Theory of rank one maps: (I)

(I) Settings

– Let $M = I \times [-1, 1]$ where I is either an interval or a circle, $\Delta_0 = (a_1, a_2) \times (0, b_1)$; Let $T_{a,b} : M \to M, (a, b) \in \Delta_0$ be a two-parameter family of 2D maps.

– Let us assume that $T_{a,b}$ assumes the general form

$$T_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} F(x, y, a) + b\ u(x, y, a, b) \\ b\ v(x, y, a, b) \end{pmatrix}. \tag{1}$$

We also assume that

(C1) For any given $(a, b) \in \Delta_0$, $T_{a,b}$ is a diffeomorphism from M to its image; and as functions in (x, y, a), the C^3-norms of $F(x, y, a)$, $u(x, y, a, b)$ and $v(x, y, a, b)$ are uniformly bounded.

(C2) Let $f_a := F(x, 0, a)$. $\{f_a\}, a \in (a_1, a_2)$ is an admissible family of 1D maps.

(C3) Let $f_{a^*}, a^* \in (a_1, a_2)$ be a Misiurewicz map ($f_{a^*} \in \mathcal{M}$), and $\mathcal{C}(f_{a^*})$ be the critical set of f_{a^*}. Then for $\widehat{x} \in \mathcal{C}(f_{a^*})$,

$$\frac{\partial}{\partial y} F(x, y, a)\bigg|_{(\widehat{x}, 0, a^*)} \neq 0.$$

Definition *Let $T_{a,b} : M \to M$ be as above. $T_{a,b}$ is an admissible family of rank one maps if it satisfies (C1)-(C3).*

– In one sentence, $T_{a,b}$ is an admissible family of rank one maps if it is a non-degenerate 2D unfolding of an admissible 1D family.

- (C1) imposes the usual 2D regularity,

• (C3) requires in particular that the unfolding is not singular in the direction of y.

• $T_{a,b,L}$ in lecture 1

$$\theta_1 = a + \theta + L \sin 2\pi\theta + r$$
$$r_1 = br + bL \sin 2\pi\theta$$

is an example of an admissible family of rank one maps if L is sufficiently large.

• Another example is the Hénon family

$$(x, y) \to (1 - ax^2 + y, bx)$$

around $a^* = 2$, and the accompanied maps with small perturbations (Hénon like maps).

– That periodic sinks are observable in both the parameter and the phase spaces follows again from the fact that the periodic sinks of 1D maps are persistent under small perturbations.

(II) Geometric structure of the attractive basin

– Objective of the theory of rank maps: to justify the observability of Scenario (b) in parameter space. That is,

to prove the existence of a parameter set $\triangle$ of positive measure, for which the attractor of $T_{a,b}, (a, b) \in \triangle$ admits no periodic sinks.

From this point on, this scenario will be referred to as *rank one chaos*.

– To prove the observability of rank one chaos in the parameter space for an admissible family, we imitate the 1D theory, and as a starting point we try to draw a corresponding version of Jakobson's theorem for $T_{a,b}$.

An immediate hurdle for us in repeating the claims for 1D maps is how to identify the set of critical points for a 2D map $T_{a,b}$, over which we wish to impose a rule of the likes of (G1) and (G2). As it turns out,

to this question there is no straight and easy answer as in the case of 1D maps.

– To find an answer we need to take a closer look at how $T_{a,b}$ acts on M.

• For $T = T_{a,b}$ let $R_0 := I \times [-Kb, Kb]$ where $K > 0$ be such that $T(R_0) \subset R_0$. Denote $R_n = T^n(R_0)$.

$\{R_n\}$ is a decreasing sequence of neighborhoods of the attractor

$$\Lambda := \cap_{n=0}^{\infty} R_n$$

.

• Let δ be a small positive number such that $d(f^n(\widehat{x}), \mathcal{C}) >> \delta$ for all $\widehat{x} \in \mathcal{C}$ and $n > 0$, where $f = f_{a^*}$ and $\mathcal{C}$ is the set of critical points for f. Define

$$\mathcal{C}^{(0)} = \{(x,y) \in R_0 : |x - \widehat{x}| < \delta \quad \text{for some } \widehat{x} \in \mathcal{C}\}.$$

