INFINITELY MANY SINKS AROUND NONUNIFORMLY EXPANDING 1D MAPS

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ABSTRACT. In this paper we study families of multi-modal 1D maps following the setting of Wang and Young [20]. Under a mild combinatoric assumption, we prove that for generic one parameter families of 1D maps containing a Misiurewicz map, parameters of non-uniformly expanding maps, the measure abundance of which was proved previously in [20], are accumulation points of paramaters admitting super-stable periodic sinks.

To motivate the main theorems of this paper, we start with a discussion on the studies of the quadratic family $\{f_a : a \in [1, 2]\}$ of 1D maps defined by $f_a(x) = 1 - ax^2$ on $I = [-1, 1]$. The dynamical properties of f_a depend sensitively on the value of the parameter a. For this family, there are two primary dynamical scenarios competing in the space of parameters. The first is the scenario of periodic sinks representing stability. The second is the scenario of positive Lyapunov exponents almost everywhere in I representing chaos. Let Δ_1 be the set of all $a \in [1, 2]$ such that f_a admits a periodic sink, and Δ_2 be the set of all $a \in [1, 2]$ such that f_a has positive Lyapunov exponents almost-everywhere in I. Notice that Δ_1 and Δ_2 are disjoint by definition. It has been proved that

(i) Δ_1 is open and dense in $[1, 2]$ ([4], [8]), and

(ii) $a = 2$ is a Lebesgue density point of Δ_2 ([6], [1]).

Statement (i) implies that the stable scenario of periodic sinks dominates parameter space in the topological sense. On the other hand, (ii) claims that, at least in the vicinity of $a = 2$, the chaotic scenario dominates in the measure-theoretic sense.

We study one parameter families of multi-modal 1D maps following the setting introduced by Wang and Young in [20]. This setting includes the quadratic family as a specific example. A correspondence of item (ii) above, in which $a = 2$ is replaced by the parameter of a Misiurewicz map, was proved in [20]. For a correspondence of item (i), the open part is obvious but the dense part has been a major challenge. The existing proofs for the quadratic family rely heavily on the specifics of the quadratic form therefore can not be modified to apply to other families of 1D maps. As a matter of fact, it is not even clear if this part of the claim remains true for multi-modal 1D families studied in [20]. In this paper we prove that, under a mild combinatoric assumption, parameters of chaotic maps such as those constructed in [20] are accumulation points of the set of parameters admitting periodic sinks.

The proofs of items (i) and (ii) above for the quadratic family have a long and celebrated history. That Δ_1 is dense in $[1, 2]$ was proved by Graczyk and Świątek [4],

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and Lyubich [8] independently. Later, Lyubich [9] also proved that $\Delta_1 \cup \Delta_2$ is a set of full measure in $[1, 2]$. The measure abundance of parameters admitting a.c.i.m. was first proved by Jakobson [6]. See also [11]. Based on techniques introduced by Benedick and Carleson [1], a proof of item (ii) for generic families of uni-modal 1D maps containing a Misiurewicz map was presented in [12]. The work of Wang and Young on multi-modal 1D maps [20] is a further generalization in the same direction. The families of 1D maps studied in [20] also serve as singular limits for *rank one maps* of higher dimensions, for which a dynamics theory has been developed by Wang and Young [17, 16] based on the theory of Benedick and Carleson on Hénon maps [2, 10, 3]. The theory of Wang and Young on rank one maps has been rigorously applied to various systems of applications [18, 19, 15, 5, 7].

This paper is organized as follows. We present the setting of [20] and we state Theorems A and B in Section 1. In Section 2, we prove Theorems A and B by introducing and applying Theorem C. Theorem C is proved in Section 3. Our proof of Theorem C is based on techniques presented in [20]. Unfortunately, what we need from [20] are scattered throughout that paper. To provide a coherent presentation, we gather what we need from [20] in one proposition. We then prove this proposition in Section 4. This paper is self-contained modulo one isolated technical result from [20].

We note that a restricted version of our Theorem B for the quadratic family around $a = 2$ was proved previously by Thunberg [13]. Ures Raúl also proved a theorem along the same line for Hénon-like maps [14].

1. Statement of results

1.1. **The class $\mathcal{E}$.** First we introduce Misiurewicz maps. Let I denote the unit interval or the circle. For $f \in C^2(I, I)$, let $C = C(f) = \{x \in I : f'(x) = 0\}$ denote the critical set of f. For $\delta > 0$, let C_δ denote the δ-neighborhood of C in I. Let $C_\delta(c)$ denote the δ-neighborhood of $c \in C$.

Definition 1.1 (**Misiurewicz map**). We say $f \in C^2(I, I)$ is a Misiurewicz map and we write $f \in \mathcal{E}$ if the following hold.

(A) There exists $\delta_0 > 0$ such that
   (1) for all $x \in C_{\delta_0}$, $f''(x) \neq 0$, and
   (2) for all $c \in C(f)$ and $n > 0$, $d(f^n(c), C(f)) \geqslant \delta_0$.

(B) There exist positive constants b_0 and λ_0 such that the following hold for all $\delta < \delta_0$ and $n > 0$.
   (1) If $f^k(x) \notin C_\delta$ for $0 \leqslant k \leqslant n-1$, then $|(f^n)'(x)| \geqslant b_0 \delta e^{\lambda_0 n}$.
   (2) If $f^k(x) \notin C_\delta$ for $0 \leqslant k \leqslant n-1$ and $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geqslant b_0 e^{\lambda_0 n}$.

This definition is equivalent to the definition of class $\mathcal{M}$ in Section 1.1 of [20] in the sense that $f \in \mathcal{E}$ if and only if $f \in \mathcal{M}$. Definition 1.1(B) asserts that derivatives grow at a uniform exponential rate (modulo a prefactor) along orbits that remain outside C_δ. For every $c \in C(f)$, the derivative $(f^n)'(f(c))$ grows exponentially by Definition 1.1(A2) and (B1).

1.2. **Admissible families.** Let $F : I \times [a_1, a_2] \to I$ be a C^2 map. The map F defines a one-parameter family $\{f_a \in C^2(I, I) : a \in [a_1, a_2]\}$ via $f_a(x) = F(x, a)$. We assume that there exists $a^* \in (a_1, a_2)$ such that $f_{a^*} \in \mathcal{E}$. For each $c \in C(f_{a^*})$, there exists a continuation $c(a) \in C(f(a))$ provided a is sufficiently close to a^* by Definition 1.1(A1).

Let $C(f_{a^*}) = \{c^{(1)}(a^*), \dots, c^{(q)}(a^*)\}$, where $c^{(i)}(a^*) < c^{(i+1)}(a^*)$ for $1 \leqslant i \leqslant q-1$. For $c(a^*) \in C(f_{a^*})$ we denote $\beta(a^*) = f_{a^*}(c(a^*))$. For all parameters a sufficiently close to a^*, there exists a unique continuation $\beta(a)$ of $\beta(a^*)$ such that the orbits

$$\{f_{a^*}^n(\beta(a^*)) : n \geqslant 0\} \text{ and } \{f_a^n(\beta(a)) : n \geqslant 0\}$$

have the same itineraries with respect to the partitions of I induced by $C(f_{a^*})$ and $C(f_a)$. This means that for all $n \geqslant 0$, $f_{a^*}^n(\beta(a^*)) \in (c^{(j)}(a^*), c^{(j+1)}(a^*))$ if and only if $f_a^n(\beta(a)) \in (c^{(j)}(a), c^{(j+1)}(a))$.[1] Moreover, the map $a \mapsto \beta(a)$ is differentiable (see Proposition 4.1 in [20]).

Definition 1.2. Let $F : I \times [a_1, a_2] \to I$ be a C^2 map. The associated one-parameter family $\{f_a : a \in [a_1, a_2]\}$ is ***admissible*** if

(i) there exists $a^* \in (a_1, a_2)$ such that $f_{a^*} \in \mathcal{E}$;

(ii) for all $c \in C(f_{a^*})$,

$$\xi(c) = \frac{d}{da}(f_a(c(a)) - \beta(a))\bigg|_{a=a^*} \neq 0. \tag{1.1}$$

1.3. **Main results.** Let $\{f_a : a \in [a_1, a_2]\}$ be an admissible family and let $a^* \in (a_1, a_2)$ be such that $f_{a^*} \in \mathcal{E}$. Set $b_1 = b_0\delta_0$ and fix $\lambda \leqslant \lambda_0/5$. For $\alpha > 0$, let $\Delta(\lambda, \alpha)$ denote the set of parameters $a \in [a_1, a_2]$ such that f_a satisfies the following: for all $c \in C(f_a)$ and for all $n > 0$,

(G1) $d(f_a^n(c), C(f_a)) \geqslant \min\{\delta_0/2, 2e^{-\alpha n}\}$, and

(G2) $|(f_a^n)'(f_a(c))| \geqslant 2b_1 e^{\lambda n}$.

