On the Homoclinic Tangle of Henri Poincaré

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King Oscar II's prized Competition

– Establish by King Oscar II of Sweden and Norway in 1888

– Prompted by Mittag-Leffler, his Majesty's science adviser

– Purposed to promote the newly lunched *Acta Mathematica*

– The proposed problem was to find global power series solution of the N-body problem

– The prize was awarded to Henri Poincaré

– After received the award, Poincaré uncovered a fatal mistake in his original submission

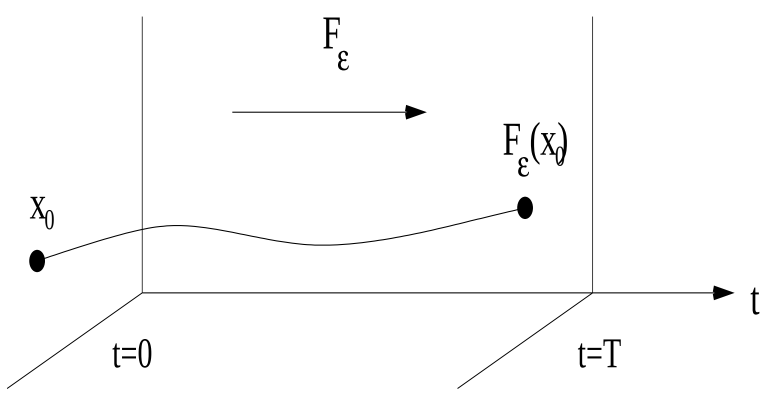
– In reckoning this mistake, Poincaré discovered homoclinic tangle, marked the beginning of the modern chaos theory.

The Discovery of Homoclinic Tangle

– Poincaré studied time-periodic equation in the form of

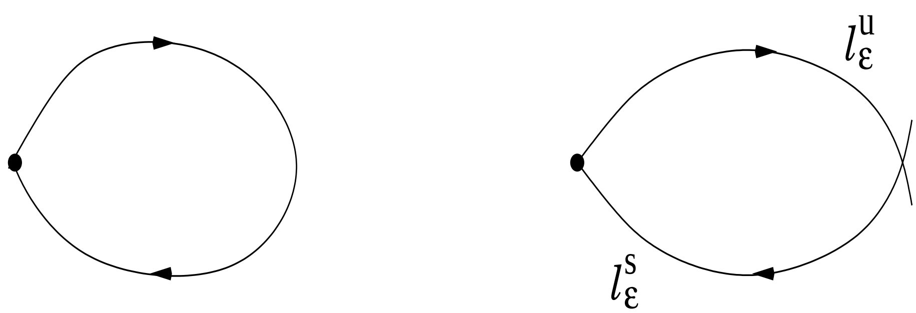
$$\frac{dx}{dt} = f(x) + \varepsilon g(x,t)$$

– He used time-T map to study this equation.



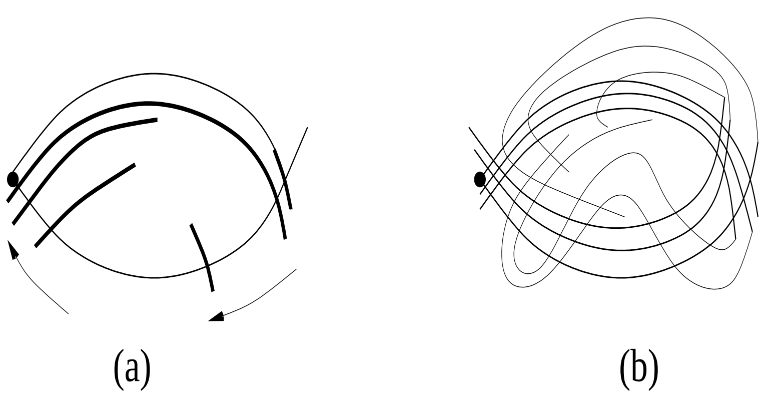
– Trajectory of a homoclinic solution is an invariant loop for the time-T map of the unperturbed equation.

– Poincaré mistakenly argued that this invariant loop remains intact under periodic perturbation.



