Theory of rank one maps: (II)

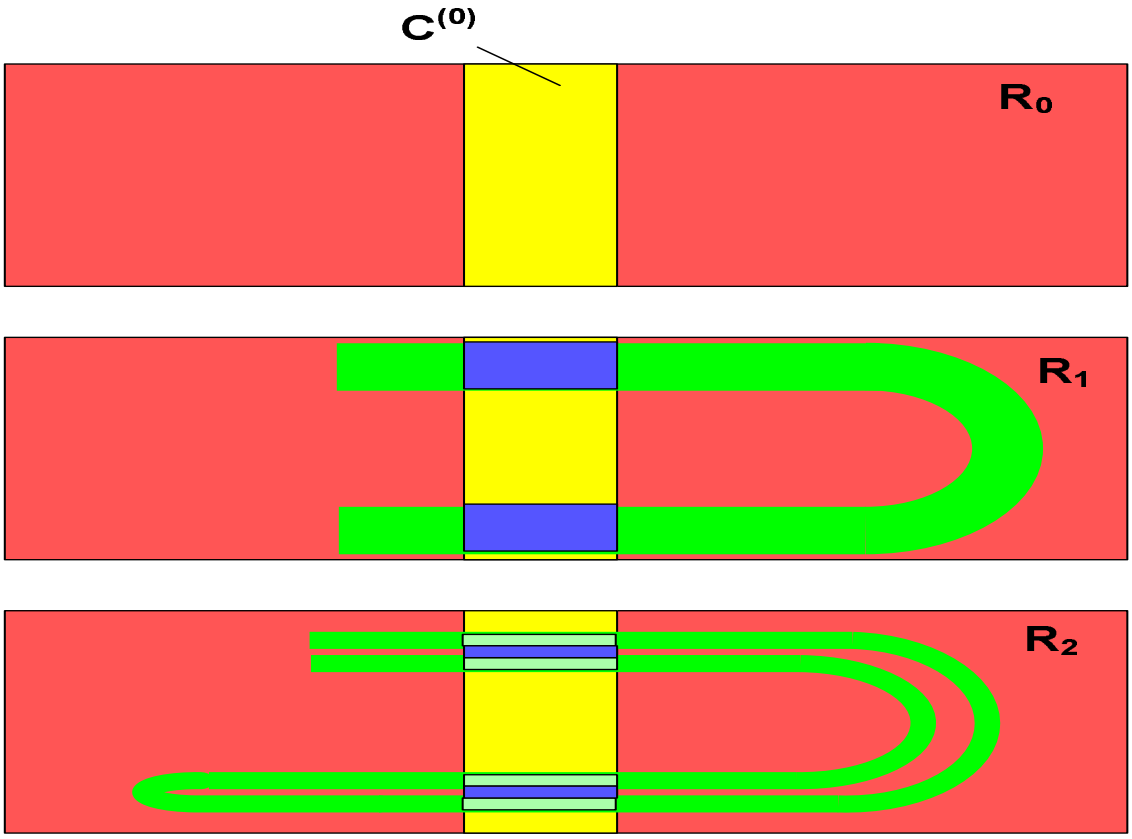
I. Inductive Assumptions

(A) Set of critical points

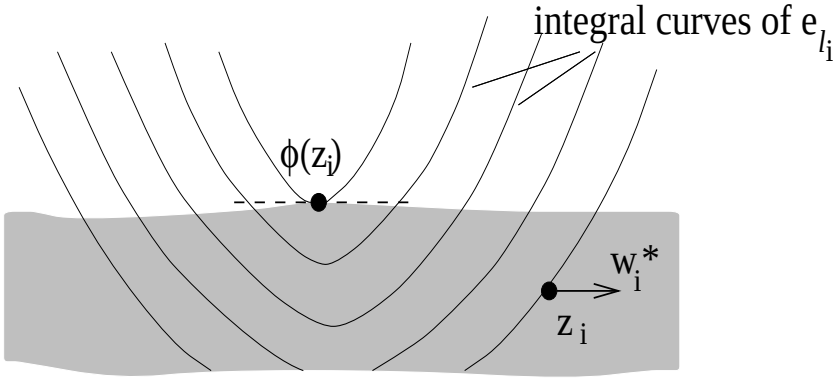
– In 1-dimension, the critical set is where all previous expansion is destroyed. Tangencies of stable and unstable manifolds play a similar role in higher dimensions.

– Let $\mathcal{F}_0$ be the foliation on R_0 with leaves $\{y = \text{constant}\}$, and let $\mathcal{F}_k$ be its image under T^k. The 0th critical region $\mathcal{C}^{(0)}$.

– Suppose that $T^i\mathcal{C}^{(0)} \cap \mathcal{C}^{(0)} = \emptyset$ for all $i \leq n_0$. Then $\mathcal{F}_{n_0}$ restricted to $\mathcal{C}^{(0)} \cap R_{n_0}$ consists of finitely many bands of roughly horizontal leaves whose tangent vectors have been expanded the previous n_0 iterates.



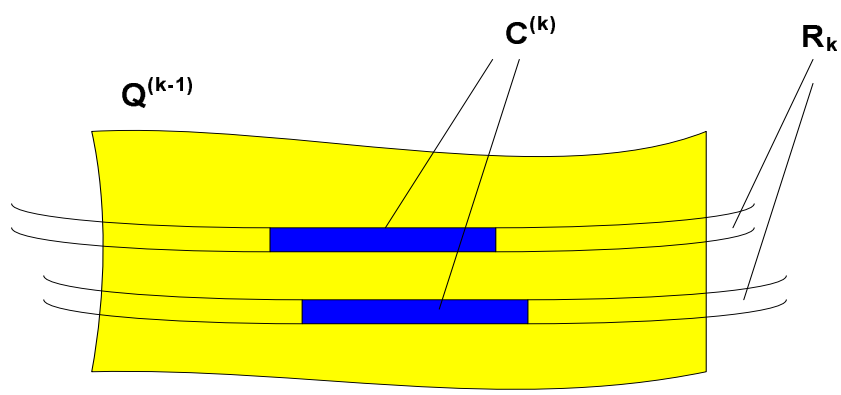
–Also in $\mathcal{C}^{(0)}$, DT^{n_0} has a well-defined field of most contracted directions, e_{n_0}, whose integral curves are roughly parabolas.



