Theory of 1D maps

(I) Setting and Jakobson's Theorem

A. Misiurewicz maps. Let I be either an interval or a circle. For $f \in C^2(I, I)$, let $\mathcal{C} = \mathcal{C}(f) = \{x \in I : \ f'(x) = 0\}$ denote the critical set of f, and let $\mathcal{C}_\delta$ denote the δ-neighborhood of $\mathcal{C}$ in I. For $x \in I$, let $d(x, \mathcal{C}) := \min_{\widehat{x} \in \mathcal{C}} |x - \widehat{x}|$.

– Conceptually, Misiurewicz maps are 1D maps with the following characteristics:

(a) the critical orbits (orbits of $x \in \mathcal{C}$) stay a fixed distance away from the critical set, and

(b) the phase space is divided into two regions, $\mathcal{C}_{\delta_0}$ (the δ_0-neighborhood of the critical set $\mathcal{C}$) and $I \setminus \mathcal{C}_{\delta_0}$; and

(i) on $I \setminus \mathcal{C}_{\delta_0}$, f is uniformly expanding;

(ii) for $x \in \mathcal{C}_{\delta_0} \backslash \mathcal{C}$, even though $|f'(x)|$ is small, the orbit of x does not return to $\mathcal{C}_{\delta_0}$ again until its derivative has regained a definite amount of exponential growth.

– A precise definition is as follows:

Definition *We say $f \in C^2(I, I)$ is a Misiurewicz map, denoting $f \in \mathcal{M}$, if the following holds for some $\delta_0 > 0$:*

(a) Outside of $\mathcal{C}_{\delta_0}$: *there exist $\lambda_0 > 0, M_0 \in \mathbb{Z}^+$ and $0 < c_0 \leq 1$ such that*

(i) for all $n \geq M_0$, if $x, f(x), \cdots, f^{n-1}(x) \notin \mathcal{C}_{\delta_0}$, then $|(f^n)'(x)| \geq e^{\lambda_0 n}$;

(ii) if $x, f(x), \cdots, f^{n-1}(x) \notin \mathcal{C}_{\delta_0}$ and $f^n(x) \in \mathcal{C}_{\delta_0}$, for any n, then

$$|(f^n)'(x)| \geq c_0 e^{\lambda_0 n}.$$

(b) Inside $\mathcal{C}_{\delta_0}$:

(i) $f''(x) \neq 0$ *for all* $x \in \mathcal{C}_{\delta_0}$*;*

(ii) for all $\hat{x} \in \mathcal{C}$ *and* $n > 0$, $d(f^n(\hat{x}), \mathcal{C}) \geq \delta_0$*;*

(iii) for all $x \in \mathcal{C}_{\delta_0} \setminus \mathcal{C}$*, there exists* $p_0(x) > 0$ *such that* $f^j(x) \notin \mathcal{C}_{\delta_0}$ *for all* $j < p_0(x)$ *and* $|(f^{p_0(x)})'(x)| \geq c_0^{-1} e^{\frac{1}{3}\lambda_0 p_0(x)}$*.*

– Condition (a) says that on $I \setminus C_{\delta_0}$, f is essentially uniformly expanding. (b)(ii) says that for $x \in \mathcal{C}_{\delta_0} \setminus \mathcal{C}$, if n is the first return time of $x \in \mathcal{C}_{\delta_0}$ to $\mathcal{C}_{\delta_0}$, then $|(f^n)'(x)| \geq e^{\frac{1}{3}\lambda_0 n}$. (To see this, use (b)(ii) followed by (a)(ii)).

Examples:

Ex 1. Let $f \in C^3(I, I)$ be such that

(i) $S_f(x) < 0$ where $S_f(x)$ denotes the Schwarzian derivative of f,

$$S_f(x) = \frac{f'''(x)f'(x) - \frac{3}{2}f''(x)^2}{f'(x)^2}.$$

(ii) $f''(\widehat{x}) \neq 0$ for all $\widehat{x} \in \mathcal{C}$,

(iii) if $f^n(x) = x$, then $|(f^n)'(x)| > 1$, and

(iv) for all $\widehat{x} \in \mathcal{C}$, $\inf_{n>0} d(f^n(\widehat{x}), \mathcal{C}) > 0$.

Then $f \in \mathcal{M}$.

(i)-(iv) are the properties used traditionally in defining Misiurewicz maps, among which (iii) is not directly checkable and (i) is often not fulfilled in applications.

