Periodic Sinks and Observable Chaos

Systems of Study: Let $M = S^1 \times \mathbb{R}$. $T_{a,b,L} : M \to M$ is a three-parameter family of maps defined by

$$\begin{aligned} \theta_1 &= a + \theta + L \sin 2\pi\theta + r \\ r_1 &= br + bL \sin 2\pi\theta \end{aligned}$$

where $\theta \in S^1$, $r \in \mathbb{R}$.

Outline of Contents:

(I) Preliminaries and Numerical Results

(II) Analytic Justifications.

Objectives: Using the maps above as a motivating example, we demonstrate (a) What the theory of rank one maps is about (b) Conclusions when applied to the maps above.

I. Preliminaries and Numerical Results

– $T: M \to M$: a diffeomorphism; DT_z be the Jacobi matrix for T at $z = (\theta, r) \in M$.

– $z \in M$ is a **periodic point** of period n if $T^n(z) = z$.

– Let z be a periodic point of period n and λ_1, λ_2 be the two eigenvalues of DT_z^n. z is a **hyperbolic periodic point** if $|\lambda_1| < 1 < |\lambda_2|$, and it is a **periodic sink** if $|\lambda_1|, |\lambda_2| < 1$.

– Let z be a hyperbolic periodic point. Stable and Unstable manifolds $W^s(z)$ and $W^u(z)$ are 1D curves immersed in M and the eigenvectors of DT_z^n for λ_1 and λ_2 are tangent to $W^s(z)$ and $W^u(z)$, respectively, at z.

– A bounded open set $U \subset M$ is a **trapping region** for $T: M \to M$ if $T(\bar{U}) \subset U$ where $\bar{U}$ is the closure of U.

– For a given trapping region $U \subset M$, let

$$\Lambda = \cap_{k \geq 0} T^k(\bar{U}).$$

Λ is a compact subset that is **invariant** under T and it is what we will refer to as an **attractor** for T.

– We also refer to the set

$$B(\Lambda) := \{z' \in M : \lim_{k \to \infty} d(T^k(z'), \Lambda) = 0\}$$

as the **basin of attraction** for Λ, where

$$d(T^k(z'), \Lambda) := \min_{z \in \Lambda} |T^k(z') - z|$$

.

– We say that $z_0 \in M$ has a *positive Lyapunov exponent* if

$$\limsup_{k \to \infty} \frac{1}{k} \ln |DT^k_{z_0} u| > 0$$

for some unit vector u in the tangent space of M at z_0.

Positive Lyapunov exponents are a trademark for *local instability* and chaotic dynamics.

Horseshoes and Axiom Solenoid

(a) Geometric description

(b) Dynamical structure

(c) Uniformly hyperbolic systems

Non-uniformly Hyperbolic Systems

(a) Non-conventional Horseshoes

(b) Rank one solenoid

(c) Homoclinic tangle

• **No** practical systems are uniformly hyperbolic,

• Much harder to gain comprehensive knowledge,

• Modern Chaos Theory: Homoclinic tangle $\rightarrow$ Horseshoes $\rightarrow$ Chaos.

Proposition *Let $T = T_{a,b,L}$. Assume that $L \geq 2$, $0 < 2\pi L|b| < 0.1$. Then T has an attractor Λ and inside Λ there exists a horseshoe. It also follows that T admits homoclinic tangles.*

Proof: Let $T = T_{a,b,L}$ and assume that $L \geq 2$ and $0 < 2\pi L|b| < 0.1$. For $z = (\theta, r) \in M$,

$$DT_z = \begin{pmatrix} 1 + 2\pi L \cos 2\pi\theta & 1 \\ 2\pi b L \cos 2\pi\theta & b \end{pmatrix},$$

from which it follows that $\det(DT_z) = b \neq 0$ so T is a diffeomorphism. Let $U = \{(\theta, r) \in M : \ |r| < 1\}$. $T(\bar{U}) \subset U$ from definition so U is a trapping region. Let $\Lambda = \cap_{k>0} T^k(\bar{U})$ be the attractor for T.

For horseshoe in Λ, we first let

$$I_1 = \{\theta : \ \theta \in [-0.2116, 0.2116]\},$$
$$I_2 = \{\theta : \ \theta \in [0.3151, 0.6849]\}.$$

$V_1, V_2 \subset M$ are such that

$$V_1 = I_1 \times [-1, 1], \quad V_2 = I_2 \times [-1, 1].$$

Let $H_1 = T(V_1)$, $H_2 = T(V_2)$. H_1 crosses both V_1 and V_2 in horizontal direction and so does H_2, creating a horseshoe.

For a rigorous proof, we need to further construct *invariant cones* in the tangent space to identify precisely the directions of contraction and expansion for all $z \in V_1 \cup V_2$.