– The set of tangencies in $\mathcal{C}^{(0)}$ between the leaves of $\mathcal{F}_{n_0}$ and the integral curves of e_{n_0} gives our first approximation of $\mathcal{C}$.

– As we continue to iterate, it is inevitable that "turns" in the leaves of $\mathcal{F}_k$ will get inside $\mathcal{C}^{(0)}$, and e_n will not be defined everywhere in $\mathcal{C}^{(0)}$ for all n. There is no chance that the simplistic picture in the last paragraph can be continued forever.

– For parameters for which these "turns" approach the critical set sufficiently slowly, we will show that this picture can be maintained by shrinking *sideways* the regions of consideration.



– In order for the contractive fields above to be defined, it is necessary that the derivative along orbits starting from $\mathcal{C}$ experience some exponential growth. This growth, which is also useful for controlling the movements of the "turns", is brought about in two ways:

(i) by arranging for n_0 in the first paragraph to be very large, growth is guaranteed for a long initial period;

(ii) when an orbit of $\mathcal{C}$ gets near a point $z \in \mathcal{C}$, it copies the initial segment of the orbit of z, thereby *replicating* the growth properties created in (i).

(B) Getting started

– The required initial growth in (i) above comes from the Misiurewicz property of f, the 1-dimensional map of which T is a perturbation.

By choosing (a, b) sufficiently near $(a^*, 0)$ and δ sufficiently small, n_0 can be arranged to be arbitrarily large.

– Let Γ_0 be the set of all critical points of order n_0 on ∂R_0. We know that each connected segment of $\partial R_0 \cap \mathcal{C}^{(0)}$ contains exactly one point of Γ_0. These are our **critical points of generation** 0.

–In order to state properly our induction hypotheses, we introduce our main system constants. They are $\theta, \alpha, \beta, \rho$, c, and n_0 and δ (which we have met):

• There are two time scales, N and a much slower one θN, where θ is chosen so that $b^\theta = \mathcal{O}(1)$.

• $e^{-\alpha n}$ and $e^{-\beta n}$, with $\alpha << \beta << 1$, represent two small length scales.

• $c > 0$ is our target Lyapunov exponent; it is $< c_1$ where c_1 is as in Lemma 8.

• Finally, $0 < \rho < K^{-1}$ is an arbitrary number of order 1. It determines the rate at which our critical regions decrease in size.

– The order in which these constants are chosen is as follows: c, ρ, α and β are first fixed; δ is then taken as small as need to be. The last constants to be determined are n_0 and θ; observe that $n_0 \to \infty$ and $\theta \to 0$ corresponds essentially to $(a, b) \to (a^*, 0)$.

(C) Inductive assumption: (IA1)

– Let $N \geq n_0$ be a large number, and let Δ_N be the set of parameters retained after N iterates. We now formulate a set of inductive assumptions that describes the desired dynamical picture for $T = T_{a,b},\ (a, b) \in \Delta_N$.

– While we will continue to provide motivations and explanations, (IA1)–(IA6) below are to be viewed as formal inductive hypotheses. As before, let $z_i = T^i z_0$.

(IA1) *For all $k \leq \theta N$, the critical regions $\mathcal{C}^{(k)}$ are defined and have the geometric properties stated in* (1)(i), (ii) *and* (iii) *of the Theorem. Moreover, on each horizontal boundary of each component of $\mathcal{C}^{(k)}$, there is a critical point of order N located within $\mathcal{O}(b^{\frac{k}{3}})$ of the midpoint of the segment.*

– Critical points on $\partial\mathcal{C}^{(k)}$ are called **critical points of generation** k. They are part of the kth approximation of $\mathcal{C}$.

– In the course of our study, it will become clear that one needs only to track a small sample of the approximate critical set from each component of $\mathcal{C}^{(k)}$. We have elected to follow

closely all orbits starting from $\Gamma_{\theta N}$, the set of critical points of generation $\leq \theta N$.

(D) Inductive assumption: (IA2)

– If Q is a component of $\mathcal{C}^{(k)}$, we let L_Q denote the vertical line midway between the two vertical boundaries of Q.

Definition *We say $z \in \mathcal{C}^{(0)}$ is* **horizontally related** *or simply* **h-related** *to $\Gamma_{\theta N}$ if there exists a component Q of $\mathcal{C}^{(k)}$, $k \leq \theta N$, such that $z \in Q$ and $dist(z, L_Q) \geq b^{\frac{k}{20}}$. When this holds, we say z is h-related to z_0 for all $z_0 \in \Gamma_{\theta N} \cap Q$.*

– This is an attempt to describe the location of a point relative to $\Gamma_{\theta N}$, which, as $N \to \infty$, converges to a fractal set. $\Gamma_{\theta N} \cap Q$ is contained in a region of width $\mathcal{O}(b^{\frac{k}{4}})$ in the middle of Q,

so that z and $\Gamma_{\theta N} \cap Q$ have a very obviously horizontal relationship.