It is proved in [20] that maps satisfying (G1) and (G2) admit invariant probability measures absolutely continuous with respect to Lebesgue measure provided α is sufficiently small and a is sufficiently close to a^*. It follows directly from the proofs in [20] that the relative measure $|\Delta(\lambda, \alpha) \cap (a^* - \epsilon, a^* + \epsilon)|/2\epsilon \to 1$ as $\epsilon \to 0$.

We now state the main theorems of this paper.

Theorem A. *Assume that $I = S^1$. Let $\{f_a : a \in [a_1, a_2]\}$ be an admissible family and let $a^* \in (a_1, a_2)$ be such that $f_{a^*} \in \mathcal{E}$. Suppose that f_{a^*} satisfies*

(1) $e^{\lambda_0} > 2$, and

(2) $f_{a^*}((c^{(i)}, c^{(i+1)})) = I$ for all $1 \leqslant i \leqslant q$.

Then for every α sufficiently small and for every $\widehat{a} \in \Delta(\lambda, \alpha)$ sufficiently close to a^*, there exists a sequence $a_n \to \widehat{a}$ such that for every $n \in \mathbb{N}$, the map f_{a_n} admits a superstable periodic sink.

Assumptions (1) and (2) in Theorem A can be dropped if $\{f_a\}$ is an admissible family of ***unimodal*** maps.

Theorem B. *Let $\{f_a : a \in [a_1, a_2]\}$ be an admissible family of unimodal maps and let $a^* \in (a_1, a_2)$ be such that $f_{a^*} \in \mathcal{E}$. Then for every α sufficiently small and for every $\widehat{a} \in \Delta(\lambda, \alpha)$ sufficiently close to a^*, there exists a sequence $a_n \to \widehat{a}$ such that for every $n \in \mathbb{N}$, the map f_{a_n} admits a superstable periodic sink.*

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[1] If I is an interval, let $c^{(0)}$ and $c^{(q+1)}$ denote the endpoints of I. If I is a circle, we use the cyclic convention $c^{(1)}(a) < \cdots < c^{(q)}(a) < c^{(q+1)}(a) = c^{(1)}(a)$.

2. Proofs of Theorems A and B

In this section we first introduce the combinatorics needed to state Theorem C. We then prove Theorems A and B assuming Theorem C. Theorem C is proved in Section 3.

For $f \in \mathcal{E}$, let $C = C(f) = \{c^{(1)}, \dots, c^{(q)}\}$ be the set of critical points of f. For $\delta > 0$ and $1 \leqslant i \leqslant q$, define $C_\delta^{(i)} = C_\delta(c^{(i)}) = \{x \in I : |x - c^{(i)}| < \delta\}$. Let $\delta < \delta_0$ be fixed.

For $1 \leqslant i \leqslant q$, let $J^{(i)}$ be a subinterval of $C_\delta^{(i)}$ and assume that there exist $n = n(i)$ and $j = j(i)$ associated with $J^{(i)}$ such that

(1) $f^k(J^{(i)}) \cap C_\delta = \emptyset$ for all $0 < k < n$, and
(2) $f^n(J^{(i)}) = C_\delta^{(j)}$.

Define the collection

$$\mathcal{J}_\delta = \{(J^{(i)}, n(i), j(i)) : 1 \leqslant i \leqslant q\}.$$

We associate a directed graph $\mathcal{P}(\mathcal{J}_\delta)$ with $\mathcal{J}_\delta$ as follows. The graph $\mathcal{P}(\mathcal{J}_\delta)$ contains q vertices $v_1, \dots, v_q$ representing $c^{(1)}, \dots, c^{(q)}$. There exists a directed edge from v_i to v_ℓ in $\mathcal{P}(\mathcal{J}_\delta)$ if and only if $j(i) = \ell$.

Definition 2.1. We say that a vertex v_{i_0} in $\mathcal{P}(\mathcal{J}_\delta)$ is ***completely accessible*** if for every $1 \leqslant i \leqslant q$, there exists a directed path from v_i to v_{i_0} in $\mathcal{P}(\mathcal{J}_\delta)$.

We now state Theorem C.

Theorem C. *Let $\{f_a : a \in [a_1, a_2]\}$ be an admissible family and let $a^* \in (a_1, a_2)$ be such that $f_{a^*} \in \mathcal{E}$. Fix $\lambda \leqslant \frac{\lambda_0}{5}$. Then for $\alpha < \lambda$ sufficiently small, there exists $\delta_1 > 0$ sufficiently small such that the following holds. If f_{a^*} admits a collection $\mathcal{J}_\delta$ such that the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ has a completely accessible vertex for some $\delta \leq \delta_1$, then for every $\widehat{a} \in \Delta(\lambda, \alpha)$ sufficiently close to a^*, there exists a sequence $a_n \to \widehat{a}$ such that for every $n \in \mathbb{N}$, the map f_{a_n} admits a superstable periodic sink.*

In the remainder of this section we prove Theorems A and B assuming Theorem C. We begin with two lemmas.

Lemma 2.2. *Let $f \in \mathcal{E}$ and let $C = C(f) = \{c^{(1)}, \dots, c^{(q)}\}$ be the set of critical points of f. The following holds provided δ is sufficiently small. For every $i \in \{1, \dots, q\}$, there exists a subinterval $J^{(i)} \subset C_\delta^{(i)}$, an integer $n(i) \leqslant 6\lambda_0^{-1} \log(\delta^{-1})$, and $j(i) \in \{1, \dots, q\}$ such that*

(1) $f^k(J^{(i)}) \cap C_\delta = \emptyset$ for all $0 < k < n(i)$, and
(2) $f^{n(i)}(J^{(i)}) = C_\delta^{(j)}$.

Proof: For $i \in \{1, \dots, q\}$, let S_i be one of the components of $C_\delta^{(i)} \setminus \{c^{(i)}\}$. We prove the existence of $J^{(i)}$, $n(i)$, and $j(i)$ in two steps. First, we iterate S_i under f until the image intersects C_δ for the first time. Let $m \geqslant 2$ be such that $f^k(S_i) \cap C_\delta = \emptyset$ for all $1 \leqslant k < m$. We have

$$\begin{aligned} |f^m(S_i)| &\geqslant |f(S_i)| \cdot \inf_{y \in f(S_i)} |(f^{m-1})'(y)| \\ &\geqslant \kappa \delta^2 \cdot b_0 \delta e^{\lambda_0 (m-1)} \end{aligned} \tag{2.1}$$

by Definition 1.1(A1) and (B1), where $\kappa = \frac{1}{2} \min_{x \in C_\delta} |f''(x)|$. This exponential growth estimate implies that the images of S_i must intersect C_δ. Let m_1 be the largest m as in the above. We have $f^{m_1}(S_i) \cap C_\delta \neq \emptyset$. Setting $m = m_1$ in (2.1), the inequality

$$\kappa\delta^3 b_0 e^{\lambda_0(m_1-1)} \leqslant 1$$

yields

$$m_1 \leqslant \frac{4}{\lambda_0}\log(\delta^{-1}). \tag{2.2}$$

If there exists $\ell \in \{1,\ldots,q\}$ such that $f^{m_1}(S_i) \supset C_\delta^{(\ell)}$, then we let $J^{(i)}$ be one of the connected components of $S_i \cap f^{-m_1}(C_\delta^{(\ell)})$ and we set $j(i) = \ell$ and $n(i) = m_1$. Otherwise, let $L_0 = f^{m_1}(S_i) \setminus C_\delta$. We have $|L_0| > \frac{1}{2}\delta_0$ if $\delta < \frac{\delta_0}{2}$ because $f \in \mathcal{E}$.