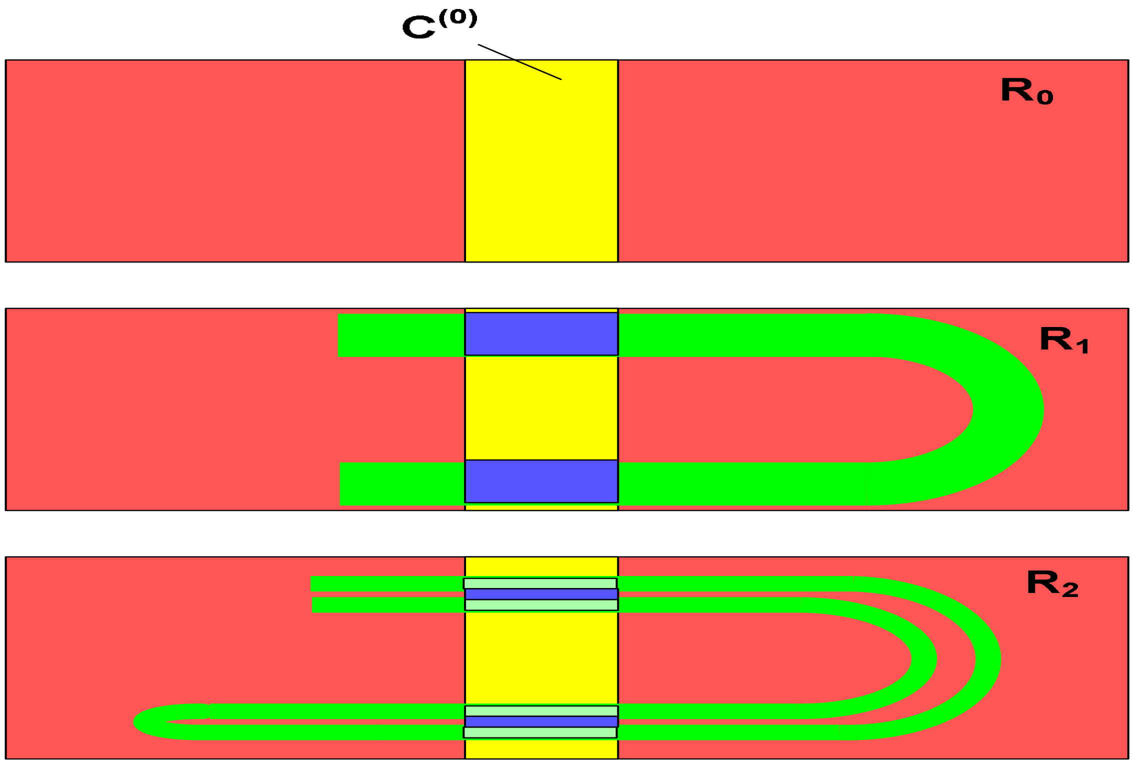
$\mathcal{C}^{(0)}$ is a collection of vertical strips of width 2δ.

– The picture of $R_1 = T(R_0)$ is rather simple.

• T maps the connected components of $R_0 \setminus \mathcal{C}^{(0)}$ to vertically compressed and horizontally stretched horizontal strips, and the components of $\mathcal{C}^{(0)}$ become small quadratic turns connecting these strips.

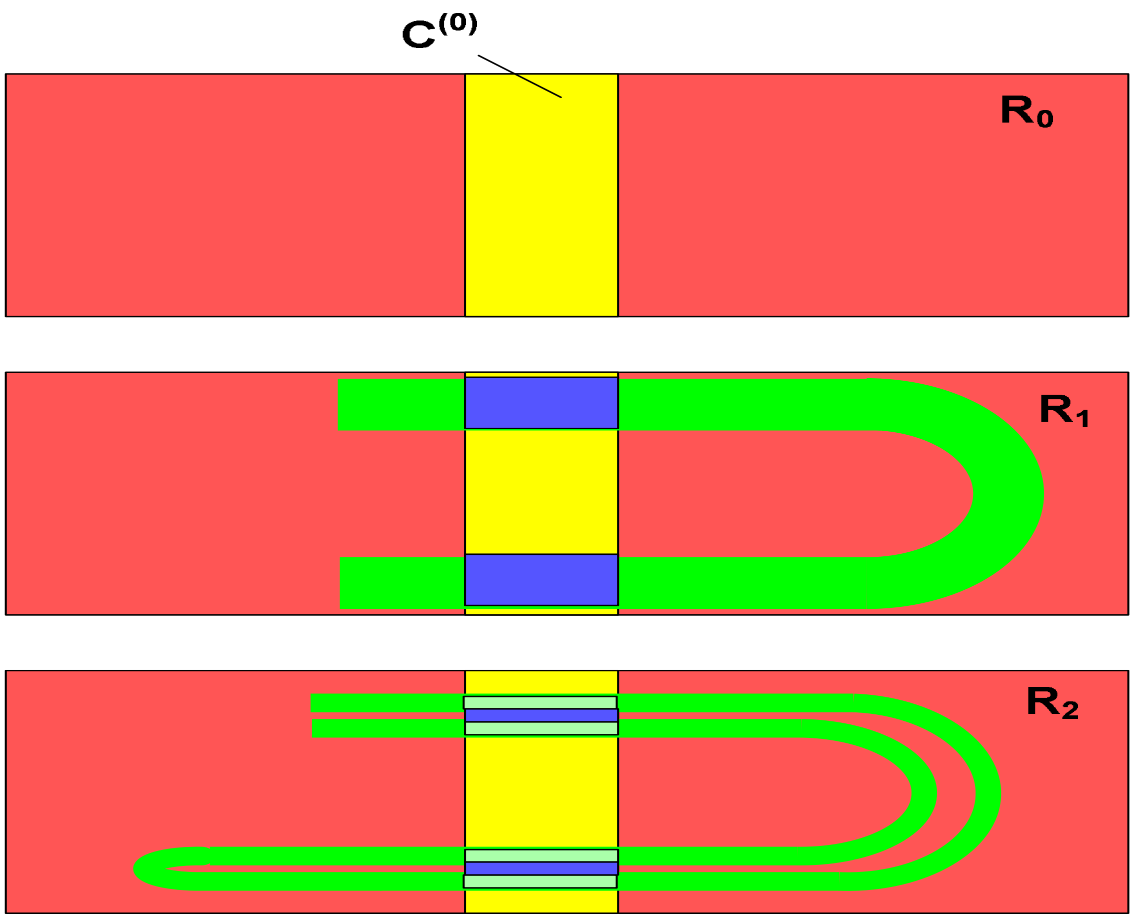
• Under one more iteration of T, new turns are created from the intersections of R_1 and $\mathcal{C}^{(0)}$, and the images of the old turns are kept away from $\mathcal{C}^{(0)}$.