– We caution, however, that there may be points in $\Gamma_{\theta N}$ that are directly above or below z, and quite possibly both to its left and to its right. Observe also that if Q' is a component of $\mathcal{C}^{(k')}$ such that $z \in Q' \subset Q$, then $dist(z, L_{Q'}) \geq b^{\frac{k'}{20}}$.

Definition *For $z \in R_0$, we define its* **distance to the critical set**, *denoted $d_{\mathcal{C}}(z)$, as follows: for $z \notin \mathcal{C}^{(0)}$, let $d_{\mathcal{C}}(z) = \delta$; for $z \in \mathcal{C}^{(0)}$, we let $d_{\mathcal{C}}(z) = dist(z, L_Q)$ where Q is the component of $\mathcal{C}^{(k)}$ containing z and k is the largest number $\leq \theta N$ with $z \in \mathcal{C}^{(k)}$. We let $\phi(z)$ be one of the two points in $\partial Q \cap \Gamma_{\theta N}$ if z is h-related to $\Gamma_{\theta N}$.*

– For $z \in \mathcal{C}^{([\theta N])}$, the definitions of $d_{\mathcal{C}}(z)$ and $\phi(z)$ are temporary and will be modified as the induction progresses.

– To secure growth properties for the orbits of $\Gamma_{\theta N}$, we forbid them to approach the critical set too closely too soon. (IA2) is a result of parameter selection.

(IA2) *For all* $z_0 \in \Gamma_{\theta N}$ *and all* $i \leq N$, $d_{\mathcal{C}}(z_i) \geq min(\delta, e^{-\alpha i})$.

– (IA2) implies that for all $z_0 \in \Gamma_{\theta N}$ and $i \leq N$, z_i is h-related to $\Gamma_{\theta N}$ whenever it is in $\mathcal{C}^{(0)}$.

(a) Let $z \in Q \subset \mathcal{C}^{(k)}$. Then $k << i$ since $\rho^k \geq e^{-\alpha i}$.

(b) We know $z_i \in R_i$. If $k < [\theta N]$, then $z_i \in Q \cap R_{k+1}$, proving $d_{\mathcal{C}}(z_i) \geq \rho^{k+1} >> b^{\frac{k}{20}}$.

(c) If $k = [\theta N]$, then $d_{\mathcal{C}}(z_i) \geq e^{-\alpha i} \geq e^{-\alpha N} >> b^{\frac{1}{20}\theta N}$ provided that b^{θ} is chosen to be $< e^{-20\alpha}$.

– (IA2) corresponds to (G1) in 1D.

(E) Inductive assumption: (IA3)

Definition *(a) For arbitrary $z \in \mathcal{C}^{(0)}$, we define its* **fold period** $\ell(z)$ *to be the nonnegative integer $\ell \geq 1$ such that $b^{\frac{\ell}{2}}$ is closest to $d_{\mathcal{C}}(z)$.*

(b) Given $z_0 \in R_0$ and unit vector w_0, we let w_i^*, $i = 0, 1, 2, \cdots$, be given by the splitting algorithm with $\ell_i = \ell(z_i)$ assuming $e_{\ell(z_i)}$ is defined at z_i.

– For $\ell \leq N$, Lemma 2 gives an estimate on the size of the neighborhood of $\Gamma_{\theta N}$ on which e_ℓ is well defined. In particular, if z is h-related to $\Gamma_{\theta N}$, then $e_{\ell(z)}$ is defined at z.

– Recall that q_1 is the slope of e_1. We fix $\varepsilon_0 > 0$ such that $\varepsilon_0 << |\frac{\partial q_1}{\partial x}|$ in $\mathcal{C}^{(0)}$. For $z \in \partial R_k$, let $\tau(z)$ denote a tangent vector to ∂R_k

at z. In the angle estimates below, τ and e_ℓ are assumed to point in roughly the same direction as w.

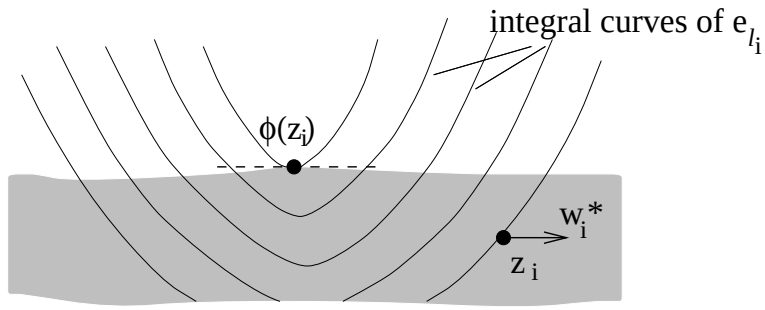
Definition *Let $z \in \mathcal{C}^{(0)}$ be h-related to $\Gamma_{\theta N}$, and let w be a vector at z. We say w* **splits correctly** *if $|\frac{w}{\|w\|} - \tau(\phi(z))| < \varepsilon_0 d_{\mathcal{C}}(z)$.*

(IA3) *For $z_0 \in \Gamma_{\theta N}, w_0 = \binom{0}{1}$ and $i \leq N$, w_i^* splits correctly whenever $z_i \in \mathcal{C}^{(0)}$.*

– The sense in which this splitting is "correct" is as follows. We wish to use Lemma 12 to understand the evolution of w_i, and (IA3) implies condition (a) of the lemma. This is because

$$
\begin{aligned}
&|e_{\ell_i}(z_i) - \frac{w_i^*}{\|w_i^*\|}| \geq |e_{\ell_i}(z_i) - e_{\ell_i}(\phi(z_i))| \\
&- |e_{\ell_i}(\phi(z_i)) - \tau(\phi(z_i))| - |\tau(\phi(z_i)) - \frac{w_i^*}{\|w_i^*\|}| \\
&\geq |\frac{\partial q_{\ell_i}}{\partial x}| d_{\mathcal{C}}(z_i) - \mathcal{O}(b^{\ell_i}) - \varepsilon_0 d_{\mathcal{C}}(z_i) \\
&\geq \ \frac{1}{2}|\frac{\partial q_1}{\partial x}| d_{\mathcal{C}}(z_i) \sim b^{\frac{\ell_i}{2}}.
\end{aligned}
$$

– Condition (b) of Lemma 12 is discussed later.



