(t, t_0, \omega, \varepsilon), \mathbb{W}^s(t, t_0, \omega, \varepsilon))$ and an unstable solution as $(\mathbb{M}^u(t, t_0, \omega, \varepsilon), \mathbb{W}^u(t, t_0, \omega, \varepsilon))$. We write

$$\mathbb{M}^s(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{M}^s_n(t, t_0, \omega); \qquad \mathbb{W}^s(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{W}^s_n(t, t_0, \omega);$$

and in dual,

$$\mathbb{M}^u(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{M}^u_n(t, t_0, \omega); \qquad \mathbb{W}^u(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{W}^u_n(t, t_0, \omega).$$

Let

$$\mathcal{D}(t_0, \omega, \varepsilon) = \mathbb{M}^s(0, t_0, \omega, \varepsilon) - \mathbb{M}^u(0, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n D_n(t_0, \omega) \tag{5.22}$$

where

$$D_n(t_0, \omega) = \mathbb{M}^s_n(0, t_0, \omega) - \mathbb{M}^u_n(0, t_0, \omega). \tag{5.23}$$

We are ready to prove a version of Proposition 5.1 for D_n instead of E_n. By letting $\mathcal{K}_d$ be the collection of all integrals functions on the right hand side of (5.18) in th form of

$$f(t) = \cos^{n_0} n_3\omega(t+t_0) \cdot b^{m_1} h^{m_2} a^{n_1} A^{-n_2} \mathbb{M}^{n_4} \mathbb{W}^{n_5}.$$

We drop $\cos^{n_0} n_3\omega(t+t_0)$ and $\mathbb{M}^{n_4}\mathbb{W}^{n_5}$ to define

$$d_f(z) = b^{m_1}(z) h^{m_2}(z) a^{n_1}(z) A^{-n_2}(z).$$

The assumption (A1) for $d_f(z)$ in Sect. 4.2 holds: (1) by the definition of A, $z=0$ is a branch point, and bA^{-1} is uniformly bounded around $z=0$ by Lemma 5.4(ii); (2) $z=\pm i\pi/2$ are poles, the order of which is bounded by $2k_2$ in Lemma 5.4(iv). Assumption (A2) for $d_f(z)$ is by Lemma 5.4(iii). The structure trees in $\mathcal{D}_n$ is obtained by taking any one of these integrals as root to branch out with n_4 many M-nodes and n_5 many W-nodes. We then recursively replace each of the child node by using the right hand side of (5.18). Observe that, after n-steps of this recursion, all new integrals added would be of degree $\geq \varepsilon^{n+1}$. We can then stop and collect all integrals of order exactly ε^n for $\mathbb{M}_n$ and $\mathbb{W}_n$. This set of $d(z)$ we take as $\mathcal{K}_d$. We also let $\mathcal{K}_M$ and $\mathcal{K}_W$ be the respective set of f from the right side of equation (5.18) for $\mathbb{M}$ and $\mathbb{W}$. We have $\mathcal{K} = \mathcal{K}_M \cup \mathcal{K}_W$. Definition 4.2(i) and (ii) follow from Lemma 5.4(v) and (vi) respectively.

To transfer this version of Proposition 5.1 for D_n to that for E_n, we derive a new integral equation for the unperturbed energy of the stable and unstable solutions. Let

$$\mathcal{E} = \varepsilon^{-1}\left(\frac{1}{2}y^2 - \frac{1}{2}x^2 - \frac{1}{4}x^4]\right).$$

We obtain

$$\frac{d\mathcal{E}}{dt} = (\varepsilon\mathbb{Y}+b)\cdot\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b)\cos j\omega(t+t_0)\right). \tag{5.24}$$

Converting to integral equations, we have for stable solution,

$$\mathcal{E}(t) = -\int_t^{+\infty} (\varepsilon\mathbb{Y}(\tau)+b(\tau))\left(\sum_{j=1}^{k_1} P_j(\varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b)\cos j\omega(t+t_0)\right) d\tau. \tag{5.25}$$

