On the Dynamics of Homoclinic Tangles

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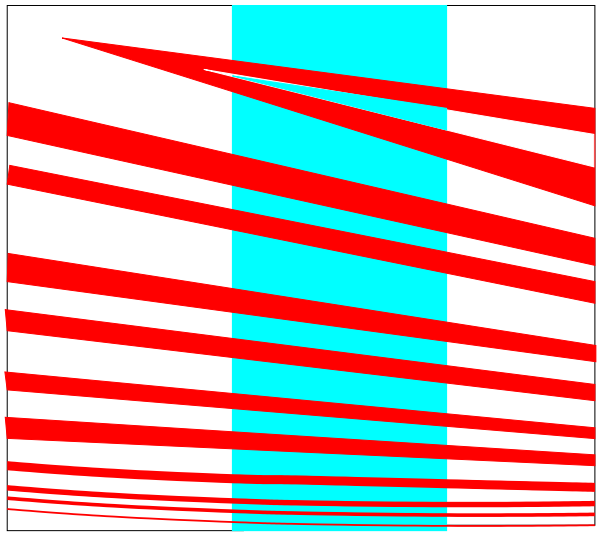
Part I

Dynamics of Homoclinic Tangles



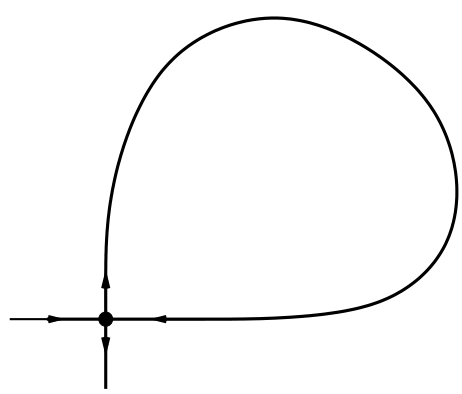
Unperturbed equation

$$\frac{dx}{dt} = -\alpha x + f(x,y),$$
$$\frac{dy}{dt} = \beta y + g(x,y)$$

- Dissipative saddle: $0 < \beta < \alpha$.

- Homoclinic solution: $(0,0)$ is with a homoclinic solution

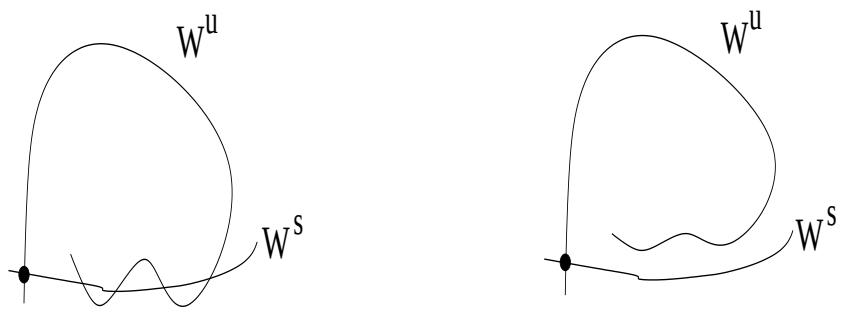
$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$



Periodically perturbed equation

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t).$$

Two Scenarios:



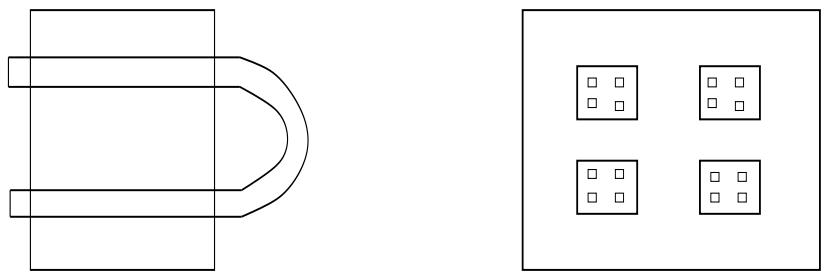
Scenario (a) Scenario (b)

— Long history dates back to Poincaré.

— Chaos theory has been focused on (a).

Smale's Horseshoe (1960)

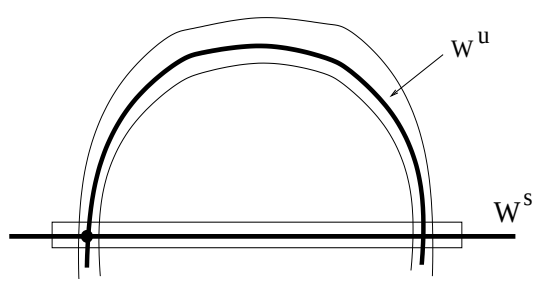
- The horseshoe map



- Horseshoe implies complicated dynamics

The dynamics of a horseshoe map conjugates to the full shift of two symbols.