(F) Inductive assumption: (IA4)

We are now ready to state the correspondences of (G2).

(IA4) *For all* $z_0 \in \Gamma_{\theta N}$ *and* $0 \leq i \leq \frac{2}{3}N$, $\|w_i^*(z_0)\| > c_0 e^{ci}$.

(G) Inductive assumption: (IA5)

– Let us start with bound period.

Definition *For arbitrary* ξ_0 *and* $\xi_0' \in \mathcal{C}^{(0)}$*, we define their* **bound period** *to be the largest integer* p *such that for all* $0 < j \leq p$*,*

$$|\xi_j - z_j| \ \leq \ e^{-\beta j}.$$

– Observe that if $\xi_0' = z_0 \in \Gamma_{\theta N}$, then for $j \leq p$, $|\xi_j - z_j| \ << d_{\mathcal{C}}(z_j)$. We may assume δ is so small and n_0 so large that $d_{\mathcal{C}}(\xi_j) > \frac{\delta}{2}$ when z_j is outside of $\mathcal{C}^{(0)}$. Our last two inductive assumptions deal with the properties z_0 passes along to ξ_0.

(IA5) *Let* $z_0 \in \Gamma_{\theta N} \cap \partial\mathcal{C}^{(k)}$*, and let* $\gamma : [0, \varepsilon] \to \mathcal{C}^{(0)}$ *be a* $C^2(b)$*-curve with* $\gamma(0) = z_0$ *and* $\gamma'(0)$ *tangent to* $\partial\mathcal{C}^{(k)}$*. We regard all* $\xi_0 \in \gamma$ *as bound to* z_0*, and let* $p(\xi_0)$ *denote their bound periods. Then:*

(a) There exists K such that for $\xi_0 \in \gamma$ with $|\xi_0 - z_0| = e^{-h}$,

$$\frac{1}{K}h \leq p(\xi_0) \leq Kh \quad provided \quad Kh < \frac{2}{3}N;$$

moreover, $p(\xi_0)$ increases monotonically with the distance between ξ_0 and z_0;

(b) for $\ell \leq j \leq min(p, \frac{2}{3}N)$,

$$|\xi_j - z_j| \approx |\xi_0 - z_0|^2 \|w_j(z_0)\|$$

where "$\approx$" means up to a factor of $(1 \pm \varepsilon_1)$ for some $\varepsilon_1 > 0$;

(c) $\|w_p(\xi_0)\| \cdot |\xi_0 - z_0| \geq e^{\frac{cp}{3}}$ provided $p < \frac{2}{3}N$.

(IA5) describes the quadratic nature of the "turn" as γ is mapped forward. (IA5) is a correspondence of (P2) in 1D.

(H) Inductive assumptions: (IA6)

– Let $w_0(\xi_0) = w_0(z_0) = \binom{0}{1}$, and let $\widehat{w}_i^*(\xi_0)$ be given by the splitting algorithm except that $e_{\ell(z_i)}$ (and not $e_{\ell(\xi_i)}$) is used for splitting at time i. (IA6) compares $w_i^*(z_0)$ and $\widehat{w}_i^*(\xi_0)$. Let $M_i(\cdot)$ and $\theta_i(\cdot)$ denote the magnitude and argument of the vectors in question. Define

$$\Delta_i(\xi_0, z_0) = \sum_{s=0}^{i} (Kb)^{\frac{s}{4}} \mid \xi_{i-s} - z_{i-s} \mid .$$

– Local distortion

(IA6) *Given $z_0 \in \Gamma_{\theta N}$ and any $\xi_0 \in \mathcal{C}^{(0)}$, we regard ξ_0 as bound to z_0 and let p be the bound period. Then for $i \leq min\{p, N\}$,*

$$\frac{M_i(z_0)}{M_i(\xi_0)}, \quad \frac{M_i(\xi_0)}{M_i(z_0)} \ \leq \ \exp\{K \sum_{j=1}^{i-1} \frac{\Delta_j}{d_{\mathcal{C}}(z_j)}\}$$

and

$$\mid \theta_i(\xi_0) - \theta_i(z_0) \mid \leq (Kb)^{\frac{1}{2}} \Delta_{i-1}.$$

The estimates above also hold with $w_i^*(z_0)$ replaced by $\widehat{w}_i^*(\xi_0')$ where ξ_0' is another point in $\mathcal{C}^{(0)}$ also thought of as bound to z_0, and p is the minimum of the two bound periods.

– From the geometry of $\mathcal{C}^{(k)}$ (see (IA1)), it is an exercise in calculus to show that if ξ_0 is h-related to $z_0 \in \Gamma_{\theta N}$, then it lies on a $C^2(b)$-curve through z_0 tangent to $\tau(z_0)$. In particular, (IA5) applies.

– From this point on, (IA1)–(IA6) are assumed up to time N.

II. Orbits Controlled by $\Gamma_{\theta N}$

A. Nested bound period

– Consider $z_0 \in \Gamma_{\theta N}$. When z_i enters $\mathcal{C}^{(0)}$, it is natural to assign to it a **bound period** $p(z_i)$ defined using $\phi(z_i)$. An unsatisfactory aspect

of this definition is that two bound periods so defined may overlap without one being completely contained in the other. The purpose of this subsection is to adjust slightly the definition of $p(z_i)$ to create a simpler binding structure.

