On the Homoclinic Tangles of Henri Poincaré

Qiudong Wang

University of Arizona

(Joint with Ali Osasaoglu)

List of Contents

Part I: Snapshots of History

- Discovery of Homoclinic Tangle
- Smale's Horseshoe
- Hénon Maps and Newhouse Tangency

Part II: Structure of Homoclinic Tangles

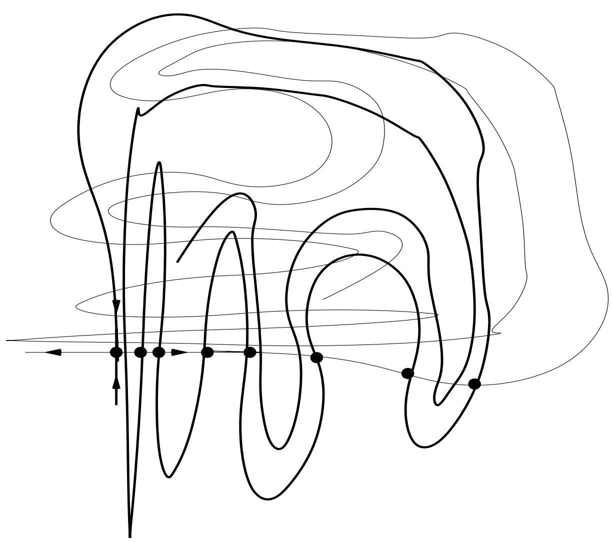
- The Separatrix Map
- Goemetric Structure
- Dynamical Consequence

Part III: Numerical Investigation

- Three Dynamical Scenarios
- Periodic Occurrence

Part I

Snapshots of History



King Oscar II's Prize Problem

- A prize for solving the N-body problem

Given a system of arbitrarily many mass points that attract each other according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

– Acta Mathematica, vol. 7, of 1885-1886

- Prize was awarded to Henri Poincaré for his work on the restricted 3-body problem

In this particular case, I have found a rigorous proof of stability and a method of placing precise limits on the elements of the third body... I now hope that I will be able to attack the general case and ... if not completely resolve the problem (of this I have little hope), then at least found sufficiently complete results to send into the competition.

– Henri Poincaré, 1887

A Prize Mess

• Hugo Gyldén informed the prize committee that the prize was wrongly bestowed.

— He published a paper two years earlier, in which Poincaré stability claim was proved.

— Poincaré said Gyldén' paper is not readable and is inconclusive.

• Poincaré then found a fatal mistake in his own essay.

— The stability conclusion he claimed was wrong.

— Acta Mathematica with the prize essay was recalled and all destroyed.

— Poincaré completely re-wrote his essay and kept the prize.

— Poincaré paid double the amount of the prize money to cover the cost of the recall.

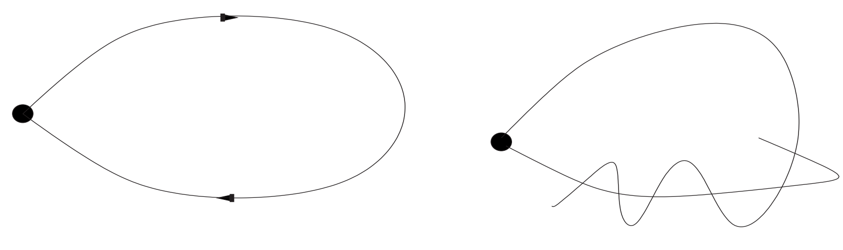
• Power series solution constructed much later.

By Sundman for the 3-body problem (1912) and by myself for all N (1985).

Homoclinic tangles of Henri Poincare

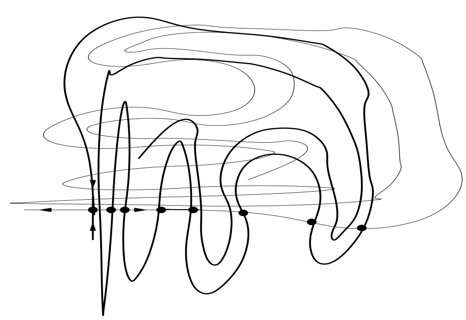
$$\frac{dx}{dt} = f(x) + \varepsilon g(x,t)$$

- The mistake Poincaré made



(a) (b)