With this integral equation, all claims on D_n are converted to that of E_n. □

5.3. **Proof of Proposition 5.2.** We prove this proposition for $\mathcal{D}_n$ in the place of $\mathcal{E}_n$. Rewrite the $\mathbb{M}$ component of (5.18) as

$$\begin{aligned}\mathbb{M}(t) =& \mathbb{M}_0(t) + \varepsilon^{-1} \sum_{f\in\mathcal{K}_M,\ n_0(f)=0} c_f \int_t^{+\infty} d_f(\tau)(\varepsilon\mathbb{M})^{n_4(f)}(\varepsilon\mathbb{W})^{n_5(f)} d\tau \\ &+ \sum_{f\in\mathcal{K}_M, n_0(f)=1} c_f \int_t^{+\infty} (\cos n_3\omega\tau) d_f(\tau)(\varepsilon\mathbb{M})^{n_4(f)}(\varepsilon\mathbb{W})^{n_5(f)} d\tau.\end{aligned} \tag{5.26}$$

Here, the two sums are to distinguish integrals from the autonomous part and integrals from the non-autonomous part of the equation (5.16). For the first sum, ε^{-1} is from the re-scaling factor in (5.15). For the second sum, this re-scaling factor is canceled by the ε in front the non-autonomous forcing function of the equation (5.11). We also note that for all integrals in the first sum, we have

$$n_4 + n_5 \geq 2 \tag{5.27}$$

because autonomous linear terms are removed from equation (5.16), and for all integrals in the second sum we have

$$n_4 + n_5 \geq 1. \tag{5.28}$$

We compute M_{n+1} by first writing

$$\begin{aligned} \varepsilon\mathbb{M} &= \varepsilon\mathbb{M}_0 + \varepsilon^2\mathbb{M}_1 + \cdots + \varepsilon^{n+1}\mathbb{M}_n; \\ \varepsilon\mathbb{W} &= \varepsilon\mathbb{W}_0 + \varepsilon^2\mathbb{W}_1 + \cdots + \varepsilon^{n+1}\mathbb{W}_n \end{aligned}$$

to obtain

$$\begin{aligned} (\varepsilon\mathbb{M})^{n_4} &= \sum_I \varepsilon^{i_0+2i_1+\cdots+(n+1)i_n} C^{n_4,I} (\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n}; \\ (\varepsilon\mathbb{W})^{n_5} &= \sum_J \varepsilon^{j_0+2j_1+\cdots+(n+1)j_n} C^{n_5,J} (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} \end{aligned} \tag{5.29}$$

where I is over all positive integers $i_0, \cdots, i_n$ satisfying

$$i_0 + i_1 + \cdots + i_n = n_4;$$

J is over all positive integers $j_0, \cdots, j_n$ satisfying

$$j_0 + j_1 + \cdots + j_n = n_5;$$

and

$$C^{n_4,I} = \frac{n_4!}{i_0! \cdots i_n!}, \qquad C^{n_5,J} = \frac{n_5!}{j_0! \cdots j_n!}.$$

We also denote

$$\kappa_I = i_0 + 2i_1 + \cdots + (n+1)i_n, \qquad \kappa_J = j_0 + 2j_1 + \cdots + (n+1)j_n$$

to write

$$(\varepsilon\mathbb{M})^{n_4}(\varepsilon\mathbb{W})^{n_5} = \sum_{I,J} C^{n_4,I} C^{n_5,J} \varepsilon^{\kappa_I+\kappa_J} (\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} \tag{5.30}$$

where $\sum_{I,J}$ are over all indexes $i_0, \cdots, i_n; j_0, \cdots, j_n$ satisfying

$$n_4 = i_0 + \cdots + i_n; \qquad n_5 = j_0 + \cdots + j_n.$$

Substituting (5.30) into (5.26), we obtain

$$\mathbb{M}_{n+1} = \sum_{f \in \mathcal{K}_M} c_f \sum_{I,J} C^{n_4,I} C^{n_5,J} \int_0^{+\infty} (\cos^{n_0} n_3\omega\tau) d_f(\tau) (\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} d\tau. \tag{5.31}$$

In the case of $n_0(f) = 0$, $\sum_{I,J}$ is over all $i_0, \cdots, i_n, j_0, \cdots, j_n$ satisfying

$$\begin{aligned} n_4 =& i_0 + \cdots + i_n; \\ n_5 =& j_0 + \cdots + j_n; \\ n+2 =& \kappa_I + \kappa_J := i_0 + j_0 + 2(i_1 + j_1) + \cdots + (n+1)(i_n + j_n) \end{aligned} \tag{5.32}$$

where $n_4 + n_5 \geq 2$. In the case $n_0(f) = 1$, $\sum_{I,J}$ is over all $i_0, \cdots, i_n, j_0, \cdots, j_n$ satisfying

$$\begin{aligned} n_4 =& i_0 + \cdots + i_n; \\ n_5 =& j_0 + \cdots + j_n; \\ n+1 =& \kappa_I + \kappa_J := i_0 + j_0 + 2(i_1 + j_1) + \cdots + (n+1)(i_n + j_n) \end{aligned} \tag{5.33}$$

where $n_4 + n_5 \geq 1$.