Ex 2. Let $f_{a,L} : S^1 \to S^1$ be given by

$$f_{a,L}(\theta) \ = \ \theta \ + \ a + \ L\Phi(\theta)$$

where $a, L \in \mathbb{R}$ and $\Phi : S^1 \to S^1$ is a Morse function (the right side is mod 1). Then there exists $L_0 > 0$ such that for all $L \geq L_0$, there exists an $\mathcal{O}(\frac{1}{L})$-dense set of a for which $f_{a,L} \in \mathcal{M}$.

Ex 3. The quadratic map $f(x) = 1 - 2x^2$ is a Misiurewicz map.

B. Admissible family of 1D maps. Assume that $F(x, a) : I \times (a_1, a_2) \mapsto I$ is C^2 and let $\{f_a \in C^2(I, I) : a \in (a_1, a_2)\}$ be the one-parameter family of one-dimensional maps defined through $f_a(x) := F(x, a)$.

– We say that f_a is an admissible family if it satisfies two conditions.

(i) First we assume that there exists $a^* \in (a_1, a_2)$ such that f_{a^*} is a Misiurewicz map.

(ii) The second compares the movement of critical points and critical orbits of f_a with respect to parameter a at a^*. This one is a little long to state:

We define the continuations of critical points as follows: For every $c \in \mathcal{C}(f_{a^*})$, continuations $c(a) \in \mathcal{C}(f_a)$ satisfying $c(a^*) = c$ is well-defined around a^*. Let $\mathcal{C}(f_{a^*}) = \{c^{(1)}(a^*) < \ldots < c^{(q)}(a^*)\}$ be the critical set for f_{a^*}. Continuation of $c^{(i)}(a^*)$ is denoted as $c^{(i)}(a)$.

Next we define the continuations of critical orbits. For $c^{(i)}(a^*) \in \mathcal{C}(f_{a^*})$, denote $\xi(a^*) = f_{a^*}(c^{(i)}(a^*))$. Then for all a that is sufficiently close to a^*, there exists $\xi(a)$, a unique continuation of $\xi(a^*)$, such that the orbits $\{f_{a^*}^n(\xi(a^*))\}_{n\geq 0}$ and $\{f_a^n(\xi(a))\}_{n\geq 0}$ have the *same itineraries*, by which we mean that, for any given $n \geq 0$, if $f_{a^*}^n(\xi(a^*)) \in (c^{(j)}(a^*), c^{(j+1)}(a^*))$ then $f_a^n(\xi(a)) \in (c^{(j)}(a),\ c^{(j+1)}(a))$. Furthermore, $a \mapsto \xi(a)$ is differentiable. Note that $\xi(a)$ is not to be confused with $f_a(c^{(i)}(a))$.

Definition *Let $F(x,a) : I \times (a_1, a_2) \mapsto I$ be C^2, and $\{f_a\}$ be such that $f_a(x) := F(x,a)$. We say that $\{f_a\}$ is an admissible family if the following holds:*

(a) There exists $a^* \in (a_1, a_1)$ such that $f_{a^*} \in \mathcal{M}$ is a Misiurewicz map.

(b) Let $c(a)$ and $\xi(a)$ be continuations of $c(a^*) \in \mathcal{C}(f_{a^*})$ and $\xi(a^*) = f_{a^*}(c(a^*))$;

$$\frac{d}{da} f_a(c(a)) \neq \frac{d}{da} \xi(a) \quad at \quad a = a^*.$$

Examples

Ex 1. $f_a = \theta + a + L\Phi(\theta)$, $L > L_0$.

Ex 2. $f_a = 1 - ax^2, \ a^* = 2.$

C. Periodic Sinks

Let $f_a, a \in (a_1, a_2)$ be an admissible 1D family, and $f_{a^*} \in \mathcal{M}$ for some $a^* \in (a_1, a_2)$. First we claim that there are many parameters of periodic sinks around a^*.

• To prove this claim we iterate critical curves. The critical curves $c_n(a)$, defined inductively by $c_n(a) = f_a(c_{n-1}(a))$, satisfying

$$\begin{aligned}\frac{d}{da}c_n(a) =& \frac{\partial F(c_{n-1}(a), a)}{\partial c_{n-1}(a)}\frac{d}{da}c_{n-1}(a) \\ &+ \frac{\partial F(c_{n-1}(a), a)}{\partial a} \\ =& f_a'(c_{n-1}(a))\frac{d}{da}c_{n-1}(a) + \partial_a f_a(c_{n-1}(a)).\end{aligned}$$

• Observe that $|\partial_a f_a|$ is uniformly bounded, therefore negligible provided that $|\frac{d}{da}c_n(a)|$ is sufficiently large (a condition guaranteed to hold for some large n through the assumption of parameter transversality). It then follows that, under the assumption that the critical curves stay out of $\mathcal{C}_{\delta_0}$,

$$\frac{d}{da}c_n \sim (f_a^n)'(c_1(a))$$

grows exponentially in magnitude.