• Keep iterating forward in time. As far as the images of all the quadratic turns are prevented from coming back to $\mathcal{C}^{(0)}$, we can maintain a relatively simple and clean picture of sharp turns connected by horizontal strips for R_n.



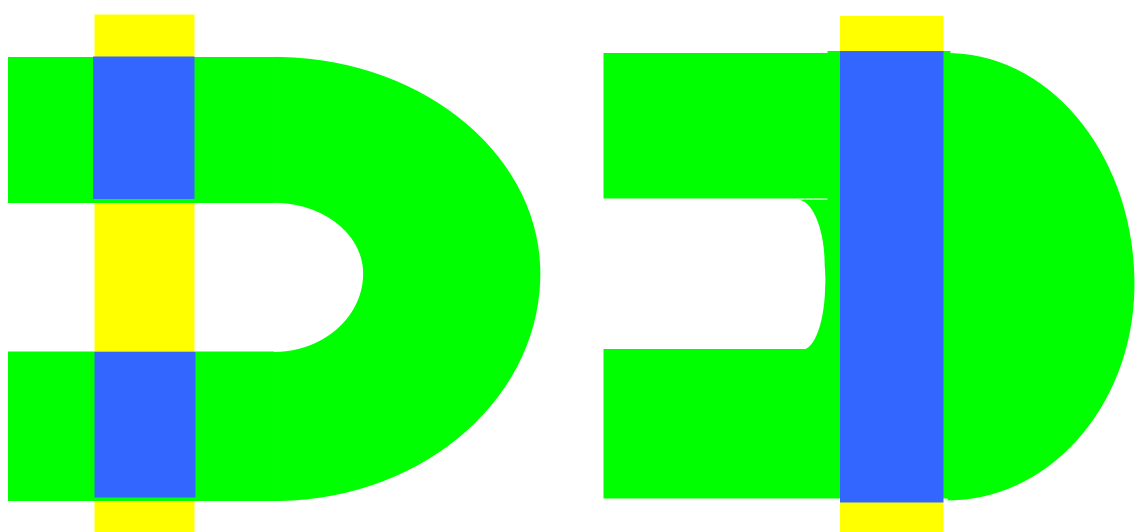
• Since T is obtained by perturbing a Misiurewicz map, this simple picture of R_n would last at least up to $n = N_0 = \mathcal{O}(log\delta)$.

• We now study the structure of $\mathcal{C}^{(0)} \cap R_n$. As long as the images of the turns are kept away from coming back to $\mathcal{C}^{(0)}$, $R_n \cap \mathcal{C}^{(0)}$ is a collection of thin, horizontal strips crossing $\mathcal{C}^{(0)}$, and the intersection of R_{n+1} with each of the strips of $R_n \cap \mathcal{C}^{(0)}$ is again a collection of thinner strips.



• However, the horizontal size of the images of the turns does grow and will eventually intersect $\mathcal{C}^{(0)}$.

The best we could hope for in terms of getting a nested structure of horizontal strips for $R_n \cap \mathcal{C}^{(0)}$ is **to make the tips of the turns on the boundary of R_n stay out of $\mathcal{C}^{(0)}$**.



• Such geometric pictures kept forever would give the correspondences of the 1D Misiurewicz maps in 2D.