– First we fix some notation. Let $Q^{(j)}$ denote the components of $\mathcal{C}^{(j)}$, and let $\widehat{Q}^{(j)}$ be the component of $R_j \cap \mathcal{C}^{(j-1)}$ containing $Q^{(j)}$. For $z \in \partial R_j$, let $\tau(z)$ be a unit vector at z tangent to ∂R_j.

Lemma *For $z, z' \in \Gamma_{\theta N} \cap Q^{(k)}$, we have*

$$|z - z'| = \mathcal{O}(b^{\frac{k}{4}}) \quad \textit{and} \quad \|\tau(z) \times \tau(z')\| = \mathcal{O}(b^{\frac{k}{4}}).$$

Proof: Let $z^{(k)}$ be a critical point in $\partial Q^{(k)}$. For $k \leq i < [\theta N]$, let $z^{(i+1)}$ be a critical point of generation $i+1$ in $Q^{(i)}(z^{(i)})$, the component of

$Q^{(i)}$ containing $z^{(i)}$. From (IA1) we know that the Hausdorff distance between the two horizontal boundaries of $Q^{(i)}(z^{(i)})$ is $\mathcal{O}(b^{\frac{i}{2}})$. Lemma 12 then tells us that $|z^{(i)} - z^{(i+1)}| = \mathcal{O}(b^{\frac{i}{4}})$. The angle estimate also follows from the proof of Lemma 11 $\square$

Lemma *Let ξ_0 be h-related to $z_0 \in \Gamma_{\theta N}$. If during their bound period z_i returns to $\mathcal{C}^{(k)}$, then $\xi_i \in \widehat{Q}^{(k)}(z_i)$.*

Proof: Let γ be a $C^2(b)$-curve joining z_0 and ξ_0. Then $T^i\gamma \subset R_i$. Since $e^{-\alpha i} \leq d_{\mathcal{C}}(z_i) \leq \rho^k$, we have $k < i$ and therefore $T^i\gamma \subset R_k$. By the monotonicity of bound periods, every point in $T^i\gamma$ is within a distance of $< e^{-\beta i}$ from z_i. This puts $\xi_i \in R_k \cap Q^{(k-1)}(z_i)$. $\square$

Lemma *Let $z_0 \in \Gamma_{\theta N}$ be such that $z_i \in \mathcal{C}^{(0)}$ at times $t_1 < t_2 < \cdots < t_r$, and that for each $j < r$*

the bound period p_j initiated at time t_j extends beyond time t_{j+1}. Then $p_j < (K\alpha)^{j-1}p_1$.

Proof: Let $\tilde{z}_0 = \phi(z_{t_1})$. We claim that $|z_{t_2} - \phi(z_{t_2})| \approx |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$, which is $> e^{-\alpha(t_2-t_1)}$. If true, this will imply, by (IA5)(a), that $p_2 < K\alpha(t_2 - t_1) < K\alpha p_1$, and the assertion in the lemma will follow inductively. Since $|z_{t_2} - \tilde{z}_{t_2-t_1}| < e^{-\beta(t_2-t_1)} << e^{-\alpha(t_2-t_1)}$, it suffices to show that $|\phi(\tilde{z}_{t_2-t_1}) - \phi(z_{t_2})| << |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$. Let k be the largest number such that $\tilde{z}_{t_2-t_1} \in \mathcal{C}^{(k)}$. By the last lemma, $z_{t_2} \in Q^{(k-1)}(\tilde{z}_{t_2-t_1})$, so $\phi(\tilde{z}_{t_2-t_1})$ and $\phi(z_{t_2})$ must both be in $Q^{(k-1)}(\tilde{z}_{t_2-t_1})$. They are $\leq b^{\frac{k-1}{4}}$ apart, and this is $<< |\tilde{z}_{t_2-t_1} - \phi(\tilde{z}_{t_2-t_1})|$. $\square$

– **Definition** *For $z_0 \in \Gamma_{\theta N}$ with $z_i \in \mathcal{C}^{(0)}$, the* **adjusted bound period** *$p^*(z_i)$ is defined to be the smallest number p^* with the property that for all j with $i \leq j < i + p^*$, if $z_j \in \mathcal{C}^{(0)}$, then $j + p(z_j) \leq i + p^*$.*

Adjusted bound periods, therefore, have a nested structure by definition.

Corollary *(a)* $p^* \leq p + K\alpha p$.

(b) For $z_i \in \mathcal{C}^{(0)}$ *with* $\phi(z_i) = \hat{z}_0$*, we have for all* $j \leq p^*$*,*

$$|z_{j+i} - \hat{z}_j| < e^{-\beta^* j}$$

for some β^* *smaller than* β *and* $>> \alpha$.

– This amended definition gives critical orbits the following simple structure of **bound** and **free states**. We call z_i a **return** if $z_i \in \mathcal{C}^{(0)}$. Then z_i is free for $i \leq n_1$ where $n_1 > 0$ is the time of the first return, and it is in bound state for $n_1 < i \leq n_1 + p_1$ where p_1 is the bound period initiated at time n_1. After time $n_1 + p_1$, z_i remains free until its next return at time n_2, is bound for the next p_2 iterates, and so on. The times n_j are called **free return** times.

A **primary bound period** begins at each n_j. Inside the time interval $[n_j, n_j + p_j]$, there may be **secondary bound periods** which comprise disjoint time intervals, and so on.

B. Nested fold period

– Next we consider fold periods ℓ. As with bound periods, if z_i enters $\mathcal{C}^{(0)}$ at times t_1 and t_2 with $t_1 < t_2 \leq N$, and if the fold period begun at t_1 remains in effect at t_2, then $\ell_{t_2} < \frac{\alpha}{\log \frac{1}{b}} \ell_{t_1}$, so that **adjusted fold periods** can be defined similarly to give a nested structure. This is condition (b) of Lemma 12.