(a) (b)

– With a broken loop, he discovered homoclinic tangle



(a) (b)

On tangle of time-periodic equation

– We present a theory on tangles of equation

$$\frac{dx}{dt} = f(x) + \varepsilon g(x, t)$$

– Assumptions:

(1) It has a saddle fixed point x_0
(2) x_0 is *dissipative*

– Two issues:

(1) The dynamics of homoclinic tangle for a fixed ε
(2) The way tangles of different ε fit together.

Conclusions

(1) Smale horseshoe, SRB measure of Benedicks-Carleson and Young, and Newhouse sinks are all participating part of homoclinic tangles of this equation

(2) Tangles of different dynamic structures are organized, asymptotic, in an infinitely refined pattern defined on parameter intervals of a fixed length of $\ln \varepsilon$, and this asymptotic pattern is repeated indefinitely as $\varepsilon \to 0$

This talk is divided into two parts

Part I: A review on participating objects

Part II: An overview on tangles of time-periodic second order equations.

(I) Participating Objects

(1) Smale's Horseshoe

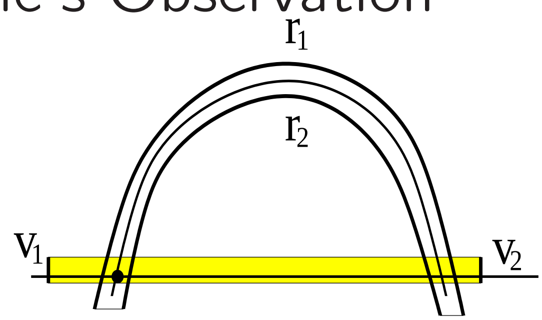
– work on ODE induced by Poincaré discovery

Cartwright and Littlewood on time-periodic ODE

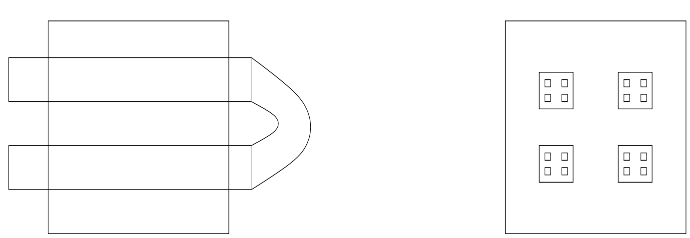
Levinson on Van der pol equation

Sitnikov and Alekseev on 3-body problem.

– Smale's Observation



– The horseshoe map



(a) (b)

(2) Anosov Diffeo and SRB Measure

Iteration of maps

– To study homoclinic tangle there is no need to start from an ODE

– All it takes is a map with one saddle

– And one transversal intersection of stable and unstable manifold

Linear map induced on 2D torus

$$x_1 = 2x + y, \qquad y_1 = x + y. \tag{1}$$

Phase space is one homoclinic tangle

– Stable and unstable manifolds of $(0, 0)$ wraps around as dense curves in $\mathbb{T}^2$

– Their intersections of are dense in $\mathbb{T}^2$

– Periodic orbits are all saddles dense on $\mathbb{T}^2$

Chaos

–All periodic orbits and their stable manifold is a set of measure zero

–On the remainder set, individual orbits are not guided by an alloted sense of *destination*

–They are jump around in a random fashion

SRB Measure

–In chaos there is a law of statistics

–Asymptotic distribution of points in phase space is the same for almost all individual orbits

Character of Anosov Chaos

–Induced entirely by topology of the phase space

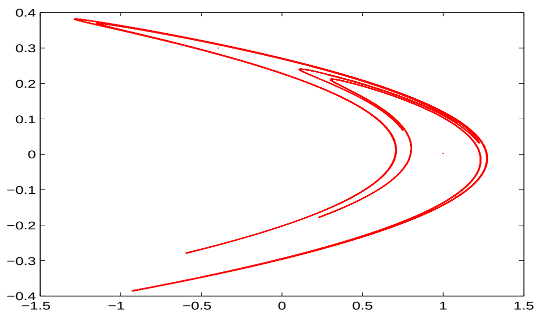
–The underlining map is linear

–Is intrinsically different from the homoclinic tangles of nonlinear ODE

–The same applies to all tangles induced by uniformly hyperbolic maps on compact manifold

(3) Hénon Map and SRB Measure

$$x_1 = 1 - ax^2 + by, \qquad y_1 = bx$$



Benedicks-Carleson Theory on Hénon Maps

– A positive Lesbegue measure set of parameters, almost all orbits of the corresponding maps are unstable.