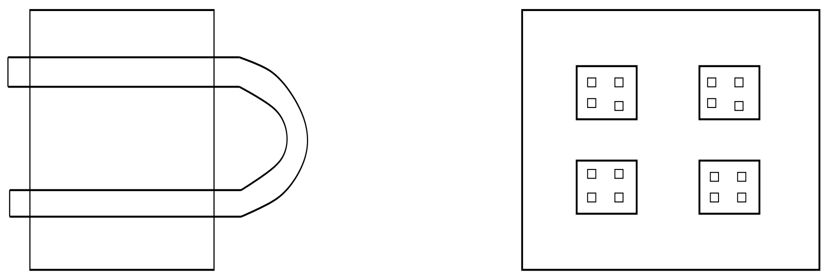
- The homoclinic tangle of Henri Poincaré



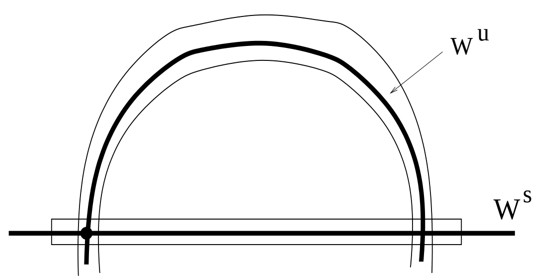
- Appeared to be an incompressible mess.

Smale's Horseshoe (1960)

- The horseshoe map



- Horseshoe embedded in homoclinic tangle



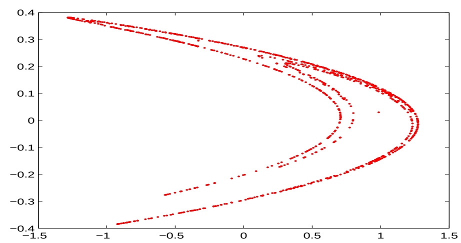
- Melnikov method

A standard computational procedure in verifying chaos in differential equations

Hénon Attractors

$$H_{a,b}: \quad x_1 = 1 - ax^2 + y, \quad y_1 = bx$$

- Numerical evidence of strange attractors

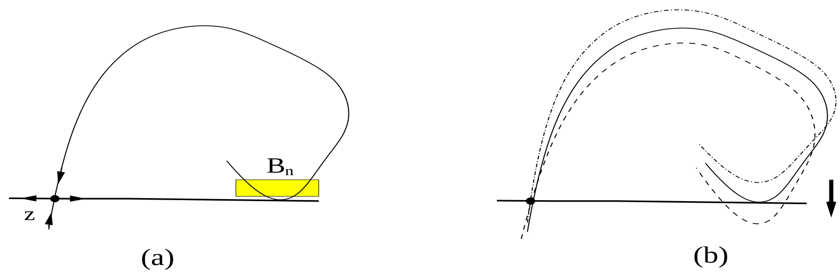


- Theory on Hénon maps

Theorem *For every b sufficiently small, there exists a positive measure set $\Delta_b \subset (1,2)$, such that for $a \in \Delta_b$, $H_{a,b}$ has a positive Lyapunov exponent Lebesgue almost everywhere on $(x,y) \in (0,2) \times (-1,1)$*

Remark: This theorem is mainly due to Benedicks and Carleson. Proof is long and hard.

Theory of Newhouse Tangency



• Return map on B_n is essentially a Hénon family after re-normalization.

— Newhouse's infinitely many sinks.

— Hénon-like attractors (Mora-Viana).

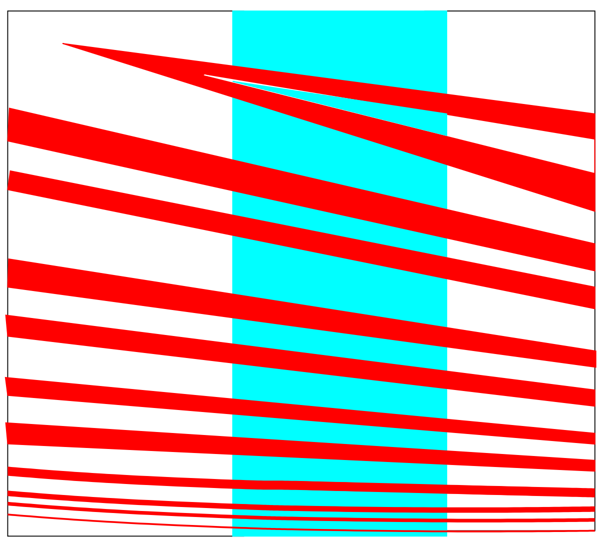
• Persistency of tangency.