• Consequently, with $c_n(a^*)$ staying out of $\mathcal{C}_{\delta_0}$, the critical curve would cross $\mathcal{C}_{\delta_0}$ repeatedly, creating parameters which admit periodic sinks.

• Since periodic sinks persist under small perturbations, they are observable in both the parameter and the phase spaces.

D. Misiuriwicz Maps

Along the same lines of thinking, we construct the set of parameter $a \in (a_1, a_2)$ for which f_a satisfies Misiuriwicz condition.

• To simplify the situation let us for the moment deal only with maps of one critical point (the uni-modal case).

• We iterate the critical curves forward in time, deleting the part that is over $\mathcal{C}_{\delta_0}$ along the way. The deletions would chop the critical curves into small pieces, each of which we iterate forward.

• Clearly, parameters surviving all deletions are those satisfying Misiuriwicz condition in (a_1, a_2). Excluding the deleted pieces on (a_1, a_2) along the way, we would construct in (a_1, a_2) a parameter set that appears very similar to a Cantor set with, say, a fixed proportion of deletions.

E. Parameters for Jakobson's Theorem

• To construct a parameter set of *positive* Lebesgue measure admitting no periodic sinks, we again follow the same lines of thinking, iterating critical curves forward in time.

• We relax the rule of deletion as follows: Instead of deleting the critical curves over $\mathcal{C}_{\delta_0}$, we delete those over $\mathcal{C}_{\delta_n}$ where

$$\delta_n = \min\{\delta_0, e^{-\alpha n}\}$$

for some $\alpha > 0$.

• By exponentially shrinking the proportions of deletion, we end up constructing a fat Cantor set of positive measure. These parameters are the ones satisfying Jakobson's Theorem.

• There are two things needs to be worked out in details:

(a) By allowing critical orbits to come back close to the critical set, we risk the much needed expansions of critical curves, and the potential losses of derivatives at close returns to $\mathcal{C}(f)$ need to be controlled with caution (the issue of derivative recovery).

(b) The mappings from $c_n(a)$ to $c_{n+1}(a)$ are obviously not linear so the proportion of deletions on (a_1, a_2) is not exactly the same as the proportion of deletions on $c_{n+1}(a)$. This nonlinearity also needs to be carefully maintained along the way of iterations (distortion estimates).

Theorem *Let $f_a, a \in (a_1, a_2)$ be an admissible family of 1D maps. Then there is a positive measure set $\Delta \subset (a_1, a_2)$, such that for all $a \in \Delta$, $f = f_a$ satisfies the following: Let $\mathcal{C}(f) = \{x : f'(x) = 0\}$ be the critical set, then for all $\hat{x} \in C(f)$,*

(a) $d(f^n(\hat{x}), \mathcal{C}(f)) > \min\{\delta_0, e^{-\alpha n}\}$ for all $n \geq 1$; and

(b) $(f^n)'(f(\hat{x})) > ce^{\lambda n}$ for some $c, \lambda > 0$ for all $n \geq 1$.

(II) Phase space analysis

We now present rigorously a proof for Jakobson's Theorem. We start with an analysis on phase space dynamics.

Lemma *Let $f \in \mathcal{M}$ be a Misiuriwicz map. Then there exists $c_0'' > 0$ depending only on f such that for all $\delta < \delta_0$ and $n > 0$:*

(a) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$, then $|(f^n)'(x)| \geq c_0''\delta e^{\frac{1}{3}\lambda_0 n}$;

(b) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$ and $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geq c_0 e^{\frac{1}{3}\lambda_0 n}$.

The set of good maps $\mathcal{G}(f_0)$

• Let $f_0 \in \mathcal{M}$ be fixed, and let δ_0, λ_0, c_1 *etc.* be the constants associated with f_0. We introduce in a neighborhood of each $f_0 \in \mathcal{M}$ an admissible set of perturbations $\mathcal{G}(f_0)$.

• For $\lambda, \alpha, \varepsilon > 0$ and $f \in C^2(I, I)$, we say $f \in \mathcal{G}(f_0; \lambda, \alpha, \varepsilon)$ if $\|f - f_0\|_{C^2} < \varepsilon$ and the following hold for all $\hat{x} \in C = C(f)$ and $n > 0$:

(G1) $d(f^n(\hat{x}), C) > \min\{\frac{1}{2}\delta_0, e^{-\alpha n}\}$;

(G2) $|(f^n)'(f(\hat{x}))| \geq c_1 e^{\lambda n}$.

