On the Theory of Chaotic Rank One Attractors

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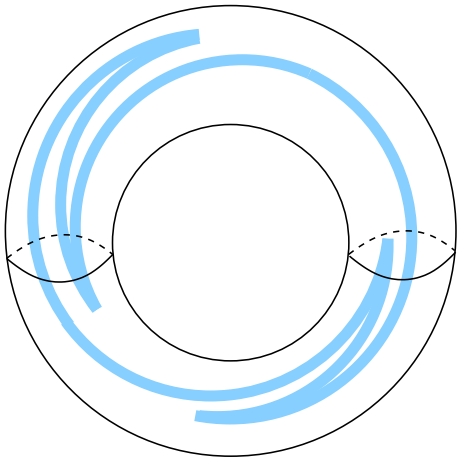
Part I

Theory On Rank One Maps



Admissible Family of Rank One Maps

- Phase Space and Parameters

— $(x, y) \in S^1 \times D_{m-1}$, $m \geq 2$.

— $(a, b) \in (a_1, a_2) \times (0, b_0)$ are parameters.

- Rank One Maps

$$\begin{aligned} x_1 &= F(x, y, a) + bu(x, y, a, b) \\ y_1 &= bv(x, y, a, b) \end{aligned} \tag{1}$$

F, u, v are C^3; $|\partial_y F(x, y, a)| \neq 0$; $\frac{\det(DT(x,y))}{\det(DT(x',y'))} < K$

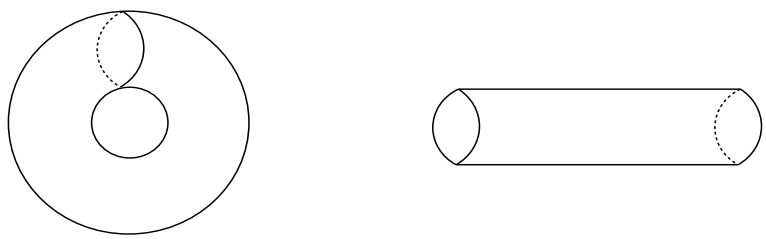
- 1D Singular Limit: $f_a(x) = F(x, 0, a)$

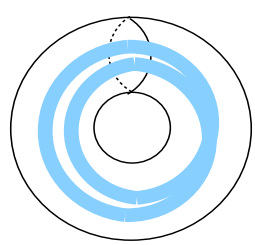
— 1D maps with critical points and expanding property.

— An example: $f_a(\theta) = \theta + a + L \sin \theta$ when L is large.

$$x_1 = F(x, y, a) + bu(x, y, a, b)$$
$$y_1 = bv(x, y, a, b)$$

If $\{f_a\}$ were uniformly expanding



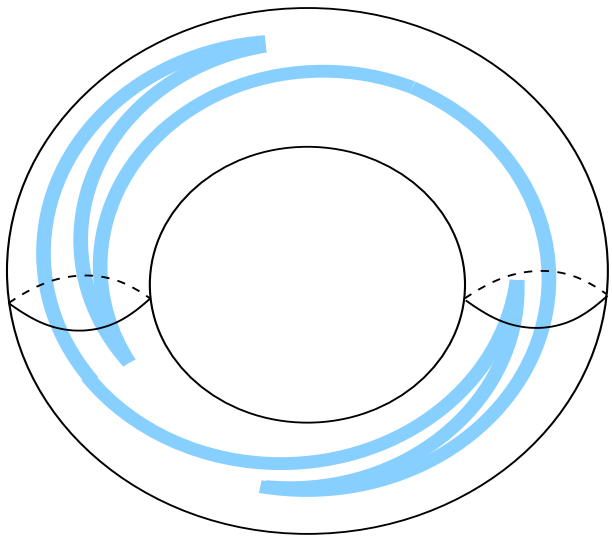


— Global splitting of stable and unstable directions

— An Axiom A solenoid

$$x_1 = F(x, y, a) + bu(x, y, a, b)$$
$$y_1 = bv(x, y, a, b)$$

But $\{f_a\}$ is with critical points



— The wrapping direction reverses at critical values of f_a

— No global splitting of stable and unstable directions

— Rank one maps are non-Axiom A solenoid

Directly Observable Chaos

- Dynamics of non-Axiom A solenoid.