– A further simplifying arrangement, which we will also adopt, is that no fold periods expire at returns to $\mathcal{C}^{(0)}$ or at the step immediately after.

– The proof of the following lemma is straightforward

Lemma *Let $z_0 \in \Gamma_{\theta N}$. Then for every $i < N$, there exist $i_1 \leq i \leq i_2$ with*

$$i_2 - i_1 < K\theta\alpha i$$

such that i_1 and i_2 are out of all fold periods.

C. Orbits controlled by $\Gamma_{\theta N}$

– We consider (z_0, w_0) where z_0 is an arbitrary point in R_0 and w_0 is a unit vector. We write $z_i = T^i z_0$ and $w_i = DT^i(z_0)w_0$.

definition *We say (z_0, w_0) is* **controlled** *by $\Gamma_{\theta N}$ up to time m (with m possibly $> N$) if the following hold.*

- Initial conditions: if $z_0 \notin \mathcal{C}^{(0)}$, then w_0 is a b-horizontal vector; if $z_0 \in \mathcal{C}^{(0)}$, then either $w_0 = \binom{0}{1}$, or z_0 is h-related to $\Gamma_{\theta N}$ and w_0 splits correctly.

- *For* $0 < i \leq m$*, if* $z_i \in \mathcal{C}^{(0)}$*, then* z_i *is h-related to* $\Gamma_{\theta N}$ *and* w_i^* *splits correctly.*

• Let (z_0, w_0) be controlled. Then the orbit of z_0 has a natural bound/free structure. Nested structure can also be assumed for the bound and fold periods of controlled orbits.

• In the language of Definition above, the situation can be summed up as follows.

– First, it follows from (IA2) and (IA3) that for all $\hat{z}_0 \in \Gamma_{\theta N}$, $(\hat{z}_0, \binom{0}{1})$ is controlled by $\Gamma_{\theta N}$ up to time N.

– Second, for (z_0, w_0) controlled by $\Gamma_{\theta N}$, (IA5) and (IA6) apply to give information during its bound periods. In particular, the orbit of (z_0, w_0) has similar bound/free structures and "derivative recovery" estimates as those of $(\hat{z}_0, \binom{0}{1})$,

$\widehat{z}_0 \in \Gamma_{\theta N}$, except that (IA2) and (IA4) need not hold.

D. w_i **and** w_i^*

Assumed that (z_0, w_0) is controlled by $\Gamma_{\theta N}$ up to time m, and all time indices are $\leq m$.

Lemma *Assume that for* (z_0, w_0), $0 < i \leq m$, z_i *is h-related to* $\Gamma_{\theta N}$ *at all returns. Then* (z_0, w_0) *is controlled up to time* m *if* w_i^* *splits correctly at all free returns.*

Lemma*Under the additional assumption that* $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ *for all* $i \leq m$, *we have*

$$K^{-\varepsilon i}\|w_i^*\| \;\leq\; \|w_i\| \;\leq\; K^{\varepsilon i}e^{\alpha i}\|w_i^*\|, \qquad \varepsilon = K\alpha\theta.$$

Lemma *There exists* $c' > 0$ *such that for every* $0 \leq k < n$,

$$\|w_n^*\| \geq K^{-1}d_{\mathcal{C}}(z_j)e^{c'(n-k)}\|w_k^*\|$$

where j is the first time $\geq k$ when a bound period extending beyond time n is initiated. If no such j exists, set $d_{\mathcal{C}}(z_j) = 1$.

Lemma *Let $k < n$ and assume z_n is free. Then*

$$\|w_n\| > K^{-K\theta(n-k)} e^{c'(n-k)} \|w_k\|.$$

III. Controlled orbits as "guides" for other orbits

– (IA2)–(IA6) are about orbits starting from $\Gamma_{\theta N}$. In the above we introduced a class of orbits that successfully use orbits from $\Gamma_{\theta N}$ as their "guides".

– We now let these orbits serve as guides for other orbits and study the properties they pass along. This is the essence of the replication process.

– We assume that

(1) $z_0 \in \mathcal{C}^{(0)}$, $w_0 = \binom{0}{1}$, and (z_0, w_0) is controlled by $\Gamma_{\theta N}$ up to time m;

(2) $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $0 < i \leq m$.

A. Local distortion

Our first order of business is to establish that for all ξ_0 bound to z_0, $\widehat{w}_i^*(\xi_0)$ copies $w_i^*(z_0)$ faithfully.

Lemma *Let (z_0, w_0) be as above, and let $\xi_0 \in \mathcal{C}^{(0)}$ be an arbitrary point which we think of as bound to z_0. Let $M_\mu(\cdot)$ and $\theta_\mu(\cdot)$ have the same meaning in (IA6). Then the estimates for*

$$\frac{M_\mu(\xi_0)}{M_\mu(z_0)}, \quad \frac{M_\mu(z_0)}{M_\mu(\xi_0)} \quad and \quad |\theta_\mu(\xi_0) - \theta_\mu(z_0)|$$

as stated in (IA6) hold for all $\mu \leq min(p, m)$. The corresponding distortion estimates for two points ξ_0 and ξ_0' bound to z_0 apply as well.

This is local distortion estimate.

B. 1D similarity and derivative recovery

– Assume that z_0 is a critical point on a $C^2(b)$-curve. We study the quadratic behavior as this curve is iterated. More precisely, let e_m be the contractive field of order m, which is defined at z_0. We assume

(3) z_0 lies on a $C^2(b)$-curve $\gamma \subset \mathcal{C}^{(0)}$, and $e_m(z_0)$ is tangent to γ.