• Note that with $\lambda < \lambda_0$, (G1) and (G2) are relaxations of the conditions on critical orbits for f_0

• The main result of this section is

Propositoin *Given $f_0 \in \mathcal{M}$, $\lambda < \frac{1}{4}\lambda_0$ and $\alpha < \frac{1}{100}\lambda$, there exists $\delta = \delta(f_0, \lambda, \alpha)$ and $\varepsilon = \varepsilon(f_0, \lambda, \alpha, \delta) > 0$ such that (P1)-(P3) below hold for all $f \in \mathcal{G}(f_0; \lambda, \alpha, \varepsilon)$.*

We now state (P1)–(P3), introducing some useful language along the way.

(P1) Outside of C_δ: (i) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$, then $|(f^n)'(x)| \geq c_1 \delta e^{\frac{1}{4}\lambda_0 n}$;

(ii) if $x, f(x), \cdots, f^{n-1}(x) \notin C_\delta$ and $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geq c_1 e^{\frac{1}{4}\lambda_0 n}$.

Bound period: Let $\hat{x} \in C$, and let $C_\delta(\hat{x}) := (\hat{x}-\delta, \hat{x}+\delta)$. For $x \in C_\delta(\hat{x}) \setminus \{\hat{x}\}$, we define $p(x)$, the *bound period* of x, to be the largest integer such that $|f^i(x) - f^i(\hat{x})| \leq e^{-2\alpha i}$ $\forall i < p(x)$.

(P2) Partial derivative recovery for $x \in C_\delta \setminus C$: For $x \in C_\delta(\hat{x}) \setminus \{\hat{x}\}$,

(i) $\frac{1}{3\ln(\max|f'|)} \log \frac{1}{|x-\hat{x}|} \leq p(x) \leq \frac{3}{\lambda} \log \frac{1}{|x-\hat{x}|}$;

(ii) $|(f^{p(x)})'(x)| > e^{\frac{\lambda}{3} p(x)}$.

Decomposition into “bound” and “free” states: For $x \in I$ such that $f^i(x) \notin C$ for all $i \geq 0$ (for example, $x = f(\hat{x})$ for $\hat{x} \in C$), let

$$t_1 < t_1 + p_1 \leq t_2 < t_2 + p_2 \leq \cdots$$

be defined as follows:

– t_1 is the smallest $j \geq 0$ such that $f^j(x) \in C_\delta$.

– For $k \geq 1$, let p_k be the bound period of $f^{t_k}(x)$, and let t_{k+1} be the smallest $j \geq t_k + p_k$ such that $f^j(x) \in C_\delta$.

(Note that an orbit may return to C_δ during its bound periods, i.e. t_i are not the only return times to C_δ.)

– This decomposes the orbit of x into segments corresponding to time intervals $(t_k, t_k + p_k)$ and $[t_k + p_k, t_{k+1}]$, during which we describe the orbit of x as being in **bound** and **free** states respectively;

– t_k are called times of **free returns**.

A standard partition

We introduce a partition $\mathcal{P}$ on I as follows:

– First let $\mathcal{P}_0 = \{I_{\mu j}\}$ be the following partition on $(-\delta, \delta)$: Assume $\delta = e^{-\mu_*}$ for some $\mu_* \in \mathbb{Z}^+$. For $\mu \geq \mu_*$, let $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$; for $\mu \leq -\mu_*$,

– Let I_μ be the reflection of $I_{-\mu}$ about 0. Each I_μ is further subdivided into $\frac{1}{\mu^2}$ subintervals of equal length called $I_{\mu j}$.

– For $\hat{x} \in C$, let $\mathcal{P}_0^{\hat{x}}$ be the partition on $C_\delta(\hat{x})$ obtained by shifting the center of $\mathcal{P}_0$ from 0 to $\hat{x}$.

– The partition $\mathcal{P}$ is defined to be $\mathcal{P}_0^{\widehat{x}}$ on $C_\delta(\widehat{x})$; on $I \setminus C_\delta$, its elements are intervals of length $\approx \delta$.

– The following shorthand is used:

(a) We refer to $\pi \in \mathcal{P}$ corresponding to (translated) $I_{\mu j}$ intervals in $\mathcal{P}_0^{\widehat{x}}$ simply as "$I_{\mu j}$".

(b) For $\pi \in \mathcal{P}$, π^+ denotes the union of π and the two elements of $\mathcal{P}$ adjacent to it.

(c) For an interval $\gamma \subset I$, we say $\gamma \approx \pi$ if $\pi \subset \gamma \subset \pi^+$.

(d) For $\gamma \subset I_{\mu j}^+$, we define the bound period of γ to be $p(\gamma) = \min_{x \in I_{\mu j}^+} \{p(x)\}$.