We now iterate L_0 forward in time and let $L_{k+1} = L_k \setminus f^{-(k+1)}(C_\delta)$ for $k \geq 0$. Let k_0 be the smallest k such that $f^k(L_{k-1}) \supset C_\delta^{(\ell)}$ for some ℓ. We argue that $k_0 \leqslant \frac{2}{\lambda_0}\log\delta^{-1}$. First, observe that for all $k < k_0$ we must have $|f^k(L_{k-1}) \cap C_\delta| \leqslant 4\delta$. Moreover, for $x \in L_{k-1}$ such that $f^k(x) \in C_\delta$, we have $|(f^k)'(x)| \geqslant b_0 e^{\lambda_0 k}$ by Definition 1.1(B2). Therefore,

$$|L_k| \geqslant |L_0| - 4\delta b_0^{-1}\sum_{\ell=1}^{k} e^{-\lambda_0\ell} \geqslant |L_0| - 4\delta b_0^{-1}e^{-\lambda_0}(1-e^{-\lambda_0})^{-1} \geqslant \frac{\delta_0}{4}$$

provided

$$\delta \leqslant \frac{\delta_0 b_0 e^{\lambda_0}(1-e^{-\lambda_0})}{16}.$$

This implies

$$|f^k(L_k)| \geqslant \frac{\delta_0}{4} b_0 \delta e^{\lambda_0 k} \tag{2.3}$$

for $k < k_0$ by Definition 1.1(B1). On the other hand, we must have

$$|f^k(L_k)| < 1,$$

inducing a contradiction to (2.3) if $k_0 > \frac{2}{\lambda_0}\log\delta^{-1}$. ∎

Lemma 2.3. *Assume that $f \in \mathcal{E}$ satisfies Assumptions (1) and (2) of Theorem A. For all δ sufficiently small, f admits a collection $\mathcal{J}_\delta$ such that the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ has a completely accessible vertex. Furthermore, for $(J^{(i)}, n(i), j(i)) \in \mathcal{J}_\delta$ we have $n(i) \leqslant K_0\log(\delta^{-1})$, where $K_0 = \frac{6}{\lambda_0 - \ln 2}$.*

Proof: Let $j_0 \in \{1,\cdots,q\}$ be fixed. We construct a directed graph $\mathcal{P}(\mathcal{J}_\delta)$ with completely accessible vertex v_{j_0}. Fix $i \in \{1,\ldots,q\}$ and let S_i be one of the components of $C_\delta^{(i)} \setminus \{c^{(i)}\}$. Arguing as in the proof of Lemma 2.2, there exists $m_1 \in \mathbb{Z}^+$ satisfying

$$m_1 \leqslant \frac{4}{\lambda_0}\log(\delta^{-1}) \tag{2.4}$$

such that $f^k(S_i) \cap C_\delta = \emptyset$ for all $1 \leqslant k < m_1$ and $f^{m_1}(S_i) \cap C_\delta \neq \emptyset$. If $f^{m_1}(S_i) \supset C_\delta^{(j_0)}$, then set $n(i) = m_1$ and let $J^{(i)}$ be one of the components of $S_i \cap f^{-m_1}(C_\delta^{(j_0)})$. If $f^{m_1}(S_i) \supset (c^{(\ell)}, c^{(\ell+1)})$ for some $1 \leqslant \ell \leqslant q$, then there exists a subinterval $\widehat{S}_i \subset S_i$ such that $f^{m_1}(\widehat{S}_i) \subset (c^{(\ell)}, c^{(\ell+1)})$ and $f^{m_1+1}(\widehat{S}_i) = C_\delta^{(j_0)}$ by assumption (2) of Theorem A. In this case, define $n(i) = m_1 + 1$ and set $J^{(i)} = \widehat{S}_i$. Otherwise, let L_0 be the component of $f^{m_1}(S_i) \setminus C_\delta$ with one end at $f^{m_1}(c^{(i)})$. We have $|L_0| \leqslant \frac{1}{2}\delta_0$ if $\delta \leqslant \frac{\delta_0}{2}$ because $f \in \mathcal{E}$.

We claim that there exists $m_2 \in \mathbb{Z}^+$ and a subinterval $L_1 \subset L_0$ such that $f^k(L_1) \cap C_\delta = \emptyset$ for $k < m_2$ and $f^{m_2}(L_1) = (c^{(\ell)}, c^{(\ell+1)})$ for some $\ell \in \{1, \ldots, q\}$. This claim is proved as follows. We iterate L_0 under f, deleting all parts that fall into C_δ. After k steps, the undeleted part of $f^k(L_0)$ is made up of finitely many segments. Suppose that for $k \leqslant n$, none of these segments contain some $(c^{(\ell)}, c^{(\ell+1)})$. This implies that the undeleted part of $f^n(L_0)$ consists of at most 2^n segments. We estimate the average length of these segments at time n. The pull-back to L_0 of all of the deleted parts has measure bounded above by

$$4\delta \sum_{k=1}^{n} 2^{k-1} b_0^{-1} e^{-\lambda_0 k} \leqslant \frac{\delta_0}{4}$$

by Definition 1.1(B2) provided

$$\delta \leqslant \frac{\delta_0 b_0 e^{\lambda_0}(1 - 2e^{-\lambda_0})}{16}.$$

The undeleted segments of $f^n(L_0)$ therefore have total length at least

$$\frac{\delta_0}{4} b_0 \delta e^{\lambda_0 n}$$

by Definition 1.1(B1) and because $|L_0| \geqslant \frac{\delta_0}{2}$. Since $f^n(L_0)$ consists of at most 2^n segments, their average length is bounded below by

$$\frac{\delta_0}{4} b_0 \delta 2^{-n} e^{\lambda_0 n}.$$

This estimate implies the claim since $e^{\lambda_0} > 2$.

Applying assumption (2) of Theorem A, there exists a subinterval $L_2 \subset L_1$ such that $f^{m_2+1}(L_2) = C_\delta^{(j_0)}$. Set $n(i) = m_1 + m_2 + 1$ and define $J^{(i)} = S_i \cap f^{-m_1}(L_2)$. The estimate

$$\frac{\delta_0}{4} b_0 \delta 2^{-(m_2-1)} e^{\lambda_0 (m_2 - 1)} \leqslant 1 \tag{2.5}$$

and (2.4) imply $n(i) \leqslant K_0 \log(\delta^{-1})$.

The vertex v_{j_0} of the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ associated with the collection $\mathcal{J}_\delta = \{(J^{(i)}, n(i), j_0), i \in \{1, \cdots, q\}\}$ is completely accessible. ■

Proof of Theorem A: (assume Theorem C) Let $\delta \leqslant \delta_1$ be small enough that Lemma 2.3 applies. By Lemma 2.3, the map f_{a^*} admits a collection $\mathcal{J}_\delta$ such that the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ has a completely accessible vertex. Theorem A is now a direct corollary of Theorem C. ■

Proof of Theorem B: (assume Theorem C) Let $\delta \leqslant \delta_1$ be small enough that Lemma 2.2 applies. By Lemma 2.2, the map f_{a^*} admits a directed graph $\mathcal{P}(\mathcal{J}_\delta)$ consisting of one vertex and one directed edge. This vertex is completely accessible, so Theorem B follows from Theorem C. ■

Remarks: (1) Theorem A and Theorem B are simply two specific propositions derived from Theorem C. One could formulate and prove variations of Theorems A and B based on Theorem C, including a version of Theorem A for interval maps.

(2) Once a directed graph $\mathcal{P} = \mathcal{P}(\mathcal{J}_\delta)$ is obtained, checking for the existence of a completely accessible vertex is straightforward. We associate a $q \times q$ matrix $D = (d_{ij})$ with $\mathcal{P}$ as follows. Set $d_{ij} = 1$ if there exists a directed edge from v_i to v_j in $\mathcal{P}$ and set $d_{ij} = 0$ otherwise. For $n > 0$, write $D^n = (d_{ij}^{(n)})$. The vertex v_{j_0}

is completely accessible if and only if for every $1 \le i \le q$, there exists $n > 0$ such that $d_{ij_0}^{(n)} \neq 0$.

3. Proof of Theorem C

3.1. **Preliminaries.** Let $\mathcal{F} = \{f_a : a \in [a_1, a_2]\}$ be an admissible family and let $a^* \in (a_1, a_2)$ be such that $f_{a^*} \in \mathcal{E}$. Let $\lambda \leqslant \frac{\lambda_0}{5}$ be fixed throughout. In what follows, three positive constants are critically important. They are α, δ, and ϵ. The constant α is as in (G1), δ represents the size of the critical interval around each critical point, and ϵ is used to define a small parameter interval $\Delta_0 = [a^* - \epsilon, a^* + \epsilon]$ around a^*. In the rest of this paper, we will only consider parameters $a \in \Delta_0$. The constants α, δ, and ϵ are small numbers chosen in the order that they are listed here. That is, δ depends on α and ϵ depends on both α and δ. All three constants are assumed to be sufficiently small and are reduced in size if necessary as we go along. The letter K represents generic constants *independent* of α, δ and ϵ. We allow the value of K to vary from line to line. Specific values of K we wish to track will be given subscripts.