— Tangency is structured as intersection of two Cantor sets.

— Small change of parameters can not do away the tangency.

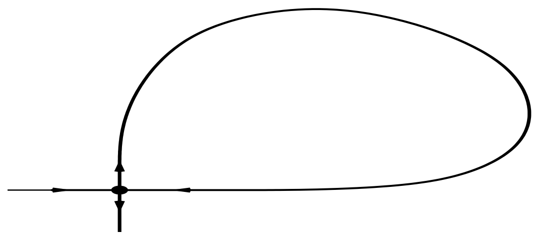
Part II

Structure of Homoclinic Tangles

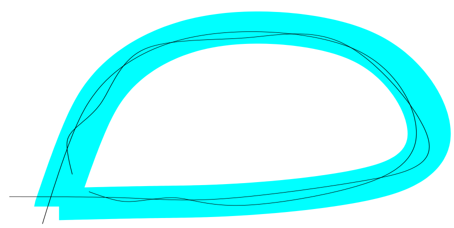


The Question of Homoclinic Tangles

$$\frac{dx}{dt} = -\alpha x + f(x,y) + \mu P(x,y,t),$$
$$\frac{dy}{dt} = \beta y + g(x,y) + \mu Q(x,y,t).$$



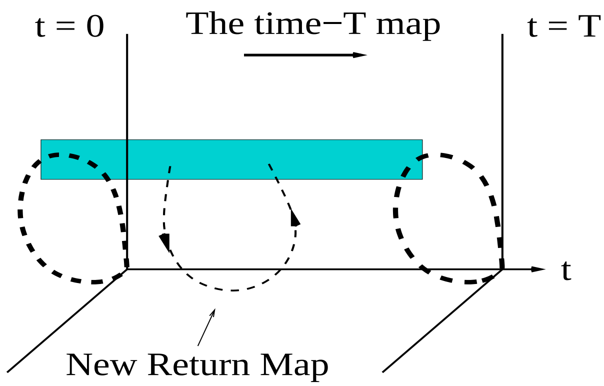
• Fundamental object: Λ = the maximum set of solutions staying around the homoclinic loop.



• Fundamental question: The geometric and dynamical structure of Λ?

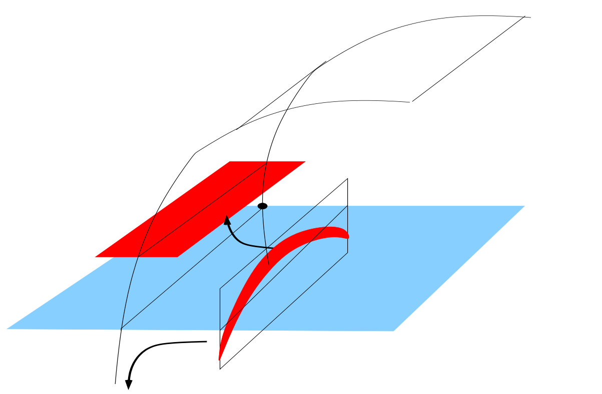
— Horseshoe is only a participating part.

The separatrix map $\mathcal{R}$



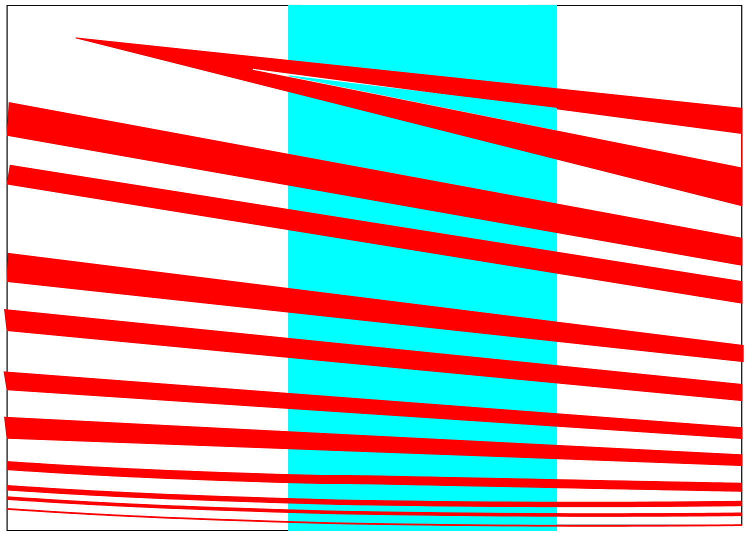
— $\mathcal{R}$ is an annulus map proposed by Afraimovich and Shilnikov.