  — Smale's Horseshoe for all parameters.

  — Periodic sinks for many parameters.

- Horseshoe is not directly observable

  — Numerically plot an orbit

  — Almost surely miss horseshoe

  — Positive chance for periodic sink.

- Existence of a directly observable chaos?

Positive Lyapunov exponent on a set of positive Lebesgue measure?

Horseshoe is not enough.

Theory of chaotic rank one attractors

Main Theorem *Let* $\{T_{a,b},\ (a,b) \in (a_1, a_2) \times (0, b_0)\}$ *be an admissible family of rank one maps. Then there exists a positive measure set of parameters* $\Delta \subset (a_1, a_2) \times (0, b_0)$*, such that for all* $(a,b) \in \Delta$*,* $T_{a,b}$ *admits a positive Lyapunov exponent Lebesgue almost everywhere in* $\mathcal{A}$*.*

History

- For $f(x) = 1 - ax^2$, Jacobson

- For Hénon maps, Benedicks-Carleson

Our Motivations: (1) Substantially improve the *tour de force* analysis of Benedicks and Carleson.

(2) To have a theory on non-uniformly hyperbolic map that could be applied to the analysis of differential equations.

Maps of good parameters

- The attractor $\Omega = \cap_{n=0}^{+\infty} T_{a,b}^n(\mathcal{A})$

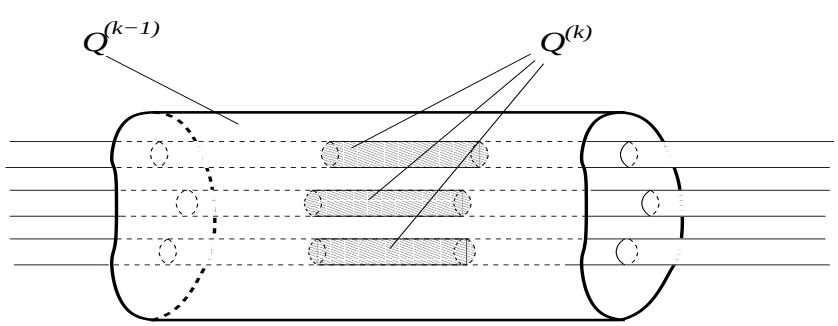
- For good parameters, $T = T_{a,b}$ possesses a well-defined critical set $\mathcal{C} \subset \Omega$.

- The critical set $\mathcal{C}$ is such that

  (i) every point in $\mathcal{C}$ is a point of quadratic tangency

  (ii) the critical orbits $\cup_n T^n(\mathcal{C})$ are all the tangency in Ω.

- $\mathcal{C}$ is well structured Cantor sets around the critical points of the 1D singular limit.



Inductive Construction of good parameters

• Good maps are constructed through an elaborated inductive process.

• The set of good parameters are constructed side by side with the critical set $\mathcal{C}$.

• Each inductive step starts with a temporary set of good parameters and a collection of small cylinders, each of which we denote as a critical region $Q^{(k)}$.

• We construct inductively new critical regions $Q^{(k+1)}$ inside of $Q^{(k)}$, deleting parameters that could potentially ruin the intended quadratic tangency

• We prove that, at each step of the induction, the measure of the deleted parameters declines exponentially.

Dynamics of Chaotic Rank One Attractors

A. Hyperbolic Structure

Theorem 1 *For any give* $\varepsilon > 0$,

$$\Lambda_\varepsilon = \{z = (x, y) \in \Omega,\ d(T^n(z), \mathcal{C}) \geq \varepsilon,\ \forall n \in \mathbb{Z}\}$$

is a uniformly hyperbolic invariant subset in Ω.

B. Symbolic Dynamics

Theorem 2 *(a) There is a natural partition of* $\Omega \backslash \mathcal{C}$ *into disjoint sets* $A_1, A_2, \cdots, A_q$ *so that* $z \in A_i$ *can be thought of as being "to the right" of* $\mathcal{C}_i$ *and "to the left" of* $\mathcal{C}_{i+1}$.