Orbits of the same itineraries:

For $x, y \in I$, $[x, y]$ denotes the segment connecting x and y. We say x and y in I have *the same itinerary* (with respect to $\mathcal{P}$) *through time* $n-1$ if

– there exist

$$t_1 < t_1 + p_1 \leq t_2 < t_2 + p_2 \leq \cdots \leq n$$

such that for every k,

$$f^{t_k}[x, y] \subset \pi^+$$

for some $\pi \subset C_\delta$, $p_k = p(f^{t_k}[x, y])$, and

– for all $i \in [0, n) \setminus \cup_k [t_k, t_k + p_k)$, $f^i[x, y] \subset \pi^+$ for some $\pi \in \mathcal{P}$ with $\pi \cap C_\delta = \emptyset$.

(P3) Distortion estimate: *There exists* $K_0 > 1$ *(depending only on* f_0 *and on* λ*) such that if* x *and* y *have the same itinerary through time* $n-1$*, then*

$$\left| \frac{(f^n)'(x)}{(f^n)'(y)} \right| \leq K_0.$$

Corollary *There exists K_1 (depending only on f_0 and on λ) such that for all $x \in I$ with $f^i(x) \notin C$ for all $0 \leq i < n$,*

$$|(f^n)'(x)| > K_1^{-1} d(f^j(x), C) \ e^{\frac{1}{4}\lambda n}$$

where j is the time of the last free return before n. The factor $d(f^j(x), C)$ may be replaced by δ if $f^n(x)$ is free.

Proofs on (P1)-(P3):

For (P1) Tricky point: When the orbit is long, need to divide into smaller segments, on each we use perturbation argument.

For (P2) Reason for derivative recovery:

– Two orbits are bound $\rightarrow$ The difference between the two orbit points are much smaller than the distance from these orbits to critical points $\rightarrow$ Derivatives along the two orbits are comparable as far as they are bound.

– Exponential growth of derivatives are copied by the orbit bound to it.

– The estimates on the length of bound period are straight forward.

For (P3) The proof is technically involved but conceptually simple.

– Recall the standard distortion estimation for orbit with uniform expansion.

– Dividing into **bound** and **free** segments, we can conceptually treat each bound segments as one iteration with exponential expansion of derivatives.

– The dividing of I_μ into smaller sub-intervals $I_{\mu,j}$ is to put the total distortion under control. This is the only place this sub-division is used.

(III) Geometry of critical curves

Equivalence of x and a derivatives For f_0, λ, α and ε as in Proposition 2.1, we define

$$\begin{aligned}\mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon) = \{f : \|f - f_0\|_{C^2} < \varepsilon \text{ and} \\ (G1), (G2) \text{ hold for all } \hat{x} \in C \\ \text{and } n \leq N\}.\end{aligned}$$

Proposition *Let λ, α and ε be fixed. Then there exist $\hat{\varepsilon} > 0$ and $\hat{i} \in \mathbb{Z}^+$ such that the following holds for all $N \in \mathbb{Z}^+$: Let $\Omega_N \subset (-\hat{\varepsilon}, \hat{\varepsilon})$ be such that $f_a \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$ for all $a \in \Omega_N$. Then for every $a \in \Omega_N$ and $\hat{x} \in C$,*

$$\frac{1}{2}|\hat{c}(\hat{x})| < \frac{|\frac{d}{da}\hat{x}_i(a)|}{|(f_a^{i-1})'(\hat{x}_1)|} < 2|\hat{c}(\hat{x})| \qquad \text{for } \hat{i} < i \leq N.$$

Proof: Writing

$$\frac{d}{da}\widehat{x}_i(a) = (f_a)'(\widehat{x}_{i-1})\frac{d}{da}\widehat{x}_{i-1}(a) + \partial_a F(\widehat{x}_{i-1}, a),$$

we obtain inductively

$$\frac{\frac{d}{da}\widehat{x}_i(a)}{(f_a^{i-1})'(\widehat{x}_1)} \;=\; \frac{d}{da}\widehat{x}_1(a) \;+\; \sum_{j=1}^{i-1} \frac{\partial_a F(\widehat{x}_j, a)}{(f_a^j)'(\widehat{x}_1)} \;.$$

Letting $I(a,i)$ denote the expression on the right side above.

We need now to recall the condition for parameter transversality (PT): For $\widehat{x} \in \mathcal{C}$,

$$\widehat{c}(\widehat{x}) = \frac{d}{da} f_a(\widehat{x}(a))) - \frac{d}{da}\xi(a)\Big|_{a=0} \neq 0.$$

Lemma $I(0,\infty) = \widehat{c}(\widehat{x})$.

Proof of this lemma is well documented. This is the only place we use (PT).