For notational simplicity, let us assume from this point on that $C(f_a) = C(f_{a^*}) = C$ where $C = \{c^{(1)}, \cdots, c^{(q)}\}$ for $a \in \Delta_0$. Let $\mathcal{Q}_0 = \{I_\mu\}$ be the following partition on $(-\delta, \delta)$. Assume $\delta = e^{-\mu_*}$ for some $\mu_* \in \mathbb{Z}^+$. For $\mu \geqslant \mu_*$, let $I_\mu = (e^{-(\mu+1)}, e^{-\mu})$. For $\mu \leqslant -\mu_*$, let I_μ be the reflection of $I_{-\mu}$ about 0. We define a partition $\mathcal{Q}(c)$ on $C_\delta(c)$ for a critical point $c \in C$ by shifting the center of $\mathcal{Q}_0$ from 0 to c. We refer to the members of $\mathcal{Q}(c)$ simply as 'I_μ'. For $I_\mu \in \mathcal{Q}(c)$, let I_μ^+ denote the union of I_μ and the two elements of $\mathcal{Q}(c)$ adjacent to it (for I_{μ^*} we use I_{μ^*-1} as one of the adjacent intervals). For an interval $\omega \subset C_\delta$, we write $\omega \approx I_\mu$ if $I_\mu \subset \omega \subset I_\mu^+$.

Recall that $\{f_a\}$ is defined by $f_a(x) = F(x, a)$ for some C^2 map $F : I \times [a_1, a_2] \to I$. Also recall that $\Delta(\lambda, \alpha)$ is defined through (G1) and (G2) in Section 1.3. First we have

Lemma 3.1. *Let α be sufficiently small and assume that $\widehat{a} \in \Delta(\lambda, \alpha)$. There exists a constant $L > 1$ independent of α, δ, and ϵ such that for all N sufficiently large, $a \in [\widehat{a} - L^{-N}, \widehat{a} + L^{-N}]$, $c \in C$, and $n \le N$, we have*

(1) $d(f_a^n(c), C) \geqslant \min\{\delta_0/2, e^{-\alpha n}\}$;
(2) $|(f_a^n)'(f_a(c))| \geqslant b_1 e^{\lambda n}$.

Proof: In this proof we denote $\hat{K} = \|F\|_{C^2}$. For $n \le N$, let $x_n = f_a^n(c)$ and $\widehat{x}_n = f_{\widehat{a}}^n(c)$. For $a \in [\widehat{a} - L^{-N}, \widehat{a} + L^{-N}]$, we have

$$\begin{aligned} |x_n - \widehat{x}_n| &\le \hat{K}(|a - \widehat{a}| + |x_{n-1} - \widehat{x}_{n-1}|) \\ &\le (\hat{K} + \hat{K}^2 + \cdots + \hat{K}^n)|a - \widehat{a}| \\ &< K_1^n |a - \widehat{a}| \end{aligned}$$

for some $K_1 > 0$ depending on $\hat{K}$. Let $L > 2K_1$. We have $|x_n - \widehat{x}_n| \ll 1$. This implies $|d(x_n, C) - d(\widehat{x}_n, C)| = |x_n - \widehat{x}_n|$, and

$$\frac{d(x_n, C)}{d(\widehat{x}_n, C)} \ge 1 - \frac{|x_n - \widehat{x}_n|}{d(\widehat{x}_n, C)} > 1 - \frac{1}{2} e^{\alpha n} K_1^n |a - \widehat{a}| > \frac{1}{2}.$$

This combined with (G1) for $d(\widehat{x}_n, C)$ implies (1) of this lemma.

To prove (2) we let $w_n = |(f_a^n)'(f_a(c))|$, $\widehat{w}_n = |(f_{\widehat{a}}^n)'(f_{\widehat{a}}(c))|$. We have

$$\begin{aligned}
\log \frac{\widehat{w}_n}{w_n} &= \sum_{i=1}^{n-1} \log\left(1 + \frac{f'_{\widehat{a}}(\widehat{x}_i) - f'_a(x_i)}{f'_a(x_i)}\right) \\
&\leq \sum_{i=1}^{n-1} \frac{\hat{K}|x_i - \widehat{x}_i|}{K^{-1}d(x_i, C)} \\
&< \left(\sum_{i=1}^{n-1} K K_1^i e^{\alpha i}\right) |a - \widehat{a}| \\
&< \log 2.
\end{aligned}$$

for L sufficiently large (independent of α, δ and ϵ). Property (2) now follows from (G2) for $\widehat{w}_n$. ∎

3.2. **Evolution of critical curves.** For $\widehat{a} \in \Delta(\lambda, \alpha)$, denote $\Delta_N(\widehat{a}) = [\widehat{a} - L^{-N}, \widehat{a} + L^{-N}]$ where L is as in Lemma 3.1. It suffices to prove that there exits N_0 sufficiently large such that for every $N > N_0$, there exists $a_N \in \Delta_N(\widehat{a})$ such that f_{a_N} admits periodic sinks. In this subsection we fix $i_0 \in \{1, \cdots, q\}$. We define $c = c^{(i_0)}$ and $\gamma_n(a) = f_a^n(c)$. First we need to study the evolutions of curves $\gamma_n : \Delta_N(\widehat{a}) \to I$. We denote

$$\psi = \frac{3\alpha}{\lambda}.$$

For the evolutions of γ_n, there are four time indices worth noting. The first is the time the derivatives of γ_n start to grow exponentially. This time will be denoted as m_0. The second is a time index $N \gg m_0$, sufficiently large such that $\Delta_N(\widehat{a}) \subset \Delta_0$. Note that N is chosen after ϵ. The third time index is $\frac{1}{\psi}N$. This is a time before which the size of γ_n must be relatively long. Observe that $\frac{1}{\psi}N \sim \frac{1}{\alpha}N \gg N$. Finally, let N_1 be the time we are about to find the parameter a_N. The time N_1 is in general larger than N but could be either smaller or larger than $\frac{1}{\psi}N$. The existence of all these time indices will soon become clear.

Let us now define the set of *good parameters* $\Pi(N_1)$ in $\Delta_N(\widehat{a})$ for the proofs of this paper. For $N < N_1 \leq \frac{1}{\psi}N$, we define $\Pi(N_1)$ as the set of all $a \in \Delta_N(\widehat{a})$ such that the rule of distance exclusion (1) in Lemma 3.1, that is

$$d(f_a^n(c), C) \geqslant \min\{\delta_0/2, e^{-\alpha n}\}, \tag{3.1}$$

holds for all $n \leq N_1 \leq \frac{1}{\psi}N$. For $N_1 > \frac{1}{\psi}N$, we define $\Pi(N_1)$ as the set of all $a \in \Delta_N(\widehat{a})$ such that the rule of distance exclusion (3.1) holds up to time $\frac{1}{\psi}N$ and for $\frac{1}{\psi}N < n \leq N_1$,

$$d(f_a^n(c), C) \geqslant e^{-\alpha \frac{1}{\psi} N}. \tag{3.2}$$

We now state the properties of the evolutions γ_n we need in proving Theorem C. In what follows,

$$\tau_n(a) = \frac{d}{da}\gamma_n(a).$$

Proposition 3.2. *Assume that α, δ, and ϵ are sufficiently small and suitably related. Then*

(D0) there exists $m_0 > 0$ such that $|\tau_{m_0}(a)| > e^{\frac{1}{3}\lambda m_0}$ for all $a \in \Delta_0$.