(III) Statement of the Main Theorem:

– To obtain the correspondences of the good maps of 1D theory, we have to allow the tips of the turns to come back to $\mathcal{C}^{(0)}$. Consequently

the nested structure of horizontal strips could not last forever.

– We will, however, impose a rule of slow return similar to (G1), and shrink the length of the horizontal strips gradually to maintain a nested structure with dwindling horizontal scale.

– The theorem below is a formulation of a correspondence of 1D theorem for $T_{a,b}$ based on these ideas.

Notation: (a) For $z_0 \in R_0$, let $z_i = T^i(z_0)$.

(b) If w_0 is a tangent vector at z_0, let $w_i = DT^i(z_0)w_0$.

(c) A curve in R_0 is called a $C^2(b)$-**curve** if the slopes of its tangent vectors are $\mathcal{O}(b)$ and its curvature is everywhere $\mathcal{O}(b)$.

Theorem *Let $T_{a,b}, (a,b) \in (a_1, a_2) \times (0, b_1)$ be an admissible family of rank one maps. Then for every $b > 0$ sufficiently small, there is a positive measure set $\Delta_b \subset (a_1, a_2)$ such that the following holds for $T_{a,b}, a \in \Delta_b$. In what follows, $\alpha, \delta, c > 0$ and $0 < \rho < 1$ are positive constants, and $b << \alpha, \delta, \rho, e^{-c}$.*

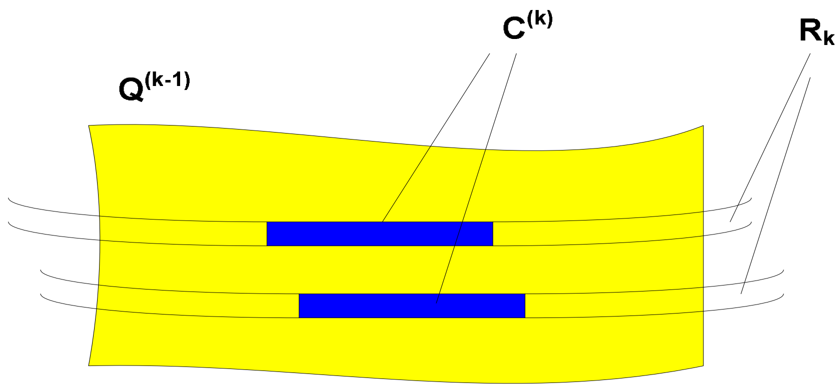
(1) **Critical regions and critical set.** *There is a Cantor set $\mathcal{C} \subset \Lambda$ called the critical set given by $\mathcal{C} = \cap_{k=0}^{\infty} \mathcal{C}^{(k)}$ where the $\mathcal{C}^{(k)}$ is a decreasing sequence of neighborhoods of $\mathcal{C}$ called critical regions.*

Geometrically:

(i) $\mathcal{C}^{(0)} = \{(x,y) \in R_0 : d(x,\mathcal{C}) < \delta\}$ *where* $\mathcal{C}$ *is the set of critical points of* $f_{a^*} \in \mathcal{M}$.

(ii) $\mathcal{C}^{(k)}$ *has a finite number of components called* $Q^{(k)}$ *each one of which is diffeomorphic to a rectangle. The boundary of* $Q^{(k)}$ *is made up of two* $C^2(b)$ *segments of* ∂R_k $(R_k = T^k(R_0))$ *connected by two vertical lines: the horizontal boundaries are* $\approx \min(2\delta, \rho^k)$ *in length, and the Hausdorff distance between them is* $\mathcal{O}(b^{\frac{k}{2}})$.

(iii) $\mathcal{C}^{(k)}$ *is related to* $\mathcal{C}^{(k-1)}$ *as follows:* $Q^{(k-1)} \cap R_k$ *has at most finitely many components, each of which lies between two* $C^2(b)$ *subsegments of* ∂R_k *that stretch across* $Q^{(k-1)}$ *as shown in the figure below. Each component of* $Q^{(k-1)} \cap R_k$ *contains exactly one component of* $\mathcal{C}^{(k)}$.



Dynamically: *On each horizontal boundary γ of $Q^{(k)}$ there is a unique point z located within $\mathcal{O}(b^{\frac{k}{4}})$ of the midpoint of γ with the property that if τ is the unit tangent vector to γ at z, then $DT^n(z)\tau$ decreases in length exponentially as n tends to ∞.*

(2) **Properties of critical orbits.** *For $z \in R_0$, let $d_{\mathcal{C}}(z)$ denote the following notion of "distance to the critical set": If $z \notin \mathcal{C}^{(0)}$, let $d_{\mathcal{C}}(z) = \delta$; if $z \in \mathcal{C}^{(0)} \setminus \mathcal{C}$, let k be the largest number with $z \in \mathcal{C}^{(k)}$, and define $d_{\mathcal{C}}(z)$ to be the horizontal distance between*

z and the midpoint of the component of $\mathcal{C}^{(k)}$ containing z. Then for all $z_0 \in \mathcal{C}$:

(i) $d_{\mathcal{C}}(z_j) \geq min\{\delta, e^{-\alpha j}\}$ *for all* $j > 0$;

(ii) $\|DT^j(z_0)\binom{0}{1}\| \geq K^{-1}e^{cj}$ *for all* $j > 0$.