- Scenario (a) implies a horseshoe map



Analysis of a Given Equation

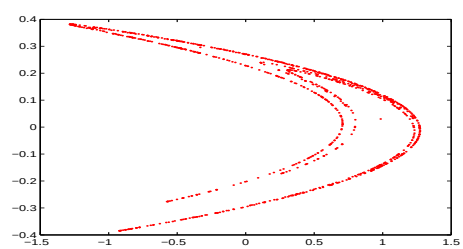
Part I Theoretic Analysis

- Compute the Melnikov function to verify that we are in scenario (a);

- Use Smale's construction to confirm the existence of complicated dynamics.

part II Numerical Evidence

- To find a few values of μ that offer numerical plots of chaos



A Gap Between Theory and the Plots:

Plots are those of physical measures, not that of horseshoes.

Physical measures:

Invariant distribution with an attractive basin of positive Lebesgue measure.

- Sinks represent stable dynamics;
- SRB measures representing chaos.
- Horseshoe is always with an attractive basin of measure zero.

A Recent Dynamical Theory

There are THREE major dynamical scenarios:

(I) Transient tangles Homoclinic tangles admits no physical measures.

(II) Tangles dominated by sinks The only physical measures admitted are sinks.

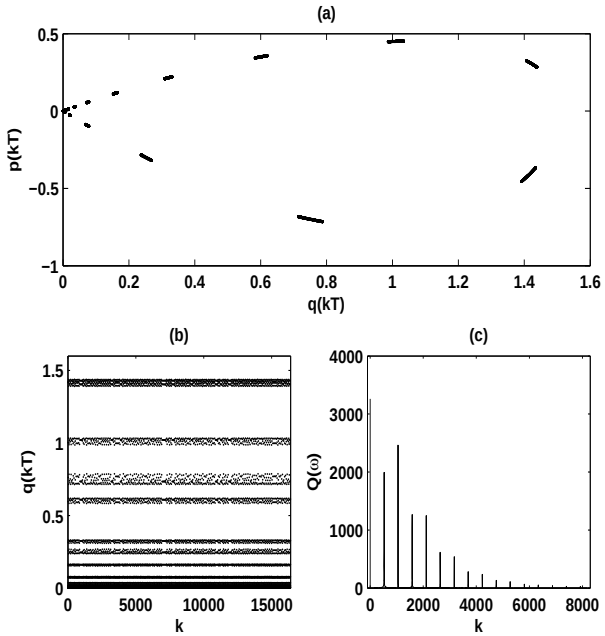
(III) Hénon-like attractors SRB measures associated to Newhouse tangency representing chaos.

As $\mu \to 0$, there is a invariant pattern of dynamical behavior that would be repeat in an accelerated fashion as $\mu \to 0$.

Multiplicative Period: $\mu_1 = e^{\beta T}\mu_0$.

Part II

Numerical Investigation



Equation of Study

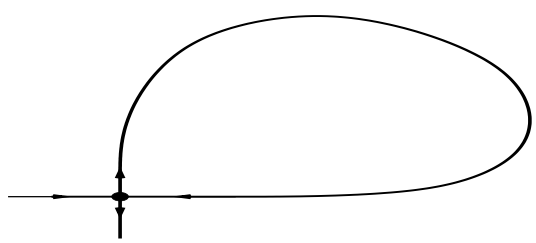
Step 1: We start with

$$\frac{d^2q}{dt^2} - q + q^2 = 0.$$

Step 2: Add non-linear damping terms

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^2 = 0.$$

For $\lambda > 0$ small, there exists γ_λ so that it has a homoclinic loops to a dissipative saddle.



Step 3: Add periodic perturbations

$$\frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2)\frac{dq}{dt} - q + q^2 = \mu \sin 2\pi t$$

Numerical Simulation

- Numerical scheme

  — Fourth-order Runge-Kutta routine

- Values of λ and γ

$$\lambda = 0.5, \quad \gamma_\lambda = 0.5770285901$$

- Initial phase position

$$(q_0,\ p_0) = (0.01,\ 0.0).$$

- Parameters varied

  — μ ranged from 10^{-3} to 10^{-7}.

  — t_0 ranged in $[0, 1)$.

— Compute the solution for each fixed combination of μ and t_0.

Simulation Results

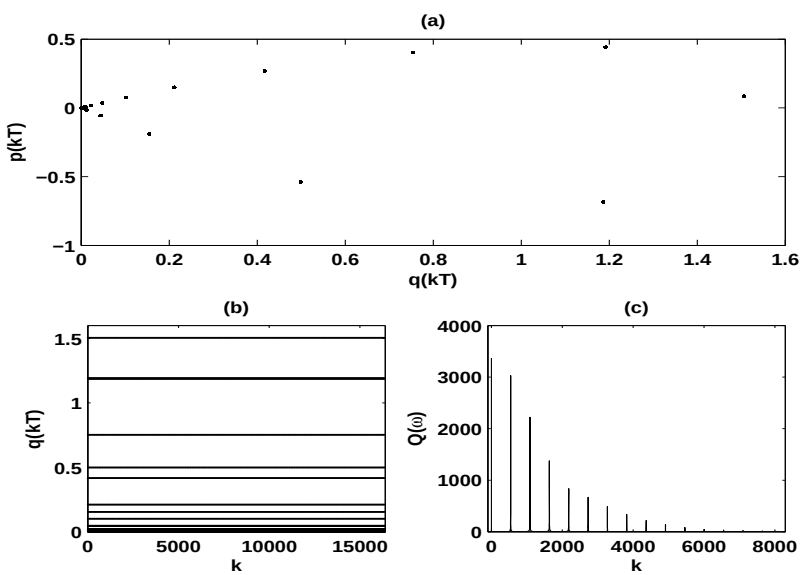
- Transient Tangle

— Let t_0 run over $[0, 1)$ with an increment of 0.001.