In addition, assume that $N \gg m_0$ is sufficiently large and let $a \in \Delta_N(\widehat{a})$ be such that $a \in \Pi(n)$ for some $n \geq m_0$. Then there exist constants $K_1, K_2 > 1$ independent of α, δ, and ϵ such that

(D1) **(Outside C_δ)** *for $m > 0$,*

> *(1) if $\gamma_{n+k}(a) \notin C_\delta$ for all $0 \leqslant k < m$, then $|\tau_{n+m}(a)| \geqslant K_1^{-1}\delta e^{\frac{1}{2}\lambda_0 m}|\tau_n(a)|$;*
> *(2) if, in addition, $\gamma_{n+m}(a) \in C_{\delta_0}$, then $|\tau_{n+m}(a)| \geqslant K_1^{-1}e^{\frac{1}{2}\lambda_0 m}|\tau_n(a)|$;*

(D2) **(Recovery)** *if $\gamma_n(a) \in I_\mu$, then there exists $K_2^{-1}|\mu| \leq p(a) \leq \frac{3}{\lambda}|\mu|$ such that $a \in \Pi(n + p(a) - 1)$. We also have*

> *(1) $|\tau_{n+p(a)}(a)| \geq e^{\frac{1}{3}\lambda p(a)}|\tau_n(a)|$;*
> *(2) if there is an interval $\omega \subset \Pi(n)$ such that $\gamma_n(\omega) \approx I_\mu$, then $|\gamma_{n+p}(\omega)| \geqslant e^{-\frac{8\alpha}{\lambda}|\mu|}$ for some $K_2^{-1}|\mu| \leqslant p \leqslant \frac{3}{\lambda}|\mu|$.*

(D0) states that there is a time m_0 when we see exponential growth of derivatives of the critical curves with respect to a for all $a \in \Delta_0$. (D1) states that along the critical orbit of a good parameter (a parameter satisfying (3.1) and (3.2)), the derivatives of the critical curves grow exponentially as long as the orbit stays out of C_δ. (D2) states that the potential drop in the derivative caused by a return to C_δ will be compensated for by growth in future iterates. In a relatively short period of time, exponential growth of derivatives will again be observed. Note that there is no need for us to put an upper bound on n in this proposition. $\Pi(n)$ is well-defined for all $n > m_0$.

Let $a \in \Pi(N_1)$. This proposition implies the following for the evolutions of the derivatives $\tau_n(a)$ for $n \leq N_1$. The derivative for the first m_0 iterates is not relevant. Let $t_1 \geq m_0$ be the first time $\gamma_{t_1}(a) \in C_\delta$. We call t_1 the first free return time. For $m_0 \leq n \leq t_1$, τ_n grows exponentially according to (D1). The derivative will drop at $\gamma_{t_1}(a)$ because $\gamma_{t_1}(a)$ is close to C. However, (D2) claims that there exists p_1 relatively small such that the derivative at $t_1 + p_1$ has regained a definite amount of exponential growth. We call the time period from t_1 to $t_1 + p_1$ a bound period. We then have the next free return time t_2 and the next bound period p_2, and so on. In this way, the time interval from m_0 to N_1 is divided into an alternating sequence of ***free intervals*** $(t_k + p_k, t_{k+1}]$ and ***bound periods*** $(t_{k+1}, t_{k+1} + p_{k+1}]$. We have the following corollary of (D0)-(D2).

Corollary 3.3. *For $a \in \Pi(n)$, let $i > t_1 + p_1$ be a free return time such that $i < n$. Then*

$$|\tau_i(a)| > e^{\frac{\lambda}{4}i}.$$

Proof: Let i be such that

$$m_0 \leq t_1 < t_1 + p_1 \leq \cdots < t_k + p_k \leqslant i,$$

where t_j are the times of free returns and p_j are the corresponding bound periods before time i. Combining (D1)(2) and (D2)(1), we have

$$\begin{aligned}|\tau_i| &> |\tau_{m_0}|K^{-1}e^{\frac{\lambda_0}{2}(t_1-m_0)} \cdot e^{\frac{\lambda}{3}p_1} \cdot K^{-1}e^{\frac{\lambda_0}{2}(t_2-t_1-p_1)} \cdot\ e^{\frac{\lambda}{3}p_2} \cdots K^{-1}e^{\frac{\lambda_0}{2}(i-t_k-p_k)}\\ &> e^{\frac{1}{4}\lambda i}.\end{aligned}$$

Note that each copy of K^{-1} is absorbed by reducing the exponent of growth from $\frac{\lambda}{3}$ in (D2)(1) to $\frac{\lambda}{4}$. It suffices to take $\delta = e^{-\mu^*}$ small enough so that $K^{-1}e^{\frac{1}{12}K_2^{-1}\lambda\mu^*} > 1$. ■

3.3. **Proof of Theorem C assuming Proposition 3.2.** Let α be small enough so that

$$e^{\frac{1}{12\alpha}} > L, \tag{3.3}$$

where L is as in Lemma 3.1. Assume that for $\delta > 0$ sufficiently small, there exists a collection $\mathcal{J}_\delta$ such that the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ admits a completely accessible vertex v_{i_0}. Denote $c = c^{(i_0)}$ and let $\gamma_n : \Delta_N(\widehat{a}) \to I$ be such that $\gamma_n(a) = f_a^n(c)$. We assume that $\delta > 0$ is small enough so that Proposition 3.2 holds. Choose N such that $N > m_0$ and $\Delta_N(\widehat{a}) \subset \Delta_0$.

Step 1. We prove that there exists a time n_0, $m_0 \ll n_0 < \frac{N}{\psi}$, such that

(i) for each $n < n_0$, either $\gamma_n(\Delta_N(\widehat{a}))$ is completely out of C_δ or there exists $\mu \in \mathbb{Z}$ satisfying $|\mu| \geq \mu^*$ ($\mu^* = \log \delta^{-1}$) such that $\gamma_n(\Delta_N(\widehat{a})) \subset I_\mu^+$;

(ii) there exists $\mu \in \mathbb{Z}$ satisfying $|\mu| \geqslant \mu^*$ such that $\gamma_{n_0}(\Delta_N(\widehat{a})) \supset I_\mu$.

If these are false, then we have (a) $|\gamma_{\frac{N}{\psi}}(\Delta_N(\widehat{a}))| < 1$, and (b) $\Delta_N(\widehat{a})) = \Pi(\frac{1}{\psi}N)$. Statement (b) holds because for $n < \frac{N}{\psi}$, when $\gamma_n(\widehat{a})$ returns to C_δ, all points of $\gamma_n(\Delta_N(\widehat{a}))$ fall into the same I_μ^+ interval where $\gamma_n(\widehat{a})$ is located. It follows that we can define identical bound periods and free time intervals for all $a \in \Delta_N(\widehat{a})$. We now argue that (b) contradicts (a) because from (b), (D1)(1), and Corollary 3.3, we have

$$|\gamma_{\frac{N}{\psi}}(\Delta_N(\widehat{a}))| > K^{-1}\delta e^{\frac{\lambda}{4}\frac{N}{\psi}} L^{-N} \gg 1$$

provided that $\frac{1}{\psi}N$ is a free time. Note that the last estimate uses (3.3). If $\frac{1}{\psi}N$ is not free, then it is inside of a bound period, say $[t_k, t_k + p_k)$. We argue that $t_k \geqslant \frac{1}{\psi}N - N$. This is because $p_k \leq N$ by (D2) and (3.1). Using Corollary 3.3 for t_k instead of $\frac{1}{\psi}N$ induces a similar contradiction.

Step 2. We prove that there exists a subinterval $\omega \subset \Delta_N(\widehat{a})$ and $N_1 > n_0$ such that $\gamma_{N_1}(\omega) \supset C_\delta(c^{(j)})$ for some $j \in \{1, \cdots, q\}$. This is proved as follows: Let $|\mu_0|$ be the smallest $|\mu| \in \mathbb{Z}^+$ such that $\gamma_{n_0}(\Delta_N(\widehat{a})) \supset I_{\mu_0}$, and let $\rho_0 \subset \Delta_N(\widehat{a})$ be an interval such that $\gamma_{n_0}(\rho_0) = I_{\mu_0}$. Since $\widehat{a} \in \Delta(\lambda, \alpha)$, the choice of μ_0 implies that $\rho_0 \subset \Pi(n_0)$.

Applying (D2)(2), there exists a recovery time $p_0 \leqslant \frac{3}{\lambda}|\mu_0|$ such that

$$|\gamma_{n_0+p_0}(\rho_0)| \geqslant e^{-\frac{8\alpha}{\lambda}|\mu_0|}. \tag{3.4}$$

Suppose that for some $\ell \geqslant 0$, $\gamma_{n_0+p_0+k}(\rho_0)$ contains no I_μ for all $0 \leqslant k \leqslant \ell$, and $n_0 + p_0 + \ell$ is a free return time. From $n_0 + p_0$ to $n_0 + p_0 + \ell$, the image of ρ_0 is either free and stays outside of C_δ, or it returns to C_δ freely with the image completely contained inside of some I_μ^+. Properties (D1) and (D2) together imply that $\rho_0 \subset \Pi(n_0 + p_0 + \ell)$, and from Corollary 3.3 we have

$$|\gamma_{n_0+p_0+\ell}(\rho_0)| \geqslant e^{\frac{1}{4}\lambda\ell - \frac{8\alpha}{\lambda}|\mu_0|}. \tag{3.5}$$

It then follows that there exists a free return time $n_1 > n_0 + p_0$ such that $\gamma_{n_1}(\rho_0) \supset I_\mu$ for some I_μ for the first time. Let I_{μ_1} be the longest I_μ inside of $\gamma_{n_1}(\rho_0)$ and let

$\rho_1 \subset \rho_0$ be a subinterval such that $\gamma_{n_1}(\rho_1) = I_{\mu_1}$. We have

$$|\mu_1| \leqslant \frac{8\alpha}{\lambda}|\mu_0| - \frac{\lambda}{4}(n_1 - (n_0 + p_0))$$

by (3.5). The choice of μ_1 implies that $\rho_1 \subset \Pi(n_1)$.