Numerical Results on $T = T_{a,b,L}$

– Previous Proposition hold for all maps used in simulations.

– An initial point $z_0 \in U$ is arbitrarily picked and $z_k = (\theta_k, r_k) := T^k(z_0)$ are numerically computed for $k = 1, 2, \cdots$ up to $k = 2^{14}$.

– (θ, r) are regarded as polar coordinates and their rectangular correspondences (x, y) are obtained through $x = (1 + r)\cos 2\pi\theta$, $y = (1 + r)\sin 2\pi\theta$.

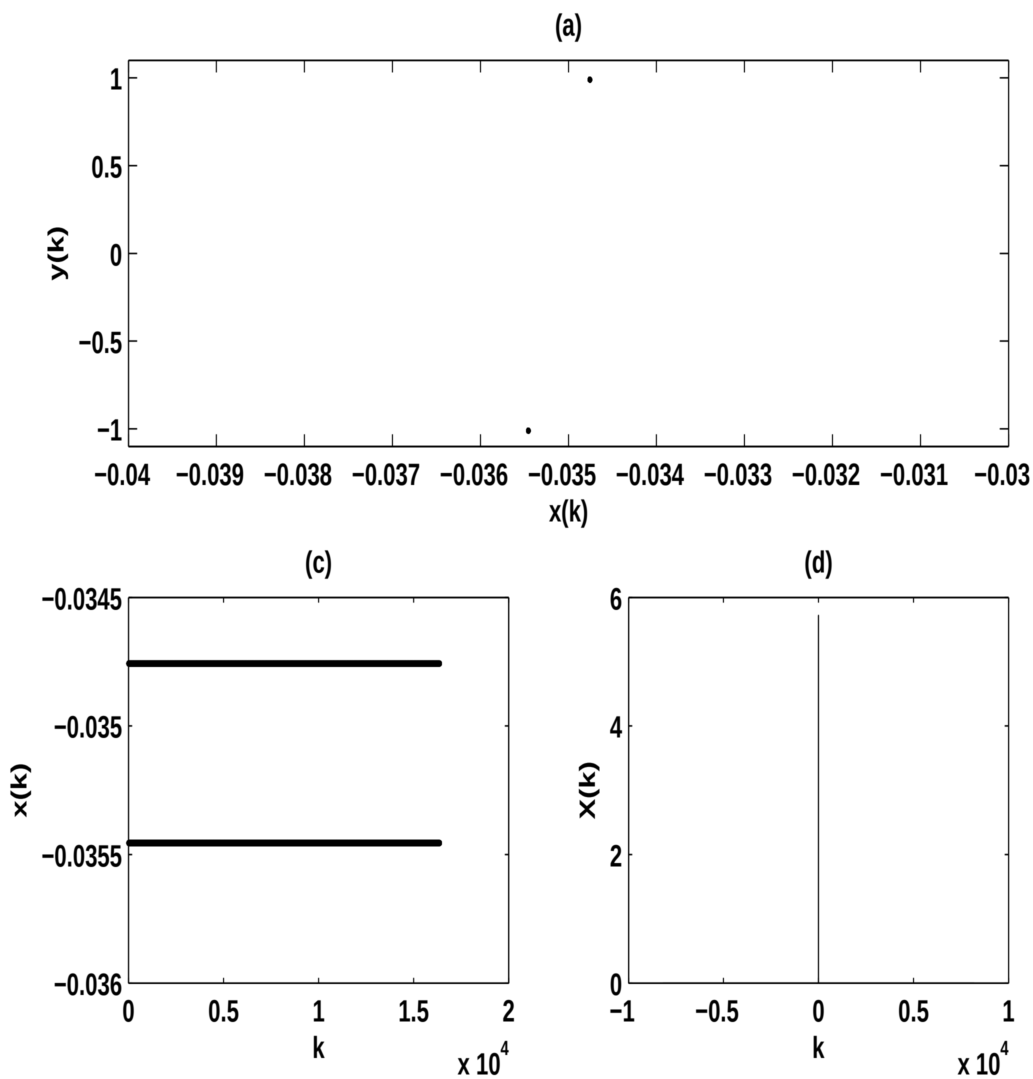
– Three pictures are plotted:

(a) the x-coordinate versus time, (k, x_k),

(b) $(x_k, y_k), k \in [2^{10}, 2^{14}]$,

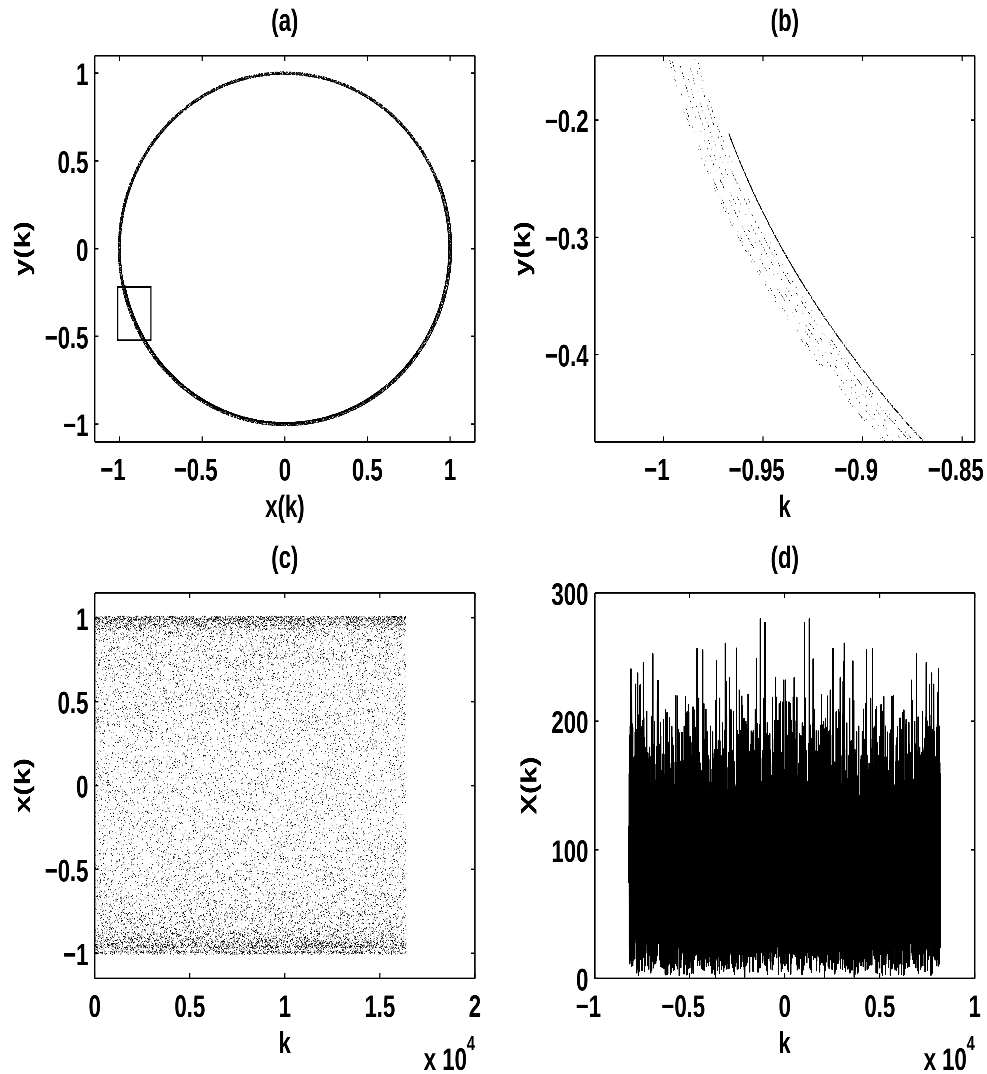
(c) The frequency spectrum of x_k.

Scenario (a): Periodic sinks



A periodic sink for $L = 2.0$, $b = 0.005$, $a = 0.5$. (a) Phase portrait $x_k - y_k$. (b) Time evolution of x_k. (c) Frequency spectrum of x_k.

Scenario (b): Observable Chaos



$L = 2.0$, $b = 0.005$, $a = 0.8$. (a) Phase portrait $x_k - y_k$. (b) Magnification. (c) Time evolution of x_k. (d) Frequency spectrum of x_k.

(1) The results of the numerical simulations above are typical. They occur frequently in the simulations of systems of various applications. What we present here is a *prototype* of behavior for systems with non-uniform expansions.

(2) Proposition above for $T_{a,b,L}$, though simple and elegant, is far from being sufficient for explaining the numerical plots of this subsection.

– It appears paradoxical for Scenario (a) since the complicated structures of horseshoes and homoclinic tangles proved to exist do not show up, i.e., not observed in simulations.

– For Scenario (b), other than the fact that the plots are complicated, there exists no valid mathematical argument in linking these plots to the horseshoes.

Objectives: Analytically justify the results of numerical simulations presented above, at least partially.

A subjective standard:

– *observability:* We regard an event in $\mathbb{R}^n$ as *observable* only if it happens on a set of positive Lesbegue measure in $\mathbb{R}^n$.