(b) There is a closed subset $\Sigma \subset \Sigma_q$ *with* $\sigma(\Sigma) \subset \Sigma$ *and a map* $\pi : \Sigma \to \Omega$ *with the property that*

- for all $\mathrm{s} = (s_i) \in \Sigma$, $\pi(\mathrm{s}) = z$ *implies that* $T^i z \in \bar{A}_{s_i}$ *for all* i;

- π *is a continuous surjection that is 1-1 except on* $\cup_{i=-\infty}^{\infty} T^i \mathcal{C}$, *where it is 2-1.*

C. Law of Statistics

- Histogram

— Dividing $\mathcal{A}$ into disjoint sub-regions.

— For a finite orbit of T, counting the proportion of points in each of the sub-regions.

- Two Limits

— Let the orbit go infinitely long

— Infinitely refined the division

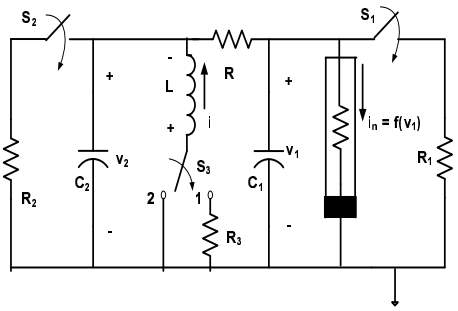
- For maps of good parameters

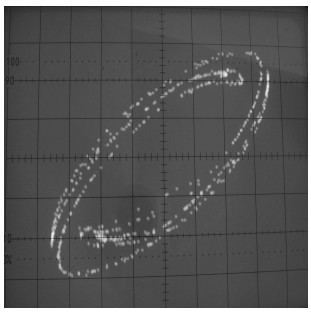
— For almost all orbits, histogram converge to a limit distribution.

— There is only finitely many pre-determined distribution for almost all orbits to converge to. They are the SRB measures.

Part II

Application to Equations





(I) Periodically perturbed Homoclinic Solution (joint with W. Ott, UH)

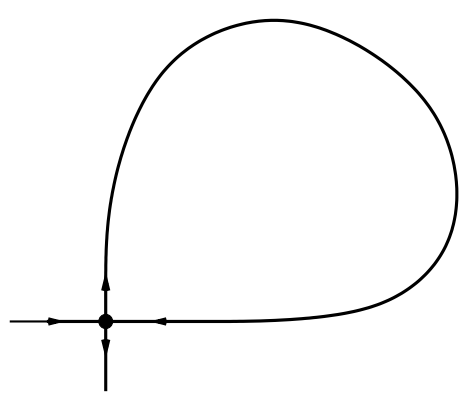
Unperturbed equation

$$\frac{dx}{dt} = -\alpha x + f(x,y),$$
$$\frac{dy}{dt} = \beta y + g(x,y)$$

- Dissipative saddle: $0 < \beta < \alpha$.

- Homoclinic solution: $(0,0)$ is with a homoclinic solution

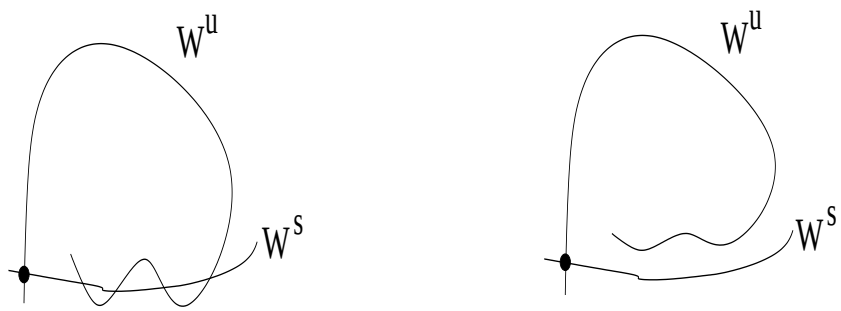
$$\ell = \{\ell(t) = (a(t), b(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R}\}.$$



Periodically perturbed equation

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t).$$

Two Scenarios:



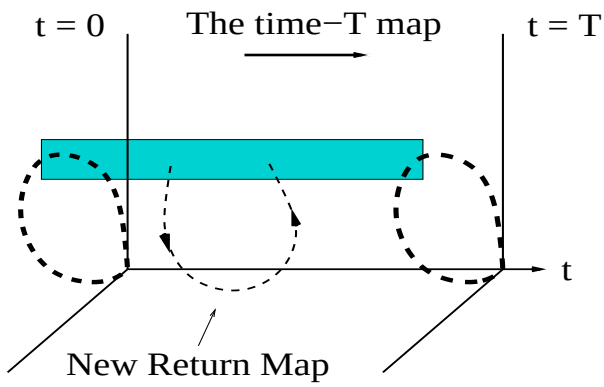
Scenario (a) Scenario (b)

— Long history dates back to Poincaré.