Inductively, suppose we have constructed $\rho_k \subset \rho_{k-1}$, $\mu_k \in \mathbb{Z}$, and $n_k \in \mathbb{Z}^+$ such that $\rho_k \subset \Pi(n_k)$ and $\gamma_{n_k}(\rho_k) = I_{\mu_k}$. Applying (D2)(2), there exists $p_k \leqslant \frac{3}{\lambda}|\mu_k|$ such that

$$|\gamma_{n_k+p_k}(\rho_k)| \geqslant e^{-\frac{8\alpha}{\lambda}|\mu_k|}. \tag{3.6}$$

Suppose that for some $\ell \geqslant 0$, $\gamma_{n_k+p_k+j}(\rho_k)$ contains no I_μ for all $0 \leqslant j \leqslant \ell$, and $n_k + p_k + \ell$ is a free return time. Then $\rho_k \subset \Pi(n_k + p_k + \ell)$ with

$$|\gamma_{n_k+p_k+\ell}(\rho_k)| \geqslant e^{-\frac{8\alpha}{\lambda}|\mu_k|+\frac{1}{4}\lambda\ell}. \tag{3.7}$$

Again, let $n_{k+1} \geqslant n_k + p_k$ be the smallest free return time such that $\gamma_{n_{k+1}}(\rho_k) \supset I_\mu$ for some I_μ. Let $I_{\mu_{k+1}}$ be the longest I_μ such that $\gamma_{n_{k+1}}(\rho_k) \supset I_\mu$ and let $\rho_{k+1} \subset \rho_k$ be a subinterval such that $\gamma_{n_{k+1}}(\rho_{k+1}) = I_{\mu_{k+1}}$. We have

$$|\mu_{k+1}| \leqslant \frac{8\alpha}{\lambda}|\mu_k| - \frac{\lambda}{4}(n_{k+1} - (n_k + p_k)) \tag{3.8}$$

by (3.7). The choice of μ_{k+1} implies that $\rho_{k+1} \subset \Pi(n_{k+1})$.

This inductive procedure must terminate after finitely many steps because of (3.8). So there exists a free return time n_k, a subinterval $\rho_k \subset \Delta_N(\widehat{a})$ and $|\mu_k| = \mu^*$ such that $\gamma_{n_k}(\rho_k) = I_{\mu_k}$. Letting the bound period for this free return be p_k, we also have

$$|\gamma_{n_k+p_k}(\rho_k)| \geqslant e^{-\frac{8\alpha}{\lambda}\mu_*} \gg \delta.$$

At this point we repeat the proof of Lemma 2.2 (regarding δ_0 in that proof as $|\gamma_{n_k+p_k}(\rho_k)|$ and putting (D1) in the position of Definition 1.1(B)). We conclude that there exists $\omega \subset \rho_k$, $N_1 > n_k + p_k$, and a critical point $c^{(j)} \in C$ such that

$$\gamma_{N_1}(\omega) = C_\delta(c^{(j)}). \tag{3.9}$$

Step 3. Recall that v_{i_0} is a completely accessible vertex of the directed graph $\mathcal{P}(\mathcal{J}_\delta)$ associated with a collection

$$\mathcal{J} = \{(J^{(i)}, n(i), j(i)) : i \in \{1, \cdots, q\}\}$$

satisfying (1) and (2) at the beginning of Section 2 for $f = f_{a^*}$. We start with the index j in (3.9). Since i_0 is completely accessible, there is a path

$$v_j = v_{j_0} \to v_{j_1} \to \cdots \to v_{j_m} = v_{i_0}$$

in $\mathcal{P}(\mathcal{J}_\delta)$. Let

$$T = n(j_0) + n(j_1) + \cdots + n(j_{m-1}).$$

We conclude that there exists an interval $J \subset J^{(j)}$ such that

$$f^T_{a^*}(J) = C^{(i_0)}_\delta. \tag{3.10}$$

We are finally ready to finish our construction of a parameter admitting a superstable periodic sink. Let $T \in \mathbb{Z}^+$ and let the subinterval J be as in (3.10). Set $\hat{K} = \|F\|_{C^1}$. Let ϵ be sufficiently small so that

$$\hat{K}^T|a - a^*| \leqslant \frac{\delta}{10}$$

for all $a \in \Delta_0$. Let $\zeta \subset \omega$ be such that $\gamma_{N_1}(\zeta) = J$. For $a \in \zeta$,

$$|\gamma_{N_1+T}(a) - f_{a^*}^T(\gamma_{N_1}(a))| = |F^T(\gamma_{N_1}(a), a) - F^T(\gamma_{N_1}(a), a^*)| \leqslant K^T|a - a^*| \leqslant \frac{\delta}{10}.$$

Since $f_{a^*}^T(J) = C_\delta^{(i_0)}$, we conclude that there exists $a \in \zeta$ such that $f_a^{N_1+T}(c^{(i_0)}) = \gamma_{N_1+T}(a) = c^{(i_0)}$. This finishes the proof of Theorem C.

4. Proof of Proposition 3.2

In this section we prove Proposition 3.2. The conclusions we gathered in this proposition are proved in [20]. These conclusions and their proofs, however, are mixed with other more complicated considerations in [20], such as estimates on global distortions, a large deviation argument and interactions of different critical curves. To make a coherent presentation, we provide a self-contained proof in this section. We hope this will save the reader the trouble of going through the entire length of [20] to achieve a complete proof of Proposition 3.2.

4.1. **Phase space dynamics.** In this subsection we fix $a \in \Delta_N(\widehat{a})$ and assume that N is large enough so that $\Delta_N(\widehat{a}) \subset \Delta_0$ where $\Delta_0 = [a^* - \epsilon, a^* + \epsilon]$. Let α, δ, and ϵ be the same as before. See the first paragraph of Section 3 for a discussion about these constants.

A. Outside of C_δ We start with exponential growth of derivatives for orbit segments staying out of C_δ. Let $f = f_a$ where $a \in \Delta_N(\widehat{a}) \subset \Delta_0$.

Lemma 4.1 (Outside of C_δ). *Let ϵ be sufficiently small depending on δ. We have*

(a) for any $n \geqslant 1$, if $f^k(x) \notin C_\delta$ for $0 \leqslant k \leqslant n-1$, then $|(f^n)'(x)| \geqslant b_0 \delta e^{\frac{1}{2}\lambda_0 n}$;

(b) if, in addition, $f^n(x) \in C_{\delta_0}$, then $|(f^n)'(x)| \geqslant b_0 e^{\frac{1}{2}\lambda_0 n}$.

Proof: By Definition 1.1(B1), there exists $M = M(\delta)$ such that for all $y \in I$, if $f_{a^*}^k(y) \notin C_{\delta/2}(f_{a^*})$ for $0 \leqslant k < M$, then

$$|(f_{a^*}^M)'(y)| > e^{\frac{3}{4}\lambda_0 M}.$$

We choose ϵ small enough so that f is sufficiently close to f_{a^*} for M iterates in the following sense. (1) If x and n are as in (a) and $n \leqslant M$, then $|(f^n)'(x) - (f_{a^*}^n)'(x)|$ is small enough that the conclusions of (a) follow from Definition 1.1(B). (2) If $f^i(y) \notin C_\delta$ for $0 \leqslant i < M$, then $|(f^M)'(y)| > e^{\frac{1}{2}\lambda_0 M}$.

For $n > M$, we let k be such that $kM \leqslant n < (k+1)M$. We estimate $|(f^n)'(x)|$ using the chain rule, comparing $(f^M)'(f^{iM}(x))$ with $(f_{a^*}^M)'(f^{iM}(x))$ for $i \leqslant k$ using (2) above, and comparing $(f^{n-kM})'(f^{kM}(x))$ with $(f_{a^*}^{n-kM})'(f^{kM}(x))$ using (1). ■

B. Bound periods and recovery Let N be fixed and let $f = f_a$, where $a \in \Delta_N(\widehat{a}) \subset \Delta_0$. Let $C(f) = \{c^{(1)}, \cdots, c^{(q)}\}$ denote the set of critical points of f. By Lemma 3.1 we have, for all $c \in C(f)$ and $n \leq N$,

(G1)' $d(f^n(c), C) \geqslant \min\{\delta_0/2, e^{-\alpha n}\}$;

(G2)' $|(f^n)'(f(c))| \geqslant b_1 e^{\lambda n}$.