We now choose $\widehat{i}$ large enough that

(i) $I(0,\hat{i}) \approx \hat{c}(\hat{x})$ and

(ii) for $i > \hat{i}$,

$$\left|\sum_{j=\hat{i}}^{i-1} \frac{\partial_a F(\hat{x}_j, a)}{(f_a^j)'(\hat{x}_1)}\right| << |\hat{c}(\hat{x})| \qquad \text{uniformly for all } a \in J.$$

(i) makes sense because $\hat{c}(\hat{x}) \neq 0$ by (PT). (ii) is because $|\partial_a F(\hat{x}_j, a)| < K$ and $|(f_a^j)'(\hat{x}_1)| > c_1 e^{\lambda j}$ from (G2). Since only a finite number of iterates are involved, we may now shrink $\hat{\varepsilon}$ sufficiently so that $|I(a,\hat{i}) - I(0,\hat{i})| << |\hat{c}(\hat{x})|$ for all $a \in (-\hat{\varepsilon}, \hat{\varepsilon})$. □

(P1)'-(P3)' for critical curves

Let $\Omega_N = \{a \in (-\hat{\varepsilon}, \hat{\varepsilon}) : f_a \in \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)\}$.

– We fix $\hat{x} \in C$. All parameters considered are assumed to be in Ω_N;

– all indices considered are assumed to be $\leq \frac{1}{\alpha^*}N$, (Why? Explain)

– (G1) is assumed to hold for $\hat{x}$ for all the indices in question.

– We use the notation $\tau_i(a) := \frac{d}{da}\hat{x}_i(a)$.

Our main results are (P1')–(P3'), three properties of $a \mapsto \hat{x}_i(a)$ that are the analogs of (P1)–(P3)

(P1') (Outside of C_δ): There exists $i_0 \geq \hat{i}$ such that the following hold for $n \geq i_0$:

(i) If $\hat{x}_n$ is free, and $\hat{x}_{n+j} \notin C_\delta$ $\forall$ $0 \leq j < j_0$, then $|\tau_{n+j}| > \frac{1}{2}c_1\delta e^{\frac{1}{4}\lambda_0 j}|\tau_n|$ for $j \leq j_0$;

(ii) if in addition $\hat{x}_{n+j_0} \in C_{\delta_0}$, then $|\tau_{n+j_0}| > \frac{1}{2}c_1 e^{\frac{1}{4}\lambda_0 j_0}|\tau_n|$.

Bound period We define the *bound period* $\widehat{p}_n(\omega)$ of $\widehat{x}_n(\omega)$ *in parameter-space dynamics* to be

$$\widehat{p}_n(\omega) := \min\{p_a : a \in \omega\}.$$

(P2') (Partial derivative recovery): *Suppose* $\widehat{x}_n(\omega) \subset I^+_{\mu j}$, *and let* $\widehat{p} = \widehat{p}_n(\omega)$. *Then*

(a) $\frac{1}{3\ln(max|f'|)}|\mu| \leq \widehat{p} \leq \frac{3}{\lambda}|\mu|$;

(b) for $a, a' \in \omega$ *and* $j < \widehat{p}$, $|\widehat{x}_{n+j}(a) - \widehat{x}_{n+j}(a')| < 2e^{-2\alpha j}$;

(c) $|\tau_{n+\widehat{p}}(a)| > e^{\frac{\lambda\widehat{p}}{4}}|\tau_n(a)|$ *for all* $a \in \omega$;

(d) if $\widehat{x}_n(\omega) \approx I_{\mu j}$, *then* $|\widehat{x}_{n+\widehat{p}}(\omega)| \geq e^{-\frac{8\alpha}{\lambda}|\mu|}$.

To state (P3'), we divide each orbit in the time interval $[i_0, n]$ into bound and free periods, and say all $a \in \omega$ have *the same itinerary* up to time n if (i) their bound and free periods coincide and (ii) whenever $\hat{x}_i(\omega)$ is free, it is $\subset \pi^+$ for some $\pi \in \mathcal{P}$.

(P3') (Global distortion): *There exist* $i_1 > i_0$ *and* $K_3 > 1$ *such that if* $n \geq i_1$ *and all points in* ω *have the same itinerary through step* $n-1$*, then for all* $a, a' \in \omega$*,*

$$K_3^{-1} < \frac{\tau_n(a)}{\tau_n(a')} < K_3.$$

Proofs of (P1)'-(P3)' A combination of (P1)-(P3) and the equivalence of x and a derivatives. Many tedious justifications. $\square$

(IV): Maintaining (G1) and (G2)

The original inductive scheme: (G1)

– Take a parameter curve covers an full sized $I_{\mu,j}$ interval at time N. Assume that all maps are inside of G_N.