– *Observability in phase space:* Let Λ be the attractor for T. For a subset $S \subset \Lambda$, let

$$B(S) := \{z \in U : \lim_{k \to \infty} d(T^k(z), S) = 0\}$$

be the basin of attraction for S. S is observable if $m(B(S)) > 0$ where $m(\cdot)$ stands for the Lebesgue measure,

– *Observability in parameter space:* For $T_{a,b,L}$, a family of maps of three parameters, there is also an issue of observability in the parameter space. A dynamical scenario is *observable in the parameter space* only if it holds for a set of parameters of positive Lebesgue measure.

(II) Analytic Justifications

Horseshoes of the previous Proposition do not occur in simulation because they are **not** observable ($m(B(D)) = 0$). In this sense *Chaos proved through the existence of horseshoes are not very relevant with regarding to the simulation results*.

Justifications for Periodic Sink: We provide two justifications for periodic sinks.

(1) *A 1D argument:* Let us start with the 1D family $f_{a,L} : S^1 \to S^1$,

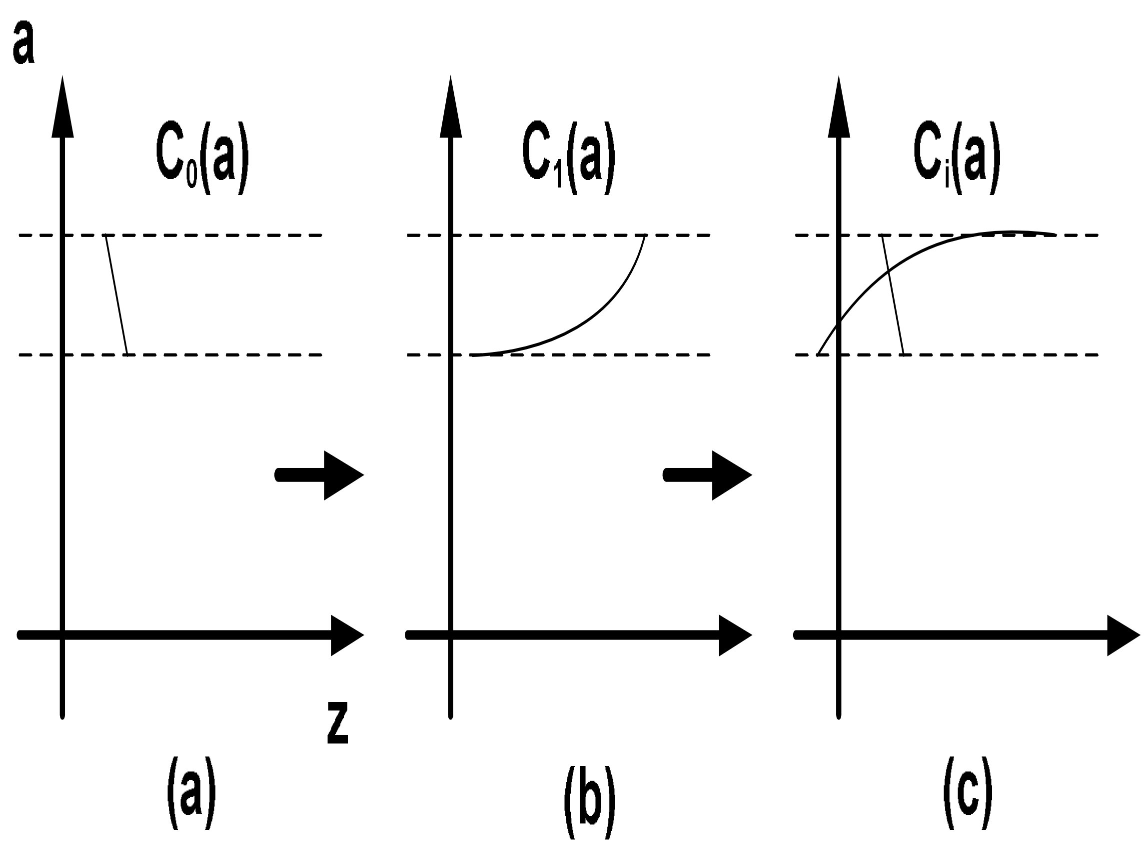
$$f_{a,L}(\theta) = a + \theta + L \sin 2\pi\theta. \qquad (1)$$

– Let $L > 2$ be fixed, and denote the 1D family $f_{a,L}$ as $f_a : S^1 \to S^1$.

– We argue that there is a set for a such that f_a admits periodic sinks.

• Let $\mathcal{C}(f_a) = \{z \in S^1 : f_a'(z) = 0\}$ be the set of critical points for f_a. For $c_0(a) \in \mathcal{C}(f_a)$ let $c_i(a) = f_a^i(c_0(a))$. For a small interval Δ_0 suitably picked for a, we call $c_i(a) : \Delta_0 \to S^1$ a critical curve

• We plot $z = c_i(a)$ on the (z, a)-plane for $i = 0, 1, \cdots$.