— It is only partially defined on the annulus.



We rigorously derived a formula for $\mathcal{R}$ from the equation.

The structure of homoclinic tangle



- $\mathcal{R}$ is defined on vertical strips.
- The action of $\mathcal{R}$ on one strip
  - — Compressing in vertical direction
  - — Stretching in horizontal direction, to infinity in length towards both end
  - — Folded, and put back into $\mathcal{A}$.
- The image moves horizontally in a constant speed with respect to $a \sim \ln \mu^{-1}$ as $\mu \to 0$.

Dynamical Consequences

(I) There exists an infinitely many disjoint open intervals for μ as $\mu \to 0$, such that Λ_μ is conjugate to a horseshoe of infinitely many branches.

(II) There exists an infinitely many disjoint open intervals for μ as $\mu \to 0$, such that Λ_μ is the union of an periodic sink and a horseshoe of infinitely many branches.

(III) There are infinitely many open intervals of μ for μ as $\mu \to 0$, such that Λ_μ admits Newhouse tangency.

— Newhouse sinks;

— Hénon attractors

(IV) As $\mu \to 0$, there is a periodic pattern of dynamical behavior Λ_μ would repeat in an accelerated fashion as $\mu \to 0$.

Part III

Numerical Investigation

Equation of Study

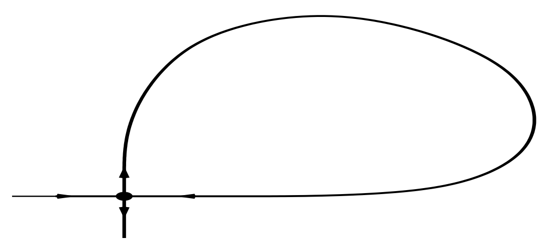
Step 1: We start with the Duffing equation

$$\frac{d^2q}{dt^2} - q + q^2 = 0.$$

Step 2: Add non-linear dumping terms

$$\frac{d^2q}{dt^2} + (\lambda - \gamma q^2)\frac{dq}{dt} - q + q^2 = 0.$$

For $\lambda > 0$ small, there exists γ_λ so that it has a homoclinic loops to a dissipative saddle.



Step 3: Add periodic perturbations

$$\frac{d^2q}{dt^2} + (\lambda - \gamma_\lambda q^2)\frac{dq}{dt} - q + q^2 = \mu \sin 2\pi t$$

Numerical Simulation

- Numerical scheme

  — Fourth-order Runge-Kutta routine

- Values of λ and γ

$$\lambda = 0.5, \quad \gamma_\lambda = 0.5770285901$$

- Initial phase position

$$(q_0,\ p_0) = (0.01,\ 0.0).$$

- Parameters varied

  — μ ranged from 10^{-3} to 10^{-7}.

  — t_0 ranged in $[0, 1)$.

  — Compute the solution for each fixed combination of μ and t_0.