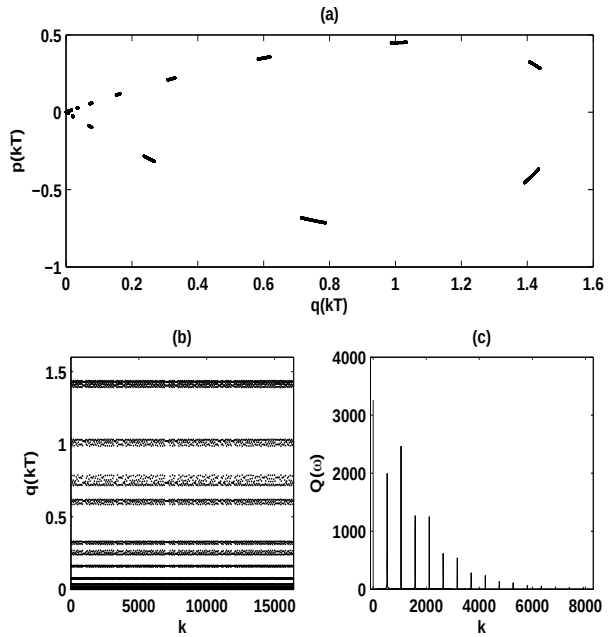
— all 1,000 solutions leave the neighborhood of the homoclinic solution.

— Homoclinic tangle contains no object *directly observable*.

- Periodic Sink



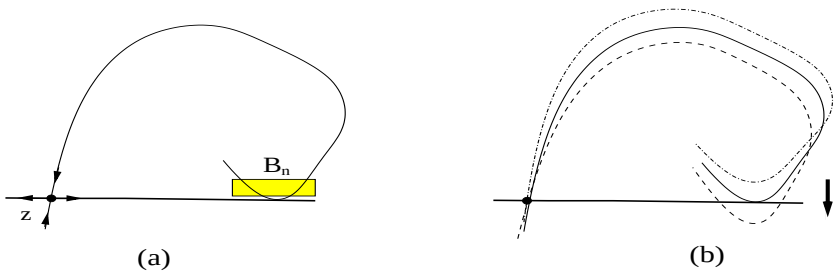
- Hénon-Like Attractor



— These are plots of a chaotic attractor close to a periodic solution.

— They represent a Hénon-like attractor associated to Newhouse tangency (Mora-Viana)

Theory of Newhouse Tangency (1970's)



• Return map on B_n is essentially a Hénon family after re-normalization.

— Newhouse's infinitely many sinks.

— Hénon-like attractors (Mora-Viana, 1990).

• Persistency of tangency.

— Tangency is structured as intersection of two Cantor sets.

— Small change of parameters can not do away the tangency.

Periodicity of Dynamical Behavior

Theoretical Multiplicity $= e^{\beta T} = 2.1831$		
μ	Dynamics	Actual Ratio
$7.041 \cdot 10^{-5}$	Transient	–
$3.415 \cdot 10^{-5}$	Hénon-like	–
$3.342 \cdot 10^{-5}$	Periodic Sink	–
$3.224 \cdot 10^{-5}$	Transient	2.1839
$1.574 \cdot 10^{-5}$	Hénon-like	2.1696
$1.504 \cdot 10^{-5}$	Periodic Sink	2.2221
$1.474 \cdot 10^{-5}$	Transient	2.1872
$7.190 \cdot 10^{-6}$	Hénon-like	2.1892
$6.931 \cdot 10^{-6}$	Periodic Sink	2.1700
$6.732 \cdot 10^{-6}$	Transient	2.1895
$3.272 \cdot 10^{-6}$	Hénon-like	2.1974
$3.149 \cdot 10^{-6}$	Periodic Sink	2.2010
$3.060 \cdot 10^{-6}$	Transient	2.2000
$1.477 \cdot 10^{-6}$	Hénon-like	2.2153
$1.448 \cdot 10^{-6}$	Periodic Sink	2.2182
$1.378 \cdot 10^{-6}$	Transient	2.2206
$6.547 \cdot 10^{-7}$	Hénon-like	2.2560
$6.500 \cdot 10^{-7}$	Periodic Sink	2.2277

Summary

(I) A Theory on Homoclinic Tangles

We provided a comprehensive description on the overall dynamical structure of homoclinic tangles from a periodically perturbed homoclinic solution.

(II) Theories on maps come together

Horseshoes, Newhouse sinks and Hénon-like attractors all fall into their places as part of a larger picture.

(III) Applications

Our results can be applied to the analysis of given equations, such as Duffing equation.

(IV) A systematic numerical investigation

The behavior of numerical solutions are entirely predictable and are consistent with the predictions of our new theory.