– This theorem, though very long to lay out, is a direct parallel of our previous 1D theorem.

• Theorem 1(2)(i) corresponds to (G1) and Theorem 1(2)(ii) to (G2).

• Theorem 1(1) is a detailed identification of the set of critical points, which is now an infinite set. This theorem ([WY1]), together with a generalization to higher dimensions ([WY2]), forms the core of what we will refer to as the theory of rank one chaos.

• Proof of Theorem 1 is like the statement itself. It is always to figure a way to repeat the 1D proof.

In appearance, things are much more complicated,

In essence, as we have seen in the above statement, we are repeating our 1D proof.

Keep these two things in mind, and with a thorough understanding of the 1D theory is the key in understanding the theory we are presenting.

(IV) Technical Preparations

(A) Linear algebra

– Let M be a 2×2 matrix. Assuming that M is not a scalar multiple of an orthogonal matrix, we say that a unit vector e defines **the most contracted direction** of M if $\|Mu\| \geq \|Me\|$ for all unit vectors u.

– For a sequence of matrices $M_1, M_2, \cdots$, we use $M^{(i)}$ to denote the matrix product $M_i \cdots M_2 M_1$ and e_i to denote the most contracted direction of $M^{(i)}$ when it makes sense.

Hypotheses The M_i are 2×2 matrices; they satisfy $|\det(M_i)| \leq b$ and $\|M_i\| \leq K_0$ where K_0 and b are fixed numbers with $K_0 > 1$ and $b << 1$.

Lemma 1 *There exists K depending only on K_0 such that if $\|M^{(i)}\| \geq \kappa^i$ and $\|M^{(i-1)}\| \geq \kappa^{i-1}$ for some $\kappa >> \sqrt{b}$, then e_i and e_{i-1} are well-defined, and*

$$\| e_i \times e_{i-1} \| \leq (\frac{Kb}{\kappa^2})^{i-1}.$$

Corollary 1 *If for $1 \leq i \leq n$, $\|M^{(i)}\| \geq \kappa^i$ for some $\kappa >> \sqrt{b}$, then:*

(a) $\| e_n - e_1 \| < \frac{Kb}{\kappa^2}$*;*

(b) $\|M^{(i)} e_n\| \leq (\frac{Kb}{\kappa^2})^i$ *for* $1 \leq i \leq n$*.*

Proof: (a) follows immediately from Lemma 1. For (b), since $\|e_n - e_i\| \leq (\frac{Kb}{\kappa^2})^i$, we have $\|M^{(i)} e_n\| \leq \|M^{(i)}(e_n - e_i)\| + \|M^{(i)} e_i\| < K_0^i \cdot (\frac{Kb}{\kappa^2})^i + (\frac{b}{\kappa})^i$. □

– Next we consider for each i a 3-parameter family of matrices $M_i(s_1, s_2, s_3)$. For the purpose of the next corollary we make the additional assumptions that

(a) for $0 < j \leq 3$, $\|\partial^j M_i(s_1, s_2, s_3)\| \leq K_0^i$ and

(b) $|\partial^j \det(M_i(s_1, s_2, s_3))| < K_0^i b$ where ∂^j represents any one of the partial derivatives of order j with respect to s_1, s_2 or s_3.

Let $\theta_i(s_1, s_2, s_3)$ denote the angle $e_i(s_1, s_2, s_3)$ makes with the positive x-axis, assuming it makes sense.

Corollary 2 *Suppose that for some $\kappa >> \sqrt{b}$, $\|M^{(i)}(s_1, s_2, s_3)\| \geq \kappa^i$ for every (s_1, s_2, s_3) and for every $1 \leq i \leq n$. Then for $j = 1, 2, 3$, $|\partial^j \theta_1| \leq K\kappa^{-(1+j)}$, and for $i \leq n$,*

$$|\partial^j(\theta_i - \theta_{i-1})| < (\frac{Kb}{\kappa^{(2+j)}})^{i-1}, \tag{2}$$

$$\|\partial^j M^{(i)} e_n\| < (\frac{Kb}{\kappa^{(2+j)}})^i. \tag{3}$$

– Our next lemma is a perturbation result. Let M_i, M_i' be two sequences of matrices, let w be a vector, and let θ_i and θ_i' denote the angles $M^{(i)}w$ and $M'^{(i)}w$ make with the positive x-axis respectively.