Proof of Proposition 5.2(1) (Order of Integrals): Let $p(n)$ be the highest order of Melnikov integrals in $\mathcal{D}_n$. We prove

$$p(n) \leq 4(n+1) - 2$$

for all $n \geq 0$. This is obviously true for $n = 0$. Assume inductively that

$$p(m) \leq 4(m+1) - 2 \tag{5.34}$$

for all $m \leq n$. We have from (5.31),

$$\begin{aligned} p(n+1) &\leq 1 + (i_0 + j_0)p(0) + (i_1 + j_1)p(1) + \cdots + (i_n + j_n)p(n) \\ &\leq 1 + 4 \cdot 1(i_0 + j_0) + 4 \cdot 2(i_1 + j_1) + \cdots + 4 \cdot (n+1)(i_n + j_n) \\ &\quad -2(i_0 + \cdots + i_n + j_0 + \cdots + j_n). \end{aligned}$$

If f is such that $n_0(f) = 0$, then we have, by using (5.32),

$$p(n+1) \leq 4(n+2) + 1 - 2(n_4 + n_5) < 4(n+2) - 2$$

where $n_4 + n_5 \geq 2$ is used to obtain the second inequality. In the case of $n_0(f) = 1$, we use (5.33) to obtain

$$p(n+1) \leq 4(n+1) + 1 - 2(n_4 + n_5) < 4(n+2) - 2.$$

This proves Proposition 5.2(1). □

Proof of Proposition 5.2(2) (Number of Integrals): Let $\mathcal{N}_n$ be the cardinality of $\mathcal{D}_n$. We prove that there is a constant $\mathbb{K}$ so that

$$\mathcal{N}_n \leq (n+1)^{-2}\mathbb{K}^{2n+1}$$

for all $n \geq 0$. Assume inductively

$$\mathcal{N}_m \leq (m+1)^{-2}\mathbb{K}^{2m+1} \tag{5.35}$$

for all $m \leq n$. To compute $\mathcal{N}_{n+1}$, we note that there are only a fixed number of $f \in \mathcal{K}$. For a given $f \in \mathcal{K}$, we let $\mathcal{N}_{n+1}^f$ be the number of integrals the sum

$$\sum_{I,J} C^{n_4,I} C^{n_5,J} \int_0^{+\infty} (\cos^{n_0} n_3\omega\tau) d_f(\tau) (\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} d\tau$$

contributes to $\mathbb{M}_{n+1}$. We have, from (5.31),

$$\mathcal{N}_{n+1} \leq \sum_{f \in \mathcal{K}} \mathcal{N}_{n+1}^f. \tag{5.36}$$

We also have

$$\mathcal{N}_{n+1}^f \leq \sum [\mathcal{N}_0]^{w_0} \cdots [\mathcal{N}_n]^{w_n} \tag{5.37}$$

where the sum is over all $w_k = i_k + j_k$ satisfying

$$\begin{aligned} n_4 + n_5 =& w_0 + w_1 + \cdots + w_n; \\ n + 2 =& w_0 + 2w_1 + \cdots + (n+1)w_n \end{aligned} \tag{5.38}$$

if $n_0(f) = 0$; but

$$\begin{aligned} n_4 + n_5 =& w_0 + w_1 + \cdots + w_n; \\ n + 1 =& w_0 + 2w_1 + \cdots + (n+1)w_n \end{aligned} \tag{5.39}$$

if $n_0(f) = 1$.