– Iterate the segment forward, deleting the part that violate (G1), dividing the rest of the curve according to standard partition, iterate each of the sub-intervals forward. Using (P1)'-(P3)', obtain the set of good parameters (satisfying (G1)) up to $\frac{1}{\alpha}N$.

– Add the total proportion of parameters deleted: exponentially small.

Two problems:

(A) At the end of this iteration process (from time N to time $\frac{1}{\alpha}N$, does the critical orbit satisfying (G2)?

Answer: Negative.

(B) If we have, say, two critical points, then we will obtain one collection of segment of critical curves satisfying (G1) for the first critical curve, and **another** collection satisfying (G1) for the second critical curve. However, since these two set of curves are obtained independently, there is no way to guarantee that a given segment from one collection satisfying (G1) foro both critical curve. How we carry the induction forward starting at $\frac{1}{\alpha}N$? At the moment let us worry about (A).

Maintaining (G2): A counting argument

– The problem: when a critical orbit is in a bound period, the exponential derivative growth recovered through (P2)' is only partial (not λ but only $\frac{1}{3}\lambda$).

– Let $T_p(a)$ be the total time from N to $\frac{1}{\alpha}N$ a critical orbit spent in bound period, (G2) would be maintained if

$$T_p(a) < \varepsilon \frac{1}{\alpha} N$$

for some ε smaller than $\lambda_0 - \lambda$. (Explain why).

– Since some parameters obviously violate this restriction on $T_p(a)$, we will **drop** them. For the parameter survived this deletion, we have (G2). (So the parameters satisfying Jakobson's theorem only go close to critical set very in-frequently, and the returns made are not very close)

– The problem is, of course, how big a set of parameters we delete following this way of maintaining (G2)?

Answer hoped for: The fraction of deletion is again exponentially small.

Unfortunately, this is not correct.

– **A correction:** If we let $T_p(a)$ be the total time a orbit spent in bounded period cause by returns to $(-\delta^2, \delta^2)$, then the answer hoped for is correct.

This is good enough because the outside Liapunov exponent is independent of δ.

Standard partitions and Stop time: Take a segment of a critical curve of full sized $I_{\mu,j}$ and let the underlining parameter interval be Ω. Assume that all maps are in $\mathcal{G}_N$ for Ω.

– Denote the time starting at N as $N, N+1, \cdots, N+n$. We iterate the critical curve forward, and at each step of iteration, we deleting according to (G1), divining according to itineraries, then iterating each of the pieces surviving the deletion forward.

– Instead of iterating reaching time n, we **Stop** when the size of a piece of critical curve in this iteration process reaches a fixed size, say, δ.

– This way we cut Ω into small pieces, and on each of these pieces the **time we stop** is well-defined. Let $S : \Omega \to \mathcal{Z}^+$ be the function of stop time.

Proposition *We have*

$$|\{a \in \Omega : S(a) > m\}| < e^{-\frac{1}{2}K^{-1}m}\ |\Omega|$$

for all $m > K \log |\mu|$.

Proof: This Proposition is proved by a detailed counting argument. It is based on:

(i) Exponential growth of critical curve guaranteed by (P1)' and (P2)'.

(ii) Right after a piece of critical curve returns to C_δ, it stays outside for a length of time $\sim |\log \delta|$, and

(iii) (P3)' afford us to pull the comparison on sizes of critical curve back to that on Ω. $\Box$

– It is helpful to view this proposition from a **probabilistic** point of view: We regard $S : \Omega \to \mathcal{Z}^+$ as a **random variable**, with the probability function defined by

$$P(S > m) = |\{a\ \Omega,\ S(a) > m\}|.$$

The this proposition claims the probability function for the random variable S has **exponential tail**.

A large deviation argument Instead of stop, we now continue to iterate the stopped pieces of critical curves, deleting, dividing, and **marking new stop times along the way**, until we reach time $N + n$. This way we obtain

– For every time index $N + i$, $0 \le i \le n$, we obtain a partition of Ω, which we denote as $\mathcal{Q}_i$, $\mathcal{Q}_{i+1}$ is a refinement of $\mathcal{Q}_i$.

– For every time index $0 \le i \le n$, we define $X_i : \Omega \to \mathbb{Z}^+$ as follows:

(a) X_i are constant in every element $Q \in \mathcal{Q}_i$;

(b) $X_i(Q) = 0$ unless Q is a freely returned piece to C_{δ^2}.

(c) For the case excluded in (b), $X_i(a)$ is the stop time starting at time $N + i$ for $a \in Q$.

– Regarding each of the $X_i, i \leq n$ as an random variable, we know, on each piece $Q \in \mathcal{Q}_i$, the **conditional distribution of** X_{i+1}.