– Since sinks persist under small perturbations, a periodic sink for $f_{\hat{a}}$ implies a periodic sink for $T_{a,b,L}$ for all b small and a sufficiently close to $\hat{a}$. Consequently,

(i) periodic sinks are observable in the phase space; and

(ii) the maps admitting periodic sinks are observable in the parameter space.

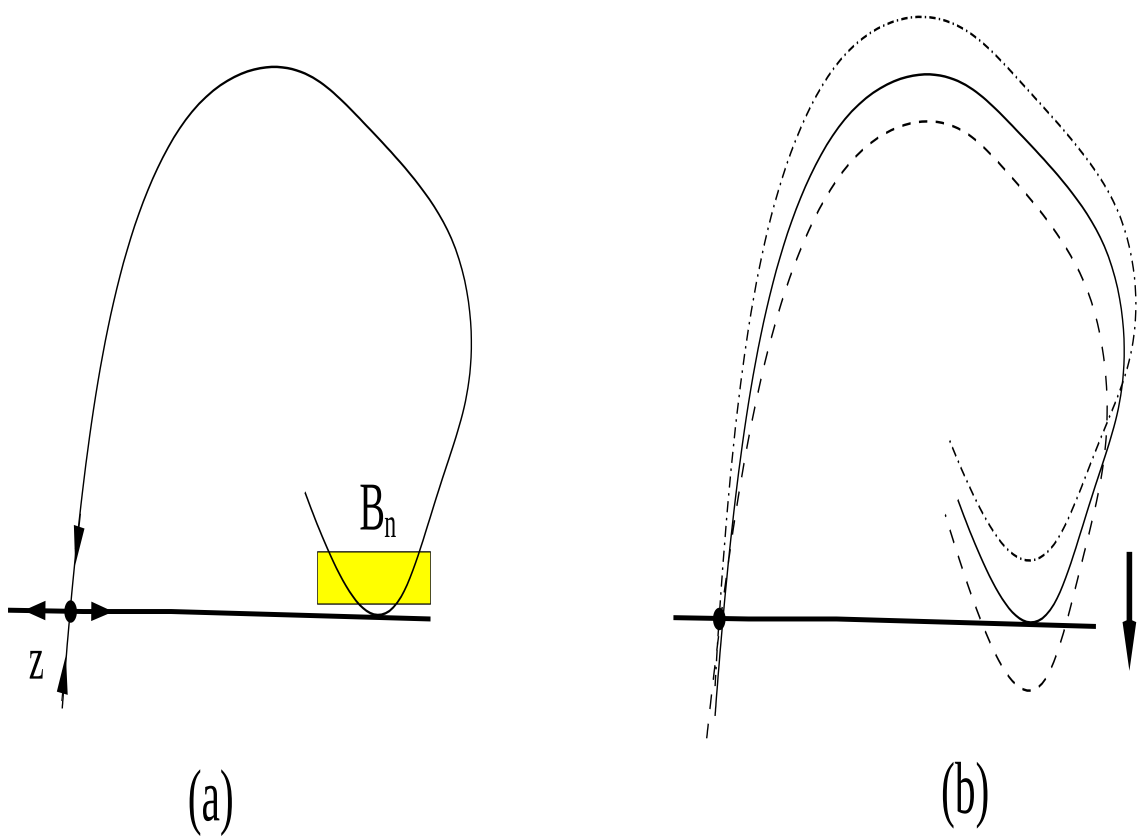
(2) *A 2D argument (Newhouse sinks)*

– Let $T_\mu : M \to M$ be a one parameter family of maps. Assume that

(i) *(Dissipative hyperbolic fixed point)* $T = T_0$ has a fixed point p_0 that is hyperbolic ($|\lambda_1| < 1 < |\lambda_2|$ where λ_1, λ_2 are the eigenvalues of DT_{p_0}) and *dissipative* ($|\lambda_1 \lambda_2| < 1$).

(ii) *(Quadratic tangency)* The stable and unstable manifolds of p_0 have a non-transversal intersection that is non-degenerate.

(iii) *(Parameter transversality)* Let p_μ be the continuations of p_0 and W^s_μ, W^u_μ be the stable and unstable manifolds of p_μ, respectively. Then, as μ moves through $\mu = 0$, W^u_μ crosses W^s_μ transversally.



– The following was observed by Newhouse.

(1) For every n sufficiently large, there exists an open region B_n (with $\mathrm{diam}(B_n) \to 0$ as $n \to \infty$) close to the point of tangency and a range of parameters μ (also depending on n) such that $T_\mu^n(B_n) \subset B_n$.