Lemma 2 *Let κ, λ be such that $\frac{Kb}{\kappa^2} < \lambda < K_0^{-12}\kappa^8$. If for $1 \le i \le n$, $\|M_i - M_i'\| \le \lambda^i$ and $\|M^{(i)}w\| \ge \kappa^i$, then*

(a) $\|M'^{(n)}w\| \ge \frac{1}{2}\kappa^n$;

(b) $|\theta_n - \theta_n'| < \lambda^{\frac{n}{4}}$.

Hypothesis for (B) and (C) below $T : A \to A$ is an embedding of the form

$$T(x, y) = (t_1(x, y), bt_2(x, y))$$

where the C^2-norms of t_1 and t_2 are $\le K_0$, and $K_0 > 1$ and $b << 1$ are fixed numbers.

Stable curves

Lemma 3 *Let κ, λ be as in Lemma 2 and $z_0 \in A$ be such that for $i = 1, \cdots, n$, $\|DT^i(z_0)\| \ge \kappa^i$. Then there is a C^1 curve γ_n passing through z_0 such that* *(a) for all $z \in \gamma_n$, $d(T^i z_0, T^i z) \leq (\frac{Kb}{\kappa^2})^i$ for all $i \leq n$;*

(b) γ_n can be extended to a curve of length $\sim \lambda$ or until it meets ∂A.

– We call γ_n a **stable curve of order** n. It will follow from this lemma that if $\|DT^i(z_0)\| \geq \kappa^i$ for all $i > 0$, then there is a **stable curve** γ_∞ passing through z_0 obtained as a limit of the γ_n's.

(C) Curvature estimates

– Let $\gamma_0 : [0,1] \to A$ be a C^2 curve, and let $\gamma_i(s) = T^i(\gamma_0(s))$. We denote the curvature of γ_i at $\gamma_i(s)$ by $k_i(s)$.

Lemma 4 *Let $\kappa > b^{\frac{1}{3}}$. We assume that for every s, $k_0(s) \leq 1$ and*

$$\|DT^j(\gamma_{n-j}(s))\gamma'_{n-j}(s)\| \geq \kappa^j \|\gamma'_{n-j}(s)\|$$

for every $j < n$. Then

$$k_n(s) \leq \frac{Kb}{\kappa^3}.$$

(D) One-dimensional dynamics

Let $f \in \mathcal{M}$ and let $C_\delta := \{x \in S^1 : d(x, C) < \delta\}$.

Lemma 5 *There exist $\hat{c}_0, \hat{c}_1 > 0$ such that the following hold for all sufficiently small $\delta > 0$: Let $x \in S^1$ be such that $x, fx, \cdots, f^{n-1}x \notin C_\delta$, any n. Then*

(i) $|(f^n)'x| \geq \hat{c}_0 \delta e^{\hat{c}_1 n}$;

(ii) if, in addition, $f^n x \in C_\delta$, then $|(f^n)'x| \geq \hat{c}_0 e^{\hat{c}_1 n}$.

Corollary 3 *Let $c_0 < \hat{c}_0$ and $c_1 < \hat{c}_1$. Then for all sufficiently small δ, there exists $\varepsilon = \varepsilon(\delta)$*

such that for all g with $\|g - f\|_{C^2} < \varepsilon$, (i) and (ii) above hold for g with c_0 and c_1 in the places of $\hat{c}_0$ and $\hat{c}_1$.

This is (P1).

Lemma 6 (Derivative recovery) *There exists K such that for g satisfying the conditions above, if $|x - \hat{x}| = e^{-\mu} < \delta$ for some $\hat{x} \in C$, then*

(i) $K^{-1}\mu \leq p(x) \leq K\mu$ *;*

(ii) $K^{-1}(x - \hat{x})^2|(g^{i-1})'(g\hat{x})| < |g^i x - g^i \hat{x}| < K(x - \hat{x})^2|(g^{i-1})'(g\hat{x})|$*;*

(iii) $|(g^p)'x| \geq K^{-1}\lambda^{\frac{p}{2}}$ *where $p = p(x)$.*

This is (P2)

(E) Dynamics outside of $\mathcal{C}^{(0)}$

From this point on

$$T_{a,b}(x,y) \;=\; (F_a(x,y) + bu_{a,b}(x,y),\; bv_{a,b}(x,y)).$$

– Assume that $b^{\frac{1}{4}} << \delta$ and let

$$\mathcal{C}^{(0)} = \{(x,y) \in R_0 : |x - \widehat{x}| < \delta \quad \text{for some } \widehat{x} \in C\}.$$

– Let $s(u)$ denote the slope of a vector u. For $z \notin \mathcal{C}^{(0)}$, if $|s(u)| < \frac{b}{\delta^4}$, then $|s(DT(z)u)| = \mathcal{O}(\frac{b}{\delta})$.