– For $\xi_0 \in \gamma$, let $p = p(\xi_0)$ denote the bound period between z_0 and ξ_0. We assume that during its bound period, the orbit of ξ_0 inherits the secondary and higher order bound structures of the orbit of z_0.

Lemma *In the part of γ where $p < m$, p increases monotinically with distance from z_0.*

– Let $s \to \xi_0(s)$ be the parametrization of γ by arc length with $\xi_0(0) = z_0$. The following lemma contains a **distance formula** for $|\xi_\mu(s) - z_\mu|$.

Lemma *Let $\varepsilon_1 > 0$ be given. Then assuming δ is sufficiently small, we have, for all $\mu \in \mathbb{Z}^+$ and $s > 0$ satisfying $\mu \leq m$, $(Kb)^{\frac{\mu}{2}} < s$ and $p(\xi_0(s)) \geq \mu$,*

$$|\xi_\mu(s) - z_\mu| \;\approx\; (1 \pm \varepsilon_1)\; \|w_\mu(0)\|\; K_1 s^2$$

where $K_1 = \frac{1}{2}|\frac{dq_1}{dx}(z_0)|$.

Corollary *Assume in addition to (1)–(3) above that $\|w_j^*(z_0)\| > e^{cj}$ for all $j \leq m$. Let $\xi_0 \in \gamma$. Suppose that $|\xi_0 - z_0| = e^{-h}$ and $p(\xi_0) \leq m$. Then*

(a) $\frac{h}{3K_2} \leq p \leq \frac{3h}{c}$ *where* $K_2 = \log \|DT\|$*;*

(b) $\|w_p(\xi_0)\| \cdot |\xi_0 - z_0| \geq e^{\frac{cp}{3}}$.

– We define for $\xi_0(s) \in \gamma$ the notion of a *fold period with respect to* z_0. This is the number ℓ such that $(Kb)^{\frac{\ell}{2}} \approx s$.

– If $\tau_0(\xi_0)$, the unit tangent vector to γ at ξ_0, is split according to this definition, then the rejoining of the E_i-vector for $\ell < i < p$ has negligible effect.

– We may assume also that as we iterate, the sub-segment of γ bound to z_0 acquires the same fold periods as z_i, and think of these as *secondary fold periods* for ξ_i .

Corollary *We let* $p = p(\xi_0)$ *where* $|\xi_0 - z_0| = e^{-h}$ *and assume that* z_p *is not in a fold period. Then*

(a) the subsegment of $T^p\gamma$ between ξ_p and z_p contains a curve $\geq e^{-K\beta h}$ in length and with b-horizontal tangent vectors;

(b)

$$\|\tau_p(\xi_0)\| \geq K^{-1}e^{h(1-\beta K)}.$$

IV. Control of $\partial R_k,\ k \leq \theta N$

– **Lemma** *Let γ be a subsegment of ∂R_k. If all the points on γ are free, then γ is a $C^2(b)$-curve.*

– For $z \in \partial R_k$, let $\tau(z)$ denote a unit tangent vector to ∂R_k at z.

Proposition *For every $\xi_0 \in \partial R_0$ and every $k \leq \theta N$, (ξ_0, τ_0) with $\tau_0 = \tau(\xi_0)$ is controlled up to time k by Γ_k.*

V. Extending control of $\Gamma_{\theta N}$-orbits to time $3N$

– We continue to assume (IA1)–(IA6), which guarantee that if $w_0 = \binom{0}{1}$, then for all $z_0 \in \Gamma_{\theta N}$, (z_0, w_0) is controlled up to time N by $\Gamma_{\theta N}$.

Proposition *If $z_0 \in \Gamma_{\theta N}$ satisfies $d_{\mathcal{C}}(z_i) > e^{-\alpha i}$ for all $i \leq 3N$, then (z_0, w_0) is automatically controlled by $\Gamma_{\theta N}$ up to time $3N$.*

– Assume that $z_k \in \mathcal{C}^{([\theta N])}$. let $j = [\theta N]$ for purposes of the following arguments.

Claim 1 *There exists j', $\frac{1}{3}j \leq j' < j$, such that if*

$$\xi_0 = z_{k-j'} \qquad and \qquad u_0 = \frac{w_{k-j'}(z_0)}{\|w_{k-j'}(z_0)\|} \ ,$$

then for $0 \leq s < j'$,

$$\|DT^s(\xi_0)u_0\| \geq \|DT\|^{-s}.$$

– Now by Lemma 3, there exists an integral curve γ of the most contracted field of order j' through ξ_0 having length $\mathcal{O}(1)$. Since γ follows roughly the direction of e_1, it has slope $> K^{-1}\delta$ outside of $\mathcal{C}^{(0)}$ and is roughly a parabola inside $\mathcal{C}^{(0)}$ (Lemma 9). In both cases, γ meets ∂R_0. Let $\xi_0' \in \gamma \cap \partial R_0$. Then

$$|\xi_s - \xi_s'| < (K^2 b)^s$$

for all $0 \leq s \leq j'$.

Claim 2 *$\xi_{j'}'$ is a free return.*

Claim 3 *With u_0 as in Claim 1, let*

$$\tau_i = DT^i(\xi_0')\tau_0, \qquad u_i = DT^i(\xi_0)u_0,$$

and let θ_i be the angle between u_i and τ_i. Then $\theta_{j'} \leq b^{\frac{j'}{2}}$.