Let $c = c^{(i)} \in C(f)$ be a critical point and let $x \in C_\delta(c^{(i)})$. Intuitively, the derivative growth of the critical orbit given by (G2)' is copied to a certain extent by the orbit of x. We make this intuition precise.

Definition 4.2. Let $c \in C(f)$ and suppose $x \in C_\delta(c)$. We define $p(x)$, the ***bound period*** of x, to be the largest positive integer j such that $|f^i(x) - f^i(c)| \leqslant e^{-2\alpha i}$ for all $i < j$.

In what follows, let $x_k = f^k(x)$. We have

Lemma 4.3 (Local Distortion)**.** *Let $c \in C(f)$ and $x \in C_\delta(c)$. Then for all $y \in [c, x]$ and $k \leqslant \min\{p(x), N\}$, we have*

$$\frac{1}{2} \leqslant \left| \frac{(f^k)'(f(y))}{(f^k)'(f(c))} \right| \leqslant 2.$$

Proof: First, we have

$$\log \left| \frac{(f^k)'(y_1)}{(f^k)'(c_1)} \right| \leqslant \sum_{j=1}^{k} \frac{|f'(y_j) - f'(c_j)|}{|f'(c_j)|} \leqslant K \sum_{j=1}^{k} \frac{|y_j - c_j|}{d(c_j, C)}.$$

We choose h_0 large enough that $e^{-\alpha h_0} < \frac{\delta_0}{2}$ and

$$\sum_{j=h_0+1}^{\infty} e^{-\alpha j} \ll 1.$$

Next, we choose δ small enough that

$$\delta \sum_{j=1}^{h_0} \frac{2}{\delta_0} \left(\max_{z \in I} |f'(z)| \right)^j \ll 1.$$

Finally, we let ϵ be small enough that $d(c_j, C) > \frac{\delta_0}{2}$ for all $j \leqslant h_0$. Then

$$\sum_{j=1} k \frac{|y_j - c_j|}{d(c_j, C)} \leqslant \delta \sum_{j=1}^{h_0} \frac{2}{\delta_0} \left(\max_{z \in I} |f'(z)| \right)^j + \sum_{j=h_0+1}^{k} \frac{e^{-2\alpha j}}{e^{-\alpha j}} \ll 1.$$

■

Our next lemma is a version of (D2) in phase space.

Lemma 4.4. *Let $c \in C(f)$ and let $x \in I_\mu \subset C_\delta(c)$ with $|\mu| < \frac{\alpha N}{\psi}$. Let $p(x)$ be the bound period. Then*

$$(3 \log \|f\|_{C^1})^{-1} |\mu| \leqslant p(x) \leqslant \frac{3}{\lambda} |\mu|. \tag{4.1}$$

We also have the following.

(a) $|(f^{p(x)})'(x)| > e^{\frac{\lambda}{3} p(x)}$.

(b) Let $\omega \approx I_\mu \subset C_\delta(c)$ be an interval such that $|\mu| < \frac{\alpha N}{\psi}$. Let

$$p(\omega) = \min_{x \in I_\mu^+} p(x).$$

Then we have

$$|f^{p(\omega)}(\omega)| \geqslant e^{-\frac{7\alpha}{\lambda}|\mu|}.$$

Proof: First we prove (4.1). Suppose $|x-c| = e^{-h}$. We first establish the upper bound on $p(x)$. For $n < \min\{p(x), N\}$, Lemma 4.3 and (G2)' imply

$$|x_n - c_n| \geqslant \frac{1}{2}|(f^{n-1})'(c_1)| \cdot |x_1 - c_1| \geqslant \frac{1}{2} b_1 e^{\lambda(n-1)} |x_1 - c_1| \geqslant K^{-1} e^{\lambda(n-1)} (x-c)^2.$$

This inequality implies that $p(x) \leqslant \frac{3h}{\lambda} \leqslant N$ provided h is sufficiently large (or δ is sufficiently small). For the lower bound, we observe that for $n \in \mathbb{N}$,

$$K \|f\|_{C^1}^{n-1} e^{-2h} \geqslant |x_n - c_n|.$$

This inequality gives

$$p(x) \geqslant \frac{h}{3 \log(\|f\|_{C^1})}$$

provided h is sufficiently large (or δ is sufficiently small).

We now prove (a). The inequality

$$K|(f^{p-1})'(c_1)|(x-c)^2 \geqslant |x_p - c_p| \geqslant e^{-2\alpha p}$$

yields

$$K|(f^{p-1})'(c_1)|^{\frac{1}{2}} |x-c| \geqslant e^{-\alpha p}. \tag{4.2}$$

Bounding $|(f^p)'(x)|$ using (4.2), we have

$$\begin{aligned} |(f^p)'(x)| &= |(f^{p-1})'(x_1)| \cdot |f'(x)| \\ &\geqslant K^{-1} |(f^{p-1})'(c_1)|^{\frac{1}{2}} |x-c| \cdot |(f^{p-1})'(c_1)|^{\frac{1}{2}} \\ &\geqslant K^{-1} e^{-\alpha p} e^{\frac{1}{2}\lambda(p-1)}. \end{aligned}$$

The final quantity is greater than $e^{\lambda p/3}$ provided that p is sufficiently large, or, equivalently, that δ is sufficiently small.

We finish with the proof of (b). Applying Lemma 4.3, we have

$$\begin{aligned} |f^{p(\omega)}(\omega)| &= \frac{|f^{p(\omega)}(\omega)|}{\left|f^{p(\omega)}\left([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]\right)\right|} \left|f^{p(\omega)}\left([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]\right)\right| \\ &\geqslant \frac{|f(\omega)|}{2\left|f\left([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]\right)\right|} \left|f^{p(\omega)}\left([c - e^{-|\mu|+1}, c + e^{-|\mu|+1}]\right)\right| \\ &\geqslant K^{-1} e^{-2\alpha p(\omega)}. \end{aligned}$$

By (4.1), we have $p(\omega) \leqslant \frac{3}{\lambda}|\mu|$. Therefore,

$$|f^{p(\omega)}(\omega)| \geqslant K^{-1} e^{-\frac{6\alpha}{\lambda}|\mu|} \geqslant e^{-\frac{7\alpha}{\lambda}|\mu|}.$$

provided $|\mu|$ is sufficiently large (or δ is sufficiently small). ■

As a direct corollary we have

Corollary 4.5. *Let $x \in I$ be such that*

$$d(f^i(x), C) > \max\{\min\{\frac{\delta_0}{2},\ e^{-\alpha i}\},\ e^{-\alpha \frac{N}{\psi}}\}$$

for all $i \leq n$. Then

$$|(f^n)'(x)| > K^{-1} \delta e^{\frac{1}{4}\lambda n}.$$

Proof: We define finite sequences $\{t_k\}$ and $\{p_k\}$ satisfying

$$t_1 < t_1 + p_1 \leqslant t_2 < t_2 + p_2 \leqslant \cdots < n$$

as follows. Let t_1 be the smallest value of $j \geqslant 0$ such that $f^j(x) \in C_\delta$. For $k \geqslant 1$, let p_k be the bound period of $f^{t_k}(x)$ and let t_{k+1} be the smallest $j \geqslant t_k + p_k$ such that $f^j(x) \in C_\delta$. This decomposition partitions the orbit of x into segments corresponding to time intervals $(t_k, t_k + p_k)$ and $[t_k + p_k, t_{k+1}]$, during which we think of the orbit as ***bound*** and ***free***, respectively.