(a) If $X_i(Q) \neq 0$, $P(X_{i+1} = 0|Q) = 1$;

(b) $X_{i+1} \neq 0$ only if $i + 1$ is a new stop time, at which the size of the image $Q \in \mathcal{Q}_i$ is $> \delta$. We again have two cases: (1) $P(a \notin C_{\delta^2}) > 1 - \delta$; and (2) $P(X_{i+1} > m,\ a \in C_{\delta^2}|Q)$ is determined by the previous proposition.

– These information is enough for us estimate the tails of the probability function for the random variable $X = X_1 + X_2 + \cdots + X_n$, and it turned out that the probability function for X also has an exponential tail (A version of the standard large deviation argument in probability).

– This is an over estimate for (G2). (Again Why? and explain)

(V): Construction of good parameter set

1. We say $\widehat{x} \in C$ satisfies (G1)$^{\#}$ and (G2)$^{\#}$ up to time N if for all $1 \le i \le N$,

(G1)$^{\#}$ $d(\widehat{x}_i, C) > \min(\frac{1}{2}\delta_0, 2e^{-\alpha i})$;

(G2)$^{\#}$ $|(f^i)'(\widehat{x}_1)| > 2c_1 e^{\lambda_1 i}$ *where* $\lambda_1 = \lambda + \frac{1}{100}\lambda_0$.

We say $f \in \mathcal{G}_N^{\#}(f_0; \lambda, \alpha, \varepsilon)$ if all $\hat{x} \in C$ satisfy (G1)$^{\#}$ and (G2)$^{\#}$ up to time N. Clearly, $\mathcal{G}_N^{\#}(f_0; \lambda, \alpha, \varepsilon) \subset \mathcal{G}_N(f_0; \lambda, \alpha, \varepsilon)$.

Lemma *There exists $K_4 > 1$ for which the following holds: If $f_{\hat{a}} \in \mathcal{G}_N^{\#}(f_0; \lambda, \alpha, \varepsilon)$, then for all $n \leq N$, $f_a \in \mathcal{G}_n(f_0; \lambda, \alpha, \varepsilon)$ for all $a \in [\hat{a} - K_4^{-n}, \hat{a} + K_4^{-n}]$.*

We fix $\lambda \leq \frac{1}{5}\lambda_0$.

For Uni-modal maps Iterate the critical curve forward, from $N \to \frac{1}{\alpha}N$, deleting according to (G1) first, then (G2). The total measure deleted is always a fraction that is exponentially small. At the end we obtain a **fat** cantor set of good parameters of positive measure.

For maps with more than one critical points There is a problem that is potentially fatal I have talked earlier (Problem (B)):

Problem (B): If we have, say, two critical points, then we will obtain one collection of segment of critical curves satisfying (G1) for the first critical curve, and **another** collection satisfying (G1) for the second critical curve. However, since these two set of curves are obtained independently, there is no way to guarantee that a given segment from one collection satisfying (G1) for both critical curve. How do we carry the induction forward starting at $\frac{1}{\alpha}N$?

2. How to solve Problem (B):

– Induction is now from $N \to 2N$.

– At time N, we are handed

(i) For every critical point, a collection of segment of critical curves, each covers a full sized $I_{\mu,j}$, satisfying (G1)$^{\#}$ and (G2)$^{\#}$ for this particular critical point.

(ii) Let Δ_N be the intersection of these two collections. All maps in Δ_N is in $\mathcal{G}_N$, but we do not know the structure of this set.

(iii) Assume inductively that every interval in (i) contains at least one point in $\Delta_{\frac{1}{2}N}$.

– **Key point**: Take an interval in (i), there might be points that is NOT in Δ_N. However, from the lemma just introduce, we know that this interval is completely in $\mathcal{G}_{2\alpha N}$. (The size of this interval, which is $\sim e^{-\lambda n}$, is much smaller than the size of the interval in $\mathcal{G}_{2\alpha N}$ around $a \in \Delta_N$, which is $\sim K^{-2\alpha N}$.)

– Good critical orbits of size $2\alpha N$ is all we need to have a free-bound structure for critical orbits from $N \to 2N$.

In short: **There is no need for all points in this interval fall in Δ_N, it suffices if it all falls in $\mathcal{G}_{2\alpha N}$.**

3. To complete the induction, we

– For each interval in the collection of the given critical point, iterating and delete from $N \to 2N$. Do it for all critical points.

– **Additional deletion:** At the end we drop all intervals that contains no point in Δ_N.

– We do not need to estimate the measure deleted in the last step: All we need is to estimate $\Delta_N \setminus \Delta_{2N}$, but these additionally deleted intervals is not in Δ_N to start with. $\square$