– If $\kappa_0 := \min \|DT(z)u\|$ where the minimum is taken over all $z \notin \mathcal{C}^{(0)}$ and unit vectors u with $|s(u)| < \frac{b}{\delta^4}$, then $\kappa_0 > K^{-1}\delta$.

– Let $K(\delta) := \frac{K}{\kappa_0^3}$, so that $K(\delta)b$ is the upper bound for k_n in Lemma 4. We call a vector u a b-**horizontal vector** if $|s(u)| < K(\delta)b$. A curve γ is called a $C^2(b)$-**curve** if

(i) its tangent vectors are b-horizontal, and

(ii) its curvature is $\leq K(\delta)b$ at every point.

Lemma 7 *(a) For $z \notin \mathcal{C}^{(0)}$, if u is b-horizontal, then so is $DT(z)u$.*

(b) If γ is a $C^2(b)$-curve outside of $\mathcal{C}^{(0)}$, then $T(\gamma)$ is again a $C^2(b)$-curve.

– Our next lemma describes the dynamics of b-horizontal vectors outside of $\mathcal{C}^{(0)}$.

Lemma 8 *There exist constants $c_0, c_1 > 0$ independent of δ such that the following holds for $T = T_{a,b}$ for all (a, b) sufficiently near $(a^*, 0)$. Let $z \in R_0$ be such that $z, Tz, \cdots, T^{n-1}z \notin \mathcal{C}^{(0)}$, and let u be a b-horizontal vector. Then*

(i) $\|DT^n(z)u\| \geq c_0 \delta e^{c_1 n}$;

(ii) if, in addition, $T^n z \in \mathcal{C}^{(0)}$, then $\|DT^n(z)u\| \geq c_0 e^{c_1 n}$.

(F) Critical points inside $\mathcal{C}^{(0)}$

– Let e_m denote the field of most contracted directions of DT^m and let q_m be the slope of e_m. When working with a curve γ parameterized by arc length, we write $q_m(s) = q_m(\gamma(s))$.

– We begin with some easy observations about e_1.

Lemma 9 *For all (a, b) sufficiently near $(a^*, 0)$, e_1 is defined everywhere on R_0, and there exists $K > 0$ such that*

(a) $|q_1| > K^{-1}\delta$ outside of $\mathcal{C}^{(0)}$, and q_1 has opposite signs on adjacent components of $R_0 \setminus \mathcal{C}^{(0)}$;

(b)

$$|\frac{dq_1}{ds}| > K^{-1}$$

on every $C^2(b)$-curve γ in $\mathcal{C}^{(0)}$.

– Let γ be a $C^2(b)$-curve in $\mathcal{C}^{(0)}$. We say that z_0 is a **critical point of order m on γ** if

(a) $\|DT^i(z_0)\| \geq 1$ for $i = 1, 2, \cdots, m$;

(b) at z_0, e_m coincides with the tangent vector to γ.

– It follows from Lemma 9 that on every $C^2(b)$-curve that stretches across a component of $\mathcal{C}^{(0)}$, there is a unique critical point of order 1. The next two lemmas are used in the "updating" of existing critical points and the creation of new ones.

Lemma 10 *Let γ be a $C^2(b)$-curve in $\mathcal{C}^{(0)}$ where $\gamma(0) = z$ is a critical point of order m. We assume that*

(a) $\|DT^i(z)\| \geq 1$ for $i = 1, 2, \cdots, 3m$;

(b) $\gamma(s)$ is defined for $s \in [-(Kb)^{\frac{m}{2}}, (Kb)^{\frac{m}{2}}]$.

Then there exists a unique critical point $\hat{z}$ of order $3m$ on γ, and $|\hat{z} - z| < (Kb)^m$.

Lemma 11 *For $\varepsilon > 0$, let γ and $\hat{\gamma}$ be two disjoint $C^2(b)$-curves in $\mathcal{C}^{(0)}$ defined for $s \in [-4K_1\sqrt{\varepsilon}, 4K_1\sqrt{\varepsilon}]$ where K_1 is the constant K in Lemma 9(b). We assume*

(a) $\gamma(0)$ is a critical point of order m;

(b) the x-coordinates of $\gamma(0)$ and $\hat{\gamma}(0)$ coincide, and $\mid \gamma(0) - \hat{\gamma}(0) \mid < \varepsilon$.

Then there exists a critical point of order $\widehat{m}$ *at* $\widehat{\gamma}(\widehat{s})$ *with* $|\widehat{s}| < 4K_1\sqrt{\varepsilon}$ *and* $\widehat{m} = min\{m, K \log \frac{1}{\varepsilon}\}$.