If $n_4(f)+n_5(f)=1$, then we must have $n_0(f)=1$ and $w_n=1$. This is to say that

$$\mathcal{N}^f_{n+1}=\mathcal{N}_n\leq\frac{(n+2)^2}{(n+1)^2}(n+2)^{-2}\mathbb{K}^{2n+1}<4(n+2)^{-2}\mathbb{K}^{2n+1}. \tag{5.40}$$

If $n_4(f)+n_5(f)=2$. Assuming $n_0(f)=0$, we have

$$\begin{aligned}2=&w_0+w_1+\cdots+w_n;\\ n+2=&w_0+2w_1+\cdots+(n+1)w_n.\end{aligned}$$

In this case, our choices for w_k are limited: we either have two non-zero w_k, which we denote as $k_1\neq k_2$ so that $w_{k_1}=w_{k_2}=1$, or we have one non-zero w_k. In the first case, we have $k_1+k_2=n$ because

$$n+2=(k_1+1)w_{k_1}+(k_2+1)w_{k_2};$$

and in the second case, we have $k=n/2$ because $2(k+1)=n+2$. Counting both cases, we obtain

$$\begin{aligned}\mathcal{N}^f_{n+1}&\leq\sum_{k=1}^{n/2}\mathcal{N}_k\mathcal{N}_{n-k}\\ &\leq\mathbb{K}^{2k+1+2(n-k)+1}\sum_{k=1}^{n/2}\frac{1}{(k+1)^2(n-k+1)^2}\\ &\leq 4n^{-2}\mathbb{K}^{2(n+1)}\sum_{k=1}^{n/2}\frac{1}{(k+1)^2}\end{aligned}$$

where (5.37) and $k_2=n-k_1$ are used for the first inequality; the inductive assumption (5.35) for the second inequality; and $n-k+1\geq n/2$ (because $k\leq n/2$) for the last inequality. In conclusion, we have

$$\mathcal{N}^f_{n+1}\leq 16(n+2)^{-2}K_0\mathbb{K}^{2(n+1)} \tag{5.41}$$

where $K_0=\sum_{k=1}^{+\infty}k^{-2}$.

We move on to consider the case $n_4(f)+n_5(f)=3$. Assuming $n_0(f)=0$, we have

$$\begin{aligned}3=&w_0+w_1+\cdots+w_n;\\ n+2=&w_0+2w_1+\cdots+(n+1)w_n.\end{aligned}$$

In this case, there are at most three non-zero w_k, which we denote as w_{k_1},w_{k_2},w_{k_3}. We consider first the case $w_{k_1}=w_{k_2}=w_{k_3}=1$. We obtain

$$k_1+k_2+k_3=n-1.$$

from

$$(k_1+1)w_{k_1}+(k_2+1)w_{k_2}+(k_3+1)w_{k_3}=n+2.$$

We also count the cases of two non-zero w_k and one non-zero w_k to obtain

$$\begin{aligned}\mathcal{N}^f_{n+1}&\leq 6\sum_{k_1\leq k_2\leq k_3\leq n-1}\mathcal{N}_{k_1}\mathcal{N}_{k_2}\mathcal{N}_{k_3}\\ &\leq 6\mathbb{K}^{2k_1+2k_2+2k_3+3}\sum_{k_1\leq k_2\leq k_3\leq n-1}(k_1+1)^{-2}(k_2+1)^{-2}(k_3+1)^{-2}\\ &\leq 6\mathbb{K}^{2n+1}\sum_{k_1\leq k_2\leq k_3\leq n-1}(k_1+1)^{-2}(k_2+1)^{-2}(k_3+1)^{-2}.\end{aligned}$$

We note that, by summing over $k_1 \le k_2 \le k_3$ (instead of $k_1 < k_2 < k_3$), the cases of two and one non-zero w_k are counted here. This implies

$$\mathcal{N}^f_{n+1} \le 108(n+2)^{-2}K_0^2\mathbb{K}^{2n+1} \tag{5.42}$$

where $K_0 = \sum_{k=1}^{+\infty} 1/k^2$. Here we used $k_3 + 1 = n - k_1 - k_2 \ge n/3$, which follows from $k_1 + k_2 \le 2n/3$. We have $k_1 + k_2 \le 2n/3$ because if $k_1 + k_2 > 2n/3$, then $k_2 > n/3$, leading to $k_3 > n/3$ hence $k_1 + k_2 + k_3 > n$, contradicting $k_1 + k_2 + k_3 = n - 1$.

The estimate for the case of $n_0(f) = 1$ is similar.