VI. Verification of (IA1)–(IA6) up to time $3N$

Step 1 *Deletion of parameters.* We delete from Δ_N all (a,b) for which there exists $z_0 \in \Gamma_{\theta N}$ and $i,\ N < i \le 3N$, such that

$$d_{\mathcal{C}}(z_i) < e^{-\alpha i} \qquad \text{or} \qquad \|w_i^*(z_0)\| < e^{ci}.$$

The set of remaining parameters is called Δ_{3N}. We do not claim in (IA1)–(IA6) that Δ_{3N} has positive measure or even that it is nonempty. Steps 2–5 below apply to $T = T_{a,b}$ for $(a,b) \in \Delta_{3N}$.

Step 2 *Updating of* $\Gamma_{\theta N}$. For each $z_0 \in \Gamma_{\theta N}$, since $\|w_i\|$ grows exponentially, there exists a unique z_0' on the component of $\partial\mathcal{C}^{(k)}$ containing z_0 that is a critical point of order $3N$ (Lemma 10). Let $\Gamma'_{\theta N}$ be the set of these z_0', i.e. $\Gamma'_{\theta N}$ is a copy of $\Gamma_{\theta N}$ updated to order $3N$.

Step 3 *Construction of* $\Gamma_{3\theta N}$ *and* $\mathcal{C}^{(k)}$, $\theta N < k \leq 3\theta N$. We establish control of ∂R_k. Assuming that all has been accomplished for $k-1$. Then R_k meets each component $Q^{(k-1)}$ of $\mathcal{C}^{(k-1)}$ in at most a finite number of strips bounded by free, and hence $C^2(b)$, curves. Let γ be one of these curves. By Lemma 11, there exists a critical point $\widehat{z}_0 \in \gamma$ of order $\widehat{m} = min\{3N, -\log d(z_0, \gamma)^{\frac{1}{2}}\}$ where $z_0 \in \Gamma'_{\theta N}$ lies on the boundary of the component $Q^{([\theta N])}$ containing γ. Since $d(z_0, \gamma) = \mathcal{O}(b^{\frac{\theta N}{2}})$, we have, assuming θ is chosen with $e^{-3N} > K^{-N} > b^{\frac{\theta N}{4}}$, that $\widehat{z}_0$ is of order $3N$.

To continue, we need to set bindings for points in ∂R_k. Technically, only $z_0 \in \Gamma_{\theta N}$ (and not the critical points on ∂R_i, $\theta N < i \leq k$) can be used. This is of no concern to us for the following reason: for k' with $k < k' \leq 3\theta N$, only those parts of $\partial R_{k'}$ that are free are involved in the construction of $\mathcal{C}^{(k')}$; and for $\xi_0 \in \partial R_k \cap$

$\mathcal{C}^{([\theta n])}$, independent of which $z_0 \in Q^{([\theta n])}(\xi_0)$ we think of it as bound to, ξ_i will remain in bound state through time $3\theta N$ because $|\xi_i - z_i| \leq K^{3\theta N}\rho^{\theta N} << e^{-3\beta\theta N}$.

The newly constructed critical points in ∂R_k, $N < k \leq 3N$, together with $\Gamma'_{\theta N}$ form $\Gamma_{3\theta N}$. We have completed the verification of (IA1) up to time $3N$.

Step 4 *Updating the definitions of $d_{\mathcal{C}}(\cdot)$ and $\phi(\cdot)$.* Using $\Gamma_{3\theta N}$ and $\mathcal{C}^{(k)}$, $k \leq [3\theta N]$, we reset these definitions for $z \in \mathcal{C}^{([\theta N])}$. Since $|\text{old}\phi(z) - \text{new}\phi(z)| = \mathcal{O}(b^{\frac{\theta N}{4}})$ and $|\tau(\text{old}\phi(z)) - \tau(\text{new}\phi(z))| = \mathcal{O}(b^{\frac{\theta N}{4}})$, these changes have essentially no effect on the correctness of splitting for points with $d_{\mathcal{C}}(\cdot) > b^{\frac{3\theta N}{20}}$. The relations in (IA5) are also not affected.

Step 5 *Verification of (IA2)–(IA6) for $i \leq 3N$.* This is carried out in 3 stages.

(1) First we argue that for $z_0 \in \Gamma_{\theta N}$ (we really mean $\Gamma_{\theta N}$, not $\Gamma'_{\theta N}$), (IA2)–(IA6) hold for $i \leq 3N$: (IA2) and (IA4) hold by design; (IA3) is given by Proposition above, and (IA5) and (IA6) are proved in (IV) with $m = 3N$.

(2) With the properties of $\Gamma_{\theta N}$ in (1) having been established, we observe that continuing to use $\Gamma_{\theta N}$ as the source of control, the material in (III) and (IV) are now valid for times up to $min(m, 3N)$.

(3) We are now ready to argue that (IA2)–(IA6) hold for all $z'_0 \in \Gamma_{3\theta N}$. For each $z'_0 \in \Gamma_{3\theta N}$, whether it is in $\Gamma'_{\theta N}$ or of generation $> \theta N$, there exists $z_0 \in \Gamma_{\theta N}$ such that $|z'_0 - z_0| = \mathcal{O}(b^{\frac{\theta N}{4}})$. This implies, for $i \leq 3N$, that $|z'_i - z_i| < b^{\frac{\theta N}{4}} \|DT\|^{3N} << e^{-\beta 3N}$ provided θ is chosen so that $b^{\frac{\theta}{4}} \|DT\|^3 < \frac{1}{2} e^{-\beta}$.

(IA2) follows immediately from the corresponding condition for z_0. Regarding z_0' as bound to z_0 for at least $3N$ iterates, (IA3) and (IA4) follow from property (IA6) of z_0. Finally, regarding $(z_0', \binom{0}{1})$ as controlled by $\Gamma_{\theta N}$ up to time $3N$, we obtain (IA5) and (IA6) from (IV)