(G) Tracking DT^n: a splitting algorithm

– Let $z_0 \in R_0$, and let w_0 be a unit vector at z_0 that is b-horizontal. We write $z_n = T^n z_0$ and $w_n = DT^n(z_0)w_0$. In the case where $z_i \notin \mathcal{C}^{(0)}$ for all i, the resemblance to 1-d is made clear in Lemmas 5 and 8.

– Consider next an orbit $z_0, z_1, \cdots$ that visits $\mathcal{C}^{(0)}$ exactly once, say at time $t > 0$. Assume:

(a) there exists $\ell > 0$ such that $\|DT^i(z_t)\binom{0}{1}\| \geq 1$ for all $i < \ell$, so that in particular e_ℓ, the most contracted direction of DT^ℓ, is defined at z_t, and

(b) $\theta(w_t, e_\ell)$, the angle between w_t and e_ℓ, is $\geq b^{\frac{\ell}{2}}$.

Then $DT^i(z_0)$ can be analyzed as follows.

• We split w_t into $w_t = \widehat{w}_t + \widehat{E}$ where $\widehat{w}_t$ is parallel to the vector $\binom{0}{1}$ and $\widehat{E}$ is parallel to e_ℓ. For $i \le t$ and $i \ge t + \ell$, let $w_i^* = w_i$. For i with $t < i < t + \ell$, let $w_i^* = DT^{i-t}(z_t)\widehat{w}_t$.

• We claim that all the w_i^* are b-horizontal vectors, so that $\{\|w_{i+1}^*\|/\|w_i^*\|\}_{i=0,1,2,\cdots}$ resemble a sequence of 1-d derivatives. In particular, $\|w_{t+1}^*\|/\|w_t^*\| \sim \theta(w_t, e_\ell)$ simulates a drop in the derivative when an orbit comes near a critical point in 1-dimension.

– To justify the statement about the slope of the w_i^*, we note that $DT(z_t)\binom{0}{1}$ is b-horizontal, so that in view of lemma 7 we need only to consider $w_{t+\ell}^*$. We have

$$\|DT^\ell(\widehat{E})\| \le b^\ell \frac{\|\widehat{w}_t\|}{\theta(w_t, e_\ell)} \le b^{\frac{\ell}{2}}\|\widehat{w}_t\| \le b^{\frac{\ell}{2}}\|DT^\ell(z_t)\widehat{w}_t\|,$$

the first and third inequalities following from (a) and the second from (b). Since the slope of $DT^\ell(z_t)\widehat{w}_t$ is smaller than $\frac{Kb}{2\delta}$, it follows that $w^*_{t+\ell} = DT^\ell(z_t)\widehat{w}_t + DT^\ell(z_t)\widehat{E}$ remains b-horizontal.

– The discussion above motivates the following splitting algorithm. Consider $\{z_i\}_{i=0}^\infty$, and let $t_1 < \cdots < t_j < \cdots$ be the times when $z_i \in \mathcal{C}^{(0)}$. We let w_0 be a b-horizontal unit vector, and assume as before that e_{ℓ_i} makes sense at z_i for $i = t_j$. Define w^*_i as follows:

1. For $0 \le i \le t_1$, let $w^*_i = DT^i(z_0)w_0$.

2. At $i = t_j$, we split w^*_i into

$$w^*_i = \widehat{w}_i + \widehat{E}_i$$

where $\widehat{w}_i$ is parallel to $\binom{0}{1}$ and $\widehat{E}_i$ is parallel to e_{ℓ_i}.

3. For $i > t_1$, let

$$w_i^* = DT(z_{i-1})\hat{w}_{i-1} \ + \sum_{j:\ t_j+\ell_{t_j}=i} DT^{\ell_{t_j}}(z_{t_j})\hat{E}_{t_j} \tag{4}$$

and let $\hat{w}_i = w_i^*$ if $i \neq t_j$ for any j.

This algorithm does not give anything meaningful in general. It does, however, in the scenario of the next lemma.

Lemma *Let z_i, w_i and w_i^* be as above. Assume*

(a) for each $i = t_j$, $\theta(w_i^*, e_{\ell_i}) \geq b^{\frac{\ell_i}{2}}$;

(b) the time intervals $I_j := [t_j, t_j + \ell_{t_j}]$ are strictly nested, i.e. for $j \neq j'$, either $I_j \cap I_{j'} = \emptyset$, $I_j \subset I_{j'}$, or $I_{j'} \subset I_j$, and $t_j + \ell_{t_j} \neq t_{j'} + \ell_{t_{j'}}$.

Then $w_i = w_i^*$ for $i \notin \cup_j I_j$, and the w_i^*'s are all b-horizontal vectors. The sequence $\{|w_i^*\|\}$ has the property that $\|w_{i+1}^*\|/\|w_i^*\| \sim \theta(w_i^*, e_{\ell_i})$ for $i = t_j$, and $\|w_{i+1}^*\| \approx \|DT(z_i)w_i^*\|$ for $i \neq t_j$.