(2) By a proper change of coordinates, $T_\mu^n : B_n \to B_n$ is transformed to become

$$\hat{T}_{a,b} : [-2,2] \times [-2,2]$$

where $\hat{T}_{a,b}$ is written as

$$\begin{aligned} x_1 &= 1 - ax^2 + y + bu \\ y_1 &= bv \end{aligned}$$

Here a, b are constants in μ, λ_i and n, and u, v are functions of (x, y) and μ, λ_i and n, the C^1-norms of which are bounded from above by constants independent of n.

(3) $b \to 0$ as $n \to \infty$, and for every n that is sufficiently large, there exists a range of μ close to $\mu = 0$, such that the corresponding values of a for $\widehat{T}_{a,b}$ covers the interval $[1, 2]$.

– For any given b that is sufficiently small, it is a triviality to find values of $a \in [1, 2]$ so that $\widehat{T}_{a,b}$ has a fixed point that is a sink. It then follows that, for every n that is sufficiently large, there exists μ_n so that $T^n_{\mu_n}$ admits a periodic sink of period n, and hence the existence of infinitely many sinks.

Existence of tangles without periodic sink

– The plots of Scenario (b) implicating a form of chaos that is observable.

– On the other hand, there remains, however unlikely, the possibility that a plot of Scenario (b) is part of a periodic sink of exceedingly long period, reducing the significance of Scenario (b) from an independent dynamical scenario to a particular case of Scenario (a).

– This possibility can never be ruled out through numerical computations. It can only be partially ruled out through mathematical analysis.

– To argue that Scenario (b) is **not** that of a periodic sink, we would need to prove that the attractor admitting no periodic sinks is **observable** in both the parameter space and the phase space for $T_{a,b,L}$.

Misiurewicz Maps Misiurewicz observed that

– In order for a 1D map to have no periodic sinks, it suffices that the forward orbits from $\mathcal{C}(f)$ stay a fixed distance away from $\mathcal{C}(f)$.

Denoting $\mathcal{C}(f) = \{c^{(1)}, \cdots, c^{(q)}\}$, this is to say that we have a $\delta_0 > 0$, so that

$$d(f^k(c^{(i)}), \mathcal{C}(f)) > \delta_0$$

for all $k > 0$ and all $1 \leq i \leq q$.

– To find maps in a 1D family $\{f_a\}$ satisfying the Misiurewicz condition let us study the evolutions of critical curves. We have

Proposition *There exists $L_0 > 2$, such that for all $L > L_0$, there exist values of parameter a such that*

(i) $f = f_{a,L}$ satisfies the Misiurewicz condition above, and

(ii) there exists $\lambda > 0$ such that for almost every $\theta \in S^1$,

$$\limsup_{k\to\infty} \frac{1}{k} \ln |(f^k)'(\theta)| > \lambda.$$

Conclusion: No periodic sinks for Misiurewicz maps f.

Question Mark: Observability in parameter space

– Conclusions of the above Proposition **do not** necessarily persist under small perturbations because the Misiurewicz condition is not an open condition maintainable in the parameter space.

– For a typical 1D family $\{f_a\}$, the set of parameter values fulfilling the Misuirewicz condition is **of Lebesgue measure zero**, therefore, it is **not observable** in the parameter space.

– To find more parameters admitting no periodic sinks, it is natural for us to relax the Misiurewicz condition to allow critical orbits approaching $\mathcal{C}(f)$ in controlled manners. This has turned out to be an extremely sophisticated mathematical task over which long theories have been built, first by Jakobson on quadratic maps, then followed by many others

– In particular, we have

Proposition *There exists $L_0 > 2$, such that for all given $L > L_0$, there is a set Δ of positive Lebesgue measure for parameter a, such that for $a \in \Delta$ and $f = f_{a,L}$, there exists $\lambda > 0$ such that for almost every $\theta \in S^1$,*

$$\lim_{k\to\infty} \frac{1}{k} \ln |(f^k)'(\theta)| > \lambda.$$

– This Proposition is significantly different from those of previous Proposition. By asserting a good parameter set of **positive** Lebesgue measure, it establishes maps admitting no sinks as a dynamical scenario observable in the parameter space.

– The likes of this Proposition are commonly referred to as versions of the Jakobson's theory.

– **Comparison of good parameter sets**

- Cantor set K_ε for Misiurewicz maps.

- *Fat Cantor set* for Jakobson's theory.