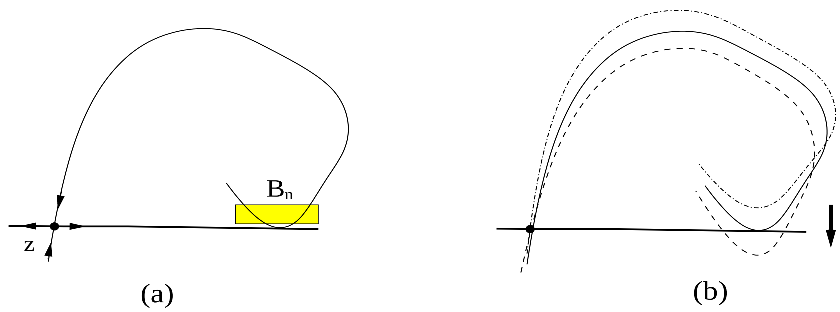
–*Positive measure* implies occurrence in numerical simulation

–*Prevalence of unstable orbits* implies chaos: individual orbits has no alloted sense of destination

SRB Measure of Benedick-Carleson and Young

There exists a governing law of statistics for good Hénon maps

(4) Homoclinic Tangency and Newhouse Sink



Newhouse Theory

– Assume saddle is *disspative*

– Return maps around the point of tangency is virtually a Hénon family

– (i) Homoclinic Tangency are persistent

– (ii) Infinitely many sinks

With the theory of Hénon maps

– (iii) SRB measures added as a third scenario

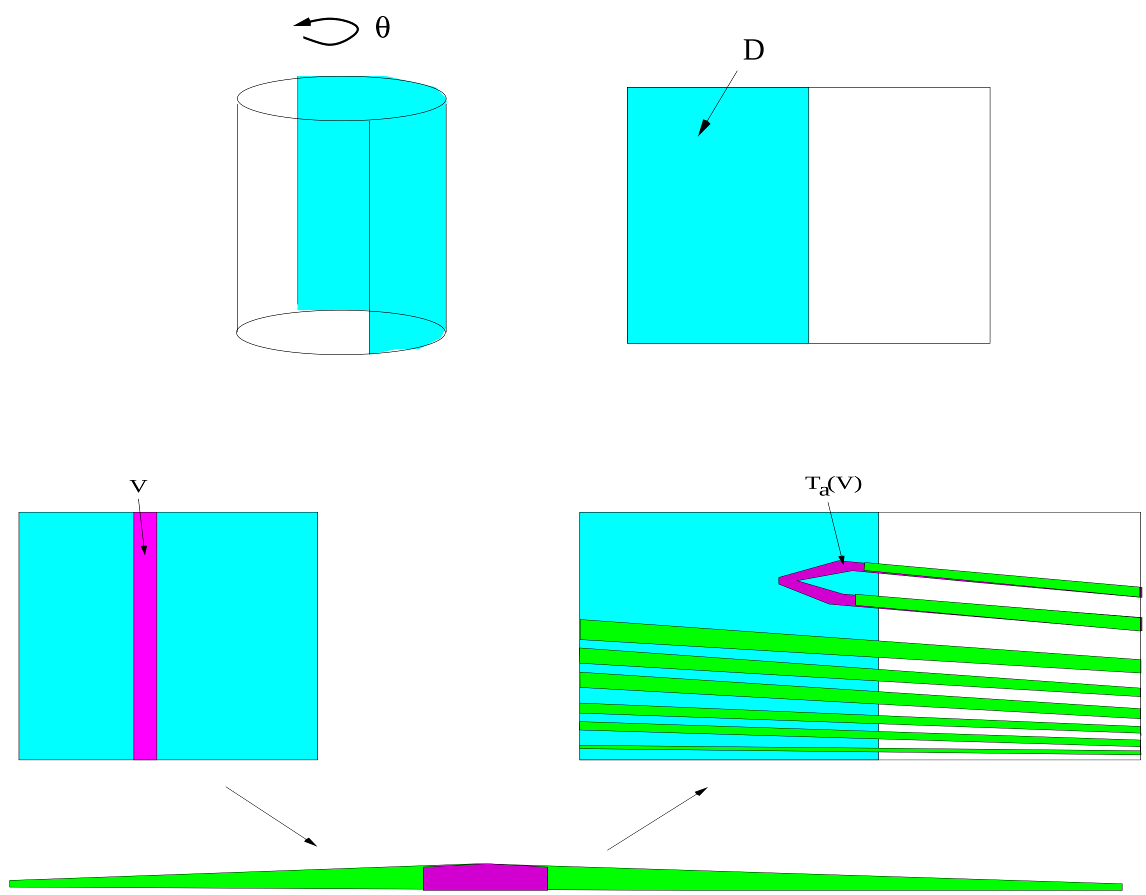
– We find SRB measure and Newhouse sink inside homoclinic tangle of ODE through Newhouse tangency