Finally, it follows from (5.36), (5.40), (5.41) and (5.42) that

$$\mathcal{N}_{n+1} \le (n+2)^{-2}\hat{K}\mathbb{K}^{2(n+1)}$$

where

$$\hat{K} = \#\{f \in \mathcal{K}\}(4 + 16K_0 + 108K_0^2).$$

We let $\mathbb{K}$ be such that $\mathbb{K} > \hat{K}$ to conclude

$$\mathcal{N}_{n+1} \le (n+2)^{-2}\mathbb{K}^{2(n+1)+1}.$$

Estimates for $n_4 + n_5 > 3$ are similar. This process ends in finite step because $n_4 + n_5$ are bounded from above by the highest degree of $P_j(x, y)$. □

Proof of Proposition 5.2(3) (Coefficients): Let $c(n)$ be the max of the magnitude of coefficients in front of an $N \in \Lambda_{M,n} \cup \Lambda_{W,n}$. We prove that there exists $K_3 > 0$ so that

$$c(n) \le c(0)K_3^n$$

for all $n \ge 0$. See (5.19). Assume inductively

$$c(m) \le c(0)K_3^m \tag{5.43}$$

for all $m \le n$. From (5.31) and

$$|c_f|,\ C^{n_4,I},\ C^{N_5,J} \le c(0),$$

we have

$$\begin{aligned} c(n+1) &\le c(0)^3 \cdot c(0)^{n_4+n_5}[c(0)]^{i_0+j_0}[c(1)]^{i_1+j_1}\cdots[c(n)]^{i_n+j_n} \\ &\le K \cdot K_3^{(i_0+j_0)+2(i_i+j_i)+(n+1)(i_n+j_n)-(i_0+j_0+i_1+j_1+\cdots+i_n+j_n)}. \end{aligned}$$

We again have the two cases of $n_0(f) = 0$ and $n_0(f) = 1$. Recall that if $n_0(f) = 0$, then $n_4(f) + n_5(f) \ge 2$. We have

$$c(n+1) \le K \cdot K_3^{n+2-(n_4+n_5)} < K_3^{n+1}$$

assuming $K_3 > K$. If $n_0(f) = 1$, we have

$$c(n+1) \le K \cdot K_3^{n+1-(n_4+n_5)} < K_3^{n+1}$$

assuming $K_3 > K$. This proves Proposition 5.2(3). □

Remarks: (1) The conclusions of Proposition 5.1 and Proposition 5.2 remain true if we replace $f(x) = x - x^3$ in (5.2) by using $f(x) = x - x^2$.

(2) We have two ways to impose conditions on an unperturbed equation, the first is to name $f(x)$ explicitly, as we just did. The second is to assume a general form on $f(x)$, but to make assumptions on the homoclinic solution $(a(t), b(t))$ of the unperturbed equation defined by $f(x)$. Though the second way sounds more general, the true value of such generality is rather superficial: all examples supplied on this subject matter in existing literature, for

which $f(x)$ can be directly related to the would-be assumptions on $(a(t), b(t))$, have been more or less restricted to the three cases of $f(x) = x - x^2$, $f(x) = x - x^3$ and $f(x) = \sin x$.

(3) The way we elected to present the theory of high order Melnikov integrals in Section 4 also imposes restrictions on the function of perturbation $P(x, y, t)$ in equation (5.2). The first is for it to be a Fourier cosine polynomial and the second is (5.4), designated for $d(-z) = -d(z)$ to hold for all $d(z) \in \mathcal{K}_d$. The assumption that $P(x, y, t)$ is a *cosine* polynomial is not essential, but allowing sine functions would induce complications on symmetry, a distraction we chose to avoid. If we let

$$P(x, y, t) = \sum_{j=1}^{k_1} \left(P_j(x, y) \cos jt + Q_j(x, y) \sin jt\right), \tag{5.44}$$

we would need to also assume $Q_j(x, -y) = -Q_j(x, y)$ and extend $\mathcal{K}_g$ and $\mathcal{K}_d$ accordingly to cover the sine terms added.

(4) Both the assumption that $P_j(x, y)$ are independent of ε, and they are polynomials in x, y are not essential, but their removal would inflict complications in notation and lengthy technical writing. Here we again chose to exchange generality for simplicity in exposition. While generality is often highly desirable, and a most general theory is worth pursuing in normal circumstances, the situation we are in is not. Unless one is compelled by circumstance, generality for the sake of generality is an unworthy object to pursue on this subject matter.

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