Simulation Results

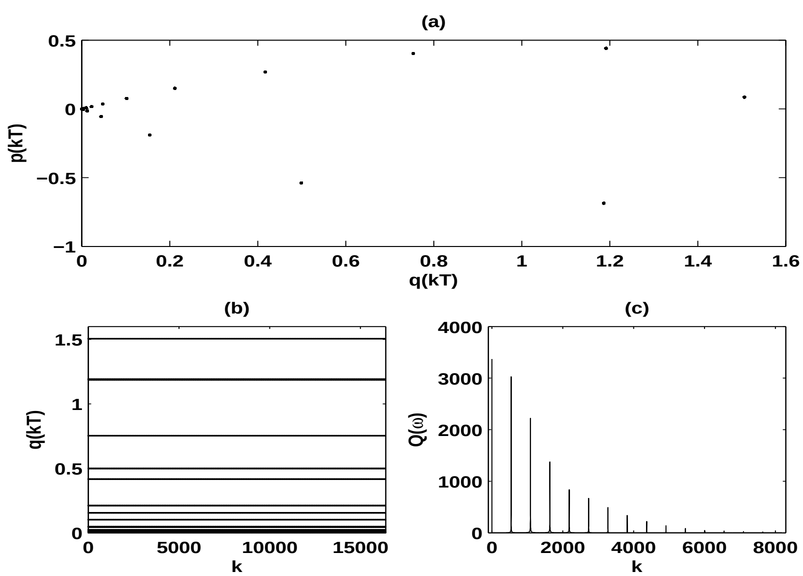
- Transient Tangle

— Let t_0 run over $[0, 1)$ with an increment of 0.001.

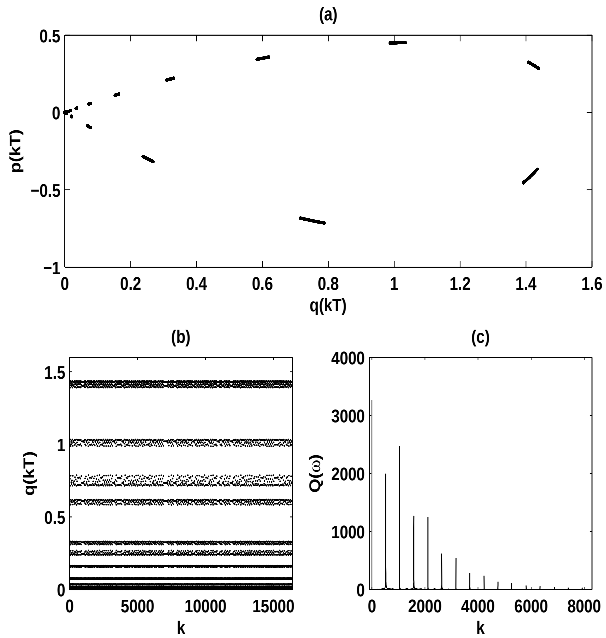
— all 1,000 solutions leave the neighborhood of the homoclinic solution.

— Homoclinic tangle contains no object *directly observable*.

- Periodic Sink



- Hénon-Like Attractor



— These are a plot of a strange attractor associated to a Newhouse tangency.

— This is the kind of Chaos predicted by Mora-Viana based on Benedick-Carleson Theory on Hénon Maps.

Periodicity of Dynamical Behavior

Theoretical Multiplicity $= e^{\beta T} = 2.1831$		
μ	Dynamics	Actual Ratio
$7.041 \cdot 10^{-5}$	Transient	2.1815
$3.415 \cdot 10^{-5}$	Chaotic	2.1977
$3.342 \cdot 10^{-5}$	Non-chaotic	2.1867
$3.224 \cdot 10^{-5}$	Transient	2.1839
$1.574 \cdot 10^{-5}$	Chaotic	2.1696
$1.504 \cdot 10^{-5}$	Non-chaotic	2.2221
$1.474 \cdot 10^{-5}$	Transient	2.1872
$7.190 \cdot 10^{-6}$	Chaotic	2.1892
$6.931 \cdot 10^{-6}$	Non-chaotic	2.1700
$6.732 \cdot 10^{-6}$	Transient	2.1895
$3.272 \cdot 10^{-6}$	Chaotic	2.1974
$3.149 \cdot 10^{-6}$	Non-chaotic	2.2010
$3.060 \cdot 10^{-6}$	Transient	2.2000
$1.477 \cdot 10^{-6}$	Chaotic	2.2153
$1.448 \cdot 10^{-6}$	Non-chaotic	2.2182
$1.378 \cdot 10^{-6}$	Transient	2.2206
$6.547 \cdot 10^{-7}$	Chaotic	2.2560
$6.500 \cdot 10^{-7}$	Non-chaotic	2.2277

Summary

(I) A Theory on Homoclinic Tangles

We provided a comprehensive description on the overall dynamical structure of homoclinic tangles from a periodically perturbed homoclinic solution.

(II) Theories on maps come together

Horseshoes, Newhouse sinks and Hénon-like attractors all fall into their places as part of a larger picture.

(III) Applications

Our results can be applied to the analysis of given equations, such as Duffing equation.

(IV) A systematic numerical investigation

We know fully what to expect in numerical simulations around a periodically perturbed homoclinic solution.