(II) An Overview on Dynamics

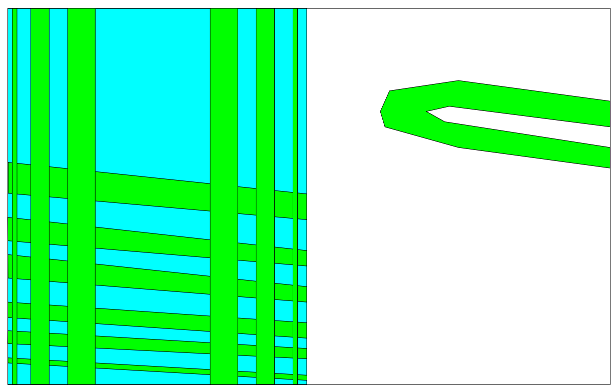
(1) Infinitely Wrapped Horseshoe Map

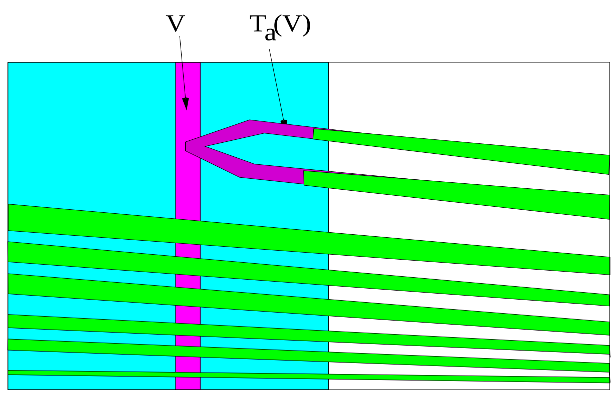
$$
\begin{aligned}
\theta_1 &= a + \theta - 10 \ln(0.001y + \sin\theta), \\
y_1 &= 0.001\sqrt{0.001y + \sin\theta}
\end{aligned}
$$