Theory of rank one maps: 2D correspondence of Jakobson's theory

Proposition *There exists $L_0 > 2$, such that for every $L > L_0$, there exists $b_0 > 0$ sufficiently small so that for all $0 < |b| < b_0$, there exists a set $\Delta_{b,L}$ of positive Lebesgue measure, such that for $a \in \Delta_{b,L}$, $T = T_{a,b,L}$ has a positive Lyapunov exponent for almost all $z \in M$.*

We will, from this point on, adopt a point of view as follows:

(1) the plots of Scenarios (a) and (b) are distinct;

(2) Scenario (a) corresponds to attractors dominated by periodic sinks;

(3) Scenario (b) corresponds to attractors with no periodic sinks;

(4) a typical orbit of Scenario (b) has a positive Lyapunov exponent.

Remember that positive Lyapunov exponents are trademark for chaos.

Question: What is plotted in scenarios (b)?

Answer: SRB measures that represent an existing *statistical law* in chaos.

A. *Borel measures* We start with some elementary measure theory.

– Let M be a smooth surface compactly embedded in $\mathbb{R}^n$, m be the Lebesgue measure induced from the surface area. Let $\mathcal{B}$ be the collection of all subsets of M that is m-measurable. $\mathcal{B}$ is the *Borel algebra* for M.

– We call a function μ from $\mathcal{B}$ to $\mathbb{R}^+$ a *Borel measure* if

(i) $\mu(\emptyset) = 0, \quad 0 < \mu(M) < \infty$ where $\emptyset$ is the empty set, and

(ii) for mutually disjoint $A_i \in \mathcal{B}, i = 1, \cdots,$ $\mu(\cup_i A_i) = \sum \mu(A_i)$.

Examples: (a) The Lebesgue measure on M is a Borel measure. If the measure of the entire space equals one, then the measure is also called a *probability* measure.

(b) For a given M in $\mathbb{R}^n$, there are infinitely many ways to define various Borel measures on M. For instances, let $M = [0, 1]$. We first take a finite set of points $P = \{p_1, \cdots, p_n\}$ in $[0, 1]$, then define $\mu(A) = \frac{1}{n} card(A \cap P)$ where $card(A \cap P)$ is the number of points in $A \cap P$ for $A \in \mathcal{B}$. Measures defined this way are *atomic*, meaning that they are positively defined on isolated points.

(c) We can also define Borel measures of a different kind. First let m be the Lebesgue measure on $[0, 1]$ and $\rho : [0, 1] \to \mathbb{R}^+$ be a Lebesgue measurable function satisfying $0 < \int_{[0,1]} \rho dm < \infty$, then $\mu : \mathcal{B} \to \mathbb{R}^+$ defined by

$$\mu(S) = \int_S \rho dm$$

for $S \in \mathcal{B}$ is a Borel measure. Observe that these Borel measures, defined by using density functions ρ, are non-atomic. In fact, for any given $S \in \mathcal{B}$ such that $m(S) = 0$ we have $\mu(S) = 0$.

– In general, let μ, ν be two Borel measures defined on the same Borel algebra. We say that μ is *absolutely continuous* with respect to ν if $\nu(A) = 0$ implies $\mu(A) = 0$. For the Borel measures defined on $[0, 1]$ above, the first kind is atomic and the second kind is non-atomic and is absolutely continuous with respect to Lebesgue.

B. *Invariant measures and ergodicity*

– Two conceptually different ways to proceed with respect to invariant measures:

(a) To fix a Borel measure μ that is predominately important, such as the Lebesgue measure m induced from a surface area, and study the properties of T with respect to such prefixed μ.

(b) To find particular Borel measures that are most interesting and suitable for the studies of a given T.

– A Borel measure μ is *an invariant measure* for T if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$. Let μ be an invariant measure for T. μ is an *ergodic* measure for T if all invariant subsets for T are either of full μ-measure or of null μ-measure. Recall that $A \in \mathcal{B}$ is an *invariant subset* for T if $T^{-1}A = A$.

– For $S \in \mathcal{B}$ let $\mathcal{X}_S$ be the characteristic function of S defined as follows:

$$\mathcal{X}_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

For any given $x \in M$, let

$$I_{S,n}(x) = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{X}_S(T^k(x))$$

$$I_S(x) = \lim_{n \to \infty} I_{S,n}(x).$$

–$I_{S,n}(x)$ is the percentage of the first n points along the orbit starting from x that fall in S and $I_S(x)$ is the limit of this percentage.

Theorem (Birkhoff's Ergodic Theorem) *Let μ be an ergodic invariant measure for $T : M \to M$. Then for all $S \in \mathcal{B}$ and for μ-almost every point $x \in M$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{X}_S(T^i(x)) = \frac{\mu(S)}{\mu(M)}.$$

– The amount of information held by an ergodic invariant measure μ about T through Birkhoff's Ergodic Theorem could be minimal.