— Chaos theory have been focused mainly on Scenario (a).

— A Standard practice is to compute the Melnikov function to exclude Scenario (b) to prove chaos.

Separatrix map $\mathcal{R}$



Fact: $\mathcal{R}$ are rank one maps and the 1D singular limit is

$$f_a(\theta) = \theta + \frac{\omega}{\beta} \ln \mu^{-1} + \frac{\omega}{\beta} \ln M(\theta)$$

where $M(\theta)$ is the classical Melnikov function.

Theorem 3 *Assume that $M(\theta)$ is a Morse function and $\min_\theta M(\theta) > 0$. Also assume the forcing frequency ω is sufficiently large. Then there is a positive measure set of μ close to $\mu = 0$, such that $\mathcal{R}$ is a chaotic rank one map.*

Duffing Equation

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^3 = \mu \sin \omega t. \qquad (2)$$

• $\forall \lambda_0 > 0$ sufficiently small, $\exists \gamma_0$ such that equation (2) has a homoclinic solution when $\mu = 0$.

• This homoclinic solution we denote as ℓ_0. Our next theorem asserts the existence of a directly observable global chaotic attractor in the neighborhood of ℓ_0.

Theorem 4 *Let $\lambda_0 > 0$ be small and γ_0, ℓ_0 be as in the above. Then there exists an positive measure set Δ for (γ, ω, μ) close to $(\gamma_0, \infty, 0)$ such that for $(\gamma, \omega, \mu) \in \Delta$, the Duffing equation (2) admits a unique chaotic rank one attractor in an open neighborhood of ℓ_0.*

Remark: This is the first rigorous theorem on the existence of a directly observable chaotic attractor for Duffing equation.

(II) Periodically Kicked Hopf Bifurcation

(Joint with Lai-Sang Young; Ali Oksasoglu, etc.)

Hopf Bifurcation

$$\frac{dx}{dt} = A_\mu x + f_\mu(x) \tag{3}$$

• $x \in \mathbb{R}^m$, $m \geq 2$; μ a parameter; A_μ is an $m \times m$ matrix; $f_\mu(x)$ is a vector valued function of order ≥ 2 at $x = 0$.

• All eigenvalues of A_μ except a conjugating pair, which we denote as $\lambda_{1,2}$, are with negative real part.

• $\lambda_{1,2} = a(\mu) \pm \omega(\mu)\sqrt{-1}$ are such that $a(0) = 0$, $\omega(0) \neq 0$.

Normal form

• Equation (3) has a 2-dimensional local center manifold W^c at $x = 0$.

• The induced-flow on W^c can be written in a complex variable z in a normal form as

$$\dot{z} = (a(\mu) + i\omega(\mu))z + k_1(\mu)z^2\bar{z} + \cdots \qquad (4)$$

— We have a generic Hopf bifurcation for equation (3) if $Re(k_1(0)) \neq 0$.

— It is sup-critical if $Re(k_1(0)) < 0$

— It is sub-critical if $Re(k_1(0)) > 0$.

• We assume $Re(k_1(0)) < 0$ so we are interested in a sup-critical Hopf bifurcation, in which a stable periodic solution comes out of $x = 0$.

Periodically kicked equation

$$\frac{dx}{dt} = A_\mu x + f_\mu(x) + \varepsilon\Phi(x) \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

ε is small; $\Phi(0) = 0$; and $\delta(t)$ is the δ-function.