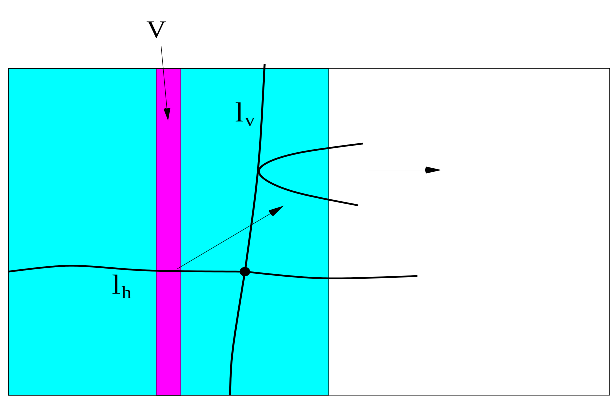
The dynamics is 2π-periodic in a



(2) Dynamic Scenarios



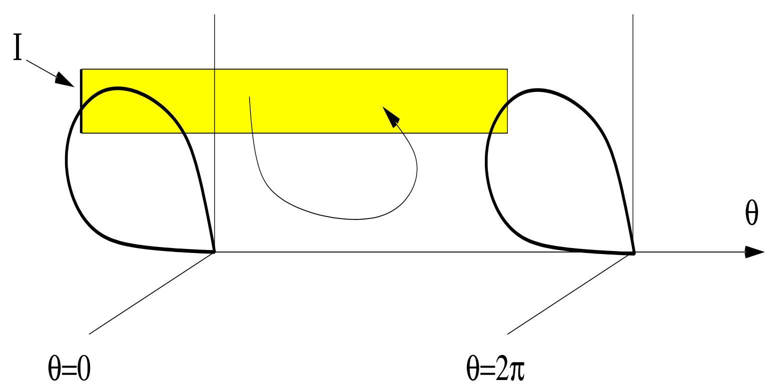




(3) Homoclinic Tangle of Time-periodic ODE

$$\frac{dx}{dt} = f(x) + \varepsilon g(x, \omega\theta)$$
$$\frac{d\theta}{dt} = \omega^{-1}$$

Separatix Map



$$\theta_1 = \frac{\omega}{\beta} \ln \varepsilon^{-1} + \theta + \frac{\omega}{\beta} \ln(k_0 y + W(\theta))$$
$$y_1 = \varepsilon^{\frac{\alpha}{\beta} - 1}(k_0 y + W(\theta))^{\frac{\alpha}{\beta}}.$$

$W(\theta)$ is the Melnikov function

Dynamics of Homoclinic Tangle

Assumption: All zeros of $W(\theta)$ are non-tangential and all critical points of $W(\theta)$ are non-degenerate.

Three Participating Components

– (i) A Horseshoe of infinitely many branches (for all ε)

–(ii) Sinks (not for all ε)

–(iii) SRB measure (not for all ε)

An Overview

–A staged show that is casted around *one* horseshoe of infinitely many branches

–The stage is the domain of the separatrix map

–The stage is darkened when the folded part is out side of the domain

–The elements of this horseshoe are lighted up, **not all at once, but one by one**, through the light shed by Newhouse tangency

–The same staged show is enacted in infinite repetition as $\varepsilon \to 0$.

Numerical Simulation

$$\frac{d^2x}{dt^2} + (\lambda - \gamma x^2)\frac{dx}{dt} - x + x^2 = \varepsilon \sin 2\pi t.$$

ε	Dynamics	Actual Ratio
$1.577 \cdot 10^{-3}$	Transient tangle	–
$7.774 \cdot 10^{-4}$	SRB measure	–
$7.637 \cdot 10^{-4}$	Sinks	–
$7.284 \cdot 10^{-4}$	Transient tangle	2.1650
$3.574 \cdot 10^{-4}$	SRB measure	2.1752
$3.449 \cdot 10^{-4}$	Sink	2.2143
$3.349 \cdot 10^{-4}$	Transient tangle	2.1750
$1.635 \cdot 10^{-4}$	SRB measure	2.1859
$1.608 \cdot 10^{-4}$	Sink	2.2143
$1.536 \cdot 10^{-4}$	Transient tangle	2.1803
$7.505 \cdot 10^{-5}$	SRB measure	2.1785
$7.308 \cdot 10^{-5}$	Sink	2.2003
$7.041 \cdot 10^{-5}$	Transient tangle	2.1815
$3.415 \cdot 10^{-5}$	SRB measure	2.1977
$3.342 \cdot 10^{-5}$	Sink	2.1867
$3.224 \cdot 10^{-5}$	Transient tangle	2.1839
$1.574 \cdot 10^{-5}$	SRB measure	2.1696
$1.504 \cdot 10^{-5}$	Sink	2.2221

$\lambda = 0.5$, $\gamma = 0.577028548901$, $\beta = 0.78077641$

Predicted Multiplicative Period $= 2.1831$