For instance, let $P = \{p_1, \cdots, p_n\}$ be a periodic orbit of period n. Then the atomic Borel measure μ supported on P by using $\mu(p_i) = \frac{1}{n}$ is an ergodic invariant measure. With respect to this invariant Borel measure, P is a set of full measure and consequently, Birkhoff's Ergodic Theorem claims that all orbits starting from P are dictated by μ, a completely trivial statement.

– Invariant measures absolutely continuous with respect to the Lebesgue measure on M are usually much more meaningful. They are supported on m-positive sets, therefore, are at the very least *observable* based on Birkhoff's Ergodic Theorem.

C. *Invariant measures and observability*

– Let μ be an invariant measure for T and $x \in M$, not necessarily in the support of μ. We want to figure out an appropriate way to say that the orbit of x is dictated by μ.

– A naive try would be to copy the conclusions of Birkhoff's Ergodic Theory to say that the orbit of x is dictated by μ if $I_S(x) = \frac{\mu(S)}{\mu(M)}$ for all $S \in \mathcal{B}$.

This would be a bad definition: Let μ be the atomic invariant measure supported on a periodic sink $P = \{p_1, \cdots, p_n\}$, and $x \in M$ be sufficiently close to p_1. The orbit of x is attracted to the periodic orbit of p_1, and it ought to be obvious that we have every intention to claim that the orbit of x is dictated by μ. The definition proposed above, however, would fail such a claim.

– A good definition is as follows: Let $x \in M$ be a starting point, and μ be an invariant measure for T. We say that the space distribution of the points of the orbit starting from x is dictated by μ, or in short, we say that x is *generic* with respect to μ, if for all *continuous* functions $\phi : M \to R$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x)) = \frac{1}{\mu(M)} \int_M \phi(x) d\mu.$$

– By replacing the characteristic functions with continuous ones, we are now allowed to reach the part of the phase space that is out of the support of μ. This definition will serve us well.

– Let m be the Lebesgue measure on M. We say that an ergodic invariant measure μ for $T : M \to M$ is *observable* if there exists a subset S that is m-positive such that every $x \in S$ is generic with respect to μ.

(a) Any periodic sink would define an observable atomic ergodic invariant measure. On the other hand, atomic measures defined by hyperbolic periodic orbits are in general not observable.

(b) According to Birkhoff's Ergodic Theorem, any ergodic invariant measure μ that is absolutely continuous with respect to m is observable.

Proposition (Jacobson's Theory) *For the good parameters* Δ*, and for any given* $a \in \Delta$*,* f_a *admits an invariant probabilistic measure that is absolutely continuous with respect to Lesbegue.*

C. *SRB measure for 2D maps*

– We now consider Scenario (b) for 2D maps $T_{a,b,L} : M \to M$. The numerical plots again point towards a non-atomic invariant measure μ for T. However, unlike the case of 1D, such invariant measures are **not** absolutely continuous with respect to the Lesbegue measure on M. This is because Λ, the attractor that supports μ, is a set of zero Lesbegue measure in M.

– It appears that the strongly dissipative nature of T crashes M into Λ along the radial direction, and the absolute continuity of the plotted measure with respect to the Lebesgue measures is preserved only in the angular direction by the expanding nature of $f_{a,L}$ in θ.

– As a rigorously defined mathematical object, the likes of such invariant measures, crashed in the contractive direction but remaining absolutely continuous with respect to the Lebesgue measures in the directions of expansion, were formally introduced by Sinai, Ruelle and Bowen originally in their studies of axiom A systems. These are the so-called *SRB measures*.

– A formal definition is as follows: A T-invariant Borel measure μ is called an **SRB measure** if

(i) T has a positive Lyapunov exponent μ-a.e.;

(ii) the conditional measures of μ on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these leaves.

– Note that (i) requires that μ is supported by a collection of orbits that are chaotic in nature. It is also proved that, in general, the set of points that are generic with respect to an SRB measure is always m-positive, therefore SRB measures are observable. This last property makes an SRB measure non-ignorable in simulations.

– SRB measures are observable for $T_{a,b,L}$ in both the parameter and the phase spaces

Recall that

$$U = \{(\theta, r),\ |r| < 1\}$$

is the trapping region.

Proposition *For the set of good parameters,*

(i) $T = T_{a,b,L}$ *admits a unique ergodic SRB measure* μ *such that* $0 < \mu(U) < \infty$*; and*

(ii) Lesbegue almost every point $z \in U$*, is generic with respect to* μ*.*

The proof of this Proposition is an application of the theory of rank one maps.

Conclusion: The plots for both Scenarios (a) and (b) are invariant measures that control the space distribution of individual trajectories for at least an observable set of orbits in the basin of attraction for Λ. For Scenario (a), the invariant measure is atomic, representing a periodic sink. For Scenario (b), it is an SRB measure representing an existing *statistical law* for chaos.