Fact: The time-T map around the stable periodic solution come out of $x = 0$ is a family of rank one maps with a singular 1D limit, as $T \to \infty$, in the form of

$$f_a(\theta) = \theta + a + L\phi(\theta)$$

— $a \approx \omega(0)T \ mod(2\pi)$;

— $\phi(\theta) = \phi_0(\theta) + \mathcal{O}(\sqrt{\mu}) + \mathcal{O}(\varepsilon)$; and $\phi_0(\theta)$ is determined by A_0 and $D\Phi(0)$;

— $L \approx \varepsilon \cdot \tau$ and

$$\tau = \left| \frac{Im(k_1(0))}{Re(k_1(0))} \right|.$$

Chaotic rank one attractors

To find chaotic rank one attractors

(i) first we compute $\phi_0(\theta)$, and verify that all its critical points are non-degenerate;

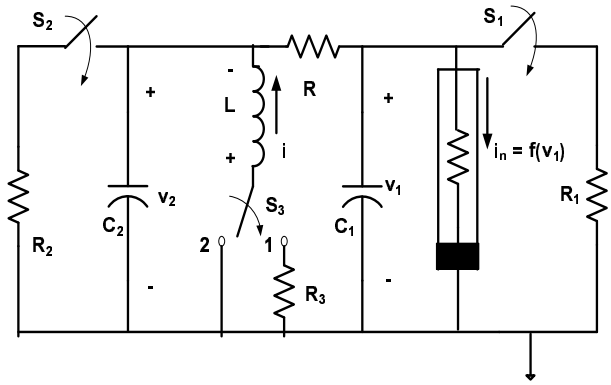
(ii) second we compute the normal form to find $k_1(0)$, and we look for equations for which τ is large;

(iii) let μ and ε be sufficiently small so that $\phi(\theta)$ is close to $\phi_0(\theta)$; we also let $L = \tau \cdot \varepsilon$ be sufficiently large;

(iv) we then conclude that chaotic rank one attractors exist for a positive measure set of T.

Switch Controlled Chua Circuit

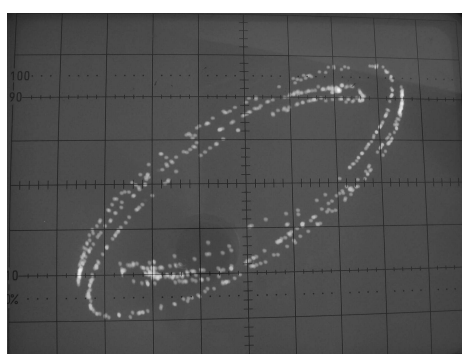
The circuit



What to do

- Follow the analysis to find good parameters.
- Build the circuit and perform lab simulation.

Chaotic rank one attractors



(III) Application to a PDE Joint with Kening Lu and Lai-Sang Young)

The Brusselator

$$u_t = d_1 \Delta u + a - (b+1)u + u^2 v,$$
$$v_t = d_2 \Delta v + bu - u^2 v.$$

The forced equation

$$u_t = d_1 \Delta u + a - (b+1)u + u^2 v + \rho \sin \pi x \; p_{T,\iota}(t),$$
$$v_t = d_2 \Delta v + bu - u^2 v \; ;$$
$$u(0,t) = u(1,t) = a, \qquad v(0,t) = v(1,t) = ba^{-1}.$$

$$p_{T,\iota}(t) = \sum_{n=-\infty}^{\infty} p_\iota(t - nT); \qquad p_\iota(t) = \begin{cases} \iota^{-1} & 0 \le t < \iota, \\ 0 & \text{elsewhere.} \end{cases}$$

— A simplified model of an autocatalytic chemical reaction with diffusion

— $u = u(x,t)$ and $v = v(x,t)$ for $x \in [0,1]$, and $\Delta = \partial_{xx}$.

— Dirichelet boundary condition is imposed.

Analysis and Conclusions

• $(u(t), v(t)) \equiv (a, ba^{-1})$ is a stationary solution.

• Hopf bifurcation occurs as the parameters a and b are varied. (A well-known fact)

• Our plan is to first develop a theory for periodically kicked sup-critical Hopf bifurcation in the context of infinite dimensional systems. We then apply the theory.

Theorem 5 *Let $d_1 = \pi^{-2}$, $0 < d_2 << 1$. Then there are open sets of a, b, ρ and ι, for which*

(i) the time-T map for the periodically forced Brusselator has a horseshoe for all large T;

(ii) the time-T map the periodically forced Brusselator has chaotic rank one attractors for a positive measure set of large T.