Let us first observe that we have $p_k \leq N$ for all k. This follows from the estimate of Lemma 4.4 on p and the restriction on $d(f^i(x), C)$ assumed by this lemma. The derivatives on time intervals $[t_k, t_k+p_k)$ and $[t_k+p_k, t_{k+1})$ are now estimated using Lemma 4.1(b) and Lemma 4.4(a) respectively, provided these orbit segments are completed before time n. We assume that p_k is sufficiently large (or δ is sufficiently small) so that

$$|(f^{t_{k+1}-t_k})'(f^{t_k}(x))| \geqslant e^{\frac{2\lambda}{7}(t_{k+1}-t_k)}.$$

If $f^n(x)$ is in a bound period initiated at time j, then Lemma 4.3 and (G2') imply

$$\begin{aligned}|(f^{n-j})'(f^j(x))| &= |(f^{n-j-1})'(f^{j+1}(x))| \cdot |f'(f^j(x))| \\ &\geqslant \frac{1}{2} b_1 e^{\lambda(n-j-1)} |f'(f^j(x))| \\ &\geqslant K^{-1} d(f^j(x), C(f)) e^{\lambda(n-j-1)} \\ &\geqslant K^{-1} e^{-\alpha n} e^{\lambda(n-j-1)}.\end{aligned}$$

If $t_k + p_k < n < t_{k+1}$ for some $k \geqslant 1$ or $n < t_1$, then we have

$$|(f^{n-(t_k+p_k)})'(f^{t_k+p_k}(x))| \geqslant K^{-1}\delta e^{\frac{1}{2}\lambda_0(n-(t_k+p_k))}$$

by Lemma 4.1(a). The chain rule therefore yields

$$|(f^n)'(x)| \geqslant K^{-1}\delta e^{\frac{1}{4}\lambda n}.$$

■

4.2. **Duality between phase space and parameter space dynamics.** Let $C = \{c^{(1)}, \cdots, c^{(q)}\}$ be the set of critical points for $\{f_a\}$ and let $c = c^{(i_0)} \in C$. We now study the evolution of critical curves $\gamma_n : \Delta_N(\widehat{a}) \to I$ where $\gamma_n(a) = f_a^n(c)$. Remember that $F(x, a) = f_a(x)$ is C^2 and $\widehat{a} \in \Delta_0 = [a^* - \epsilon, a^* + \epsilon]$ is such that $\widehat{a} \in \Delta(\lambda, \alpha)$. For $n > 0$ and $a \in \Delta_N$, we say that $a \in \Pi(n)$ if (3.1) and (3.2) hold for $\gamma_i(a)$, $i \leq n$. That is, we have

$$d(\gamma_i(a), C) > \max\{\min\{\frac{1}{2}\delta_0,\ e^{-\alpha i}\},\ e^{-\alpha\frac{N}{\psi}}\}$$

for $i \leq n$, where $\psi = \frac{3\alpha}{\lambda}$.

Recall that there are two conditions for $\{f_a\}$ to be admissible. The first is the existence of a Misiurewicz map f_{a^*}, and the second is a parameter transversality condition $\xi(c) \neq 0$. For $c \in C$, let $\beta(a)$ and $\xi(c)$ be as in Section 1.2. We have

Lemma 4.6 (Corollary 4.2, [20])**.** *Let ϵ be sufficiently small and let $c \in C$. Then $\beta(a)$ is well-defined on Δ_0 and $a \mapsto \beta(a)$ is differentiable. In addition, we have*

$$\xi(c) = \frac{d}{da}(f_a(c) - \beta(a))\Big|_{a=a^*} = \frac{dc_1}{da}(a^*) + \sum_{i=1}^{\infty} \frac{\partial_a F(c_i(a^*), a^*)}{(f_{a^*}^i)'(c_1(a^*))}.$$

For a proof of this lemma we refer the reader to Section 4.2 of [20]. This is a proof the reader can pick up directly from [20] without interference from other parts of that paper. We skip it here because the arguments used in proving this lemma are not related to the techniques developed so far and they are not used anywhere else in this paper.

The assumption $\xi(c) \neq 0$ implies the equivalence of spatial and parametric derivatives.

Proposition 4.7 (Derivative Equivalence)**.** *Let ϵ be sufficiently small. Then there exists $m_0 \in \mathbb{Z}^+$ such that the following holds. For all $n > m_0$, $a \in \Delta_N$, and under the assumption that $a \in \Pi(n)$, we have*

$$\frac{1}{2}|\xi(c)| \leqslant \frac{\left|\frac{d}{da}\gamma_i(a)\right|}{|(f_a^{i-1})'(c_1(a))|} \leqslant 2|\xi(c)| \tag{4.3}$$

for $m_0 < i \leqslant n$, where $c_1(a) = f_a(c)$.

Proof: Computing the parametric derivative of $\gamma_i(a)$, we have

$$\frac{d}{da}\gamma_i(a) = (f_a)'(\gamma_{i-1}(a))\frac{d}{da}\gamma_{i-1}(a) + \partial_a F(\gamma_{i-1}(a), a).$$

Inductively, we obtain

$$\frac{\frac{d}{da}\gamma_i(a)}{(f_a^{i-1})'(c_1(a))} = \frac{d}{da}c_1(a) + \sum_{j=1}^{i-1} \frac{\partial_a F(\gamma_j(a), a)}{(f_a^j)'(c_1(a))}. \tag{4.4}$$

Let $W(a, i)$ denote the expression on the right side of (4.4). We choose m_0 large enough that the following hold.

(1) $e^{-\alpha m_0/\psi} < \delta$;
(2) $|W(a^*, m_0) - \xi(c)| \ll |\xi(c)|$;
(3) for $m_0 < i \leqslant n$, we have

$$\left| \sum_{j=m_0}^{i-1} \frac{\partial_a F(\gamma_j(a), a)}{(f_a^j)'(c_1(a))} \right| \ll |\xi(c)|.$$

Condition (2) follows from Lemma 4.6 and $\xi(c) \neq 0$. Condition (3) is achievable because for $m_0 \leqslant j \leqslant i-1$, we have $|\partial_a F(\gamma_j(a), a)| \leqslant K$ and $|(f_a^j)'(c_1(a))| \geqslant K^{-1}\delta e^{\frac{\lambda}{4}j}$ by Corollary 4.5. Finally, let $\epsilon > 0$ be sufficiently small so that $|W(a, m_0) - W(a^*, m_0)| \ll |\xi(c)|$ for all $a \in \Delta_0$. We conclude that for $m_0 < i \leqslant n$,

$$\begin{aligned} |W(a,i) - \xi(c)| &\leqslant |W(a,i) - W(a,m_0)| + |W(a,m_0) - W(a^*,m_0)| \\ &\quad + |W(a^*,m_0) - \xi(c)| \ll |\xi(c)|. \end{aligned}$$

■

We now prove Proposition 3.2.

Proof of (D0). Observe that the condition $a \in \Delta_N(\widehat{a})$ can be replaced by $a \in \Delta_0$ if n in Proposition 4.7 is $m_0 + 1$. (D0) follows from Proposition 4.7 and Lemma 4.1(a) because by making ϵ sufficiently small, we keep $\gamma_k(\Delta_0)$ out of C_δ for $1 \leq k \leq m_0$. ■

Proof of (D1). Property (D1) follows from Proposition 4.7 and Lemma 4.1. ■

Proof of (D2). Property (D2)(1) follows from Proposition 4.7 and Lemma 4.4(a). For (D2)(2) we assume α satisfies

$$(\hat{K})^{\frac{3\alpha}{\lambda}} < e^{\frac{1}{8}\lambda} \tag{4.5}$$

where $\hat{K} = \|F\|_{C^1}$. For $a \in \omega$, let p_a denote the bound period of I_μ with respect to the fixed map f_a. Define

$$p = \min_{a \in \omega} p_a.$$

For all $a, \overline{a} \in \omega$ and $j \leqslant p$, we have

$$\begin{aligned} |\gamma_{n+j}(\overline{a}) - f_a^j(\gamma_n(\overline{a}))| &= |F^j(\gamma_n(\overline{a}), \overline{a}) - F^j(\gamma_n(\overline{a}), a)| \\ &\leqslant \hat{K}^j |\overline{a} - a| \leqslant \hat{K}^j |\omega|. \end{aligned}$$

Observe that $|\omega| \leqslant K e^{-\lambda n/4}$ by Corollary 4.5 and $j \leqslant p \leqslant \frac{3}{\lambda}|\mu|$ by (4.1). Since α satisfies (4.5),

$$\hat{K}^j |\omega| \leqslant K e^{-\frac{1}{4}\lambda n} \hat{K}^{\frac{3}{\lambda}|\mu|} \leqslant K e^{-\frac{1}{4}\lambda n} \hat{K}^{\frac{3}{\lambda}\alpha n} < K e^{-\frac{1}{4}\lambda n} e^{\frac{1}{8}\lambda n} < e^{-\frac{1}{9}\lambda n}.$$

Therefore, for $j \leqslant p$ we have

$$|\gamma_{n+j}(\overline{a}) - f_a^j(\gamma_n(\overline{a}))| < e^{-\frac{1}{9}\lambda n}. \tag{4.6}$$

The estimate $|\gamma_{n+p}(\omega)| \geqslant e^{-\frac{8\alpha}{\lambda}|\mu|}$ now follows from Lemma 4.4(b). ■

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