The Hill's Region of The Four-Body Problem

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ABSTRACT. In this article we generalize and apply the geometric analysis made in [**9**] to study the integral manifold of the spatial four-body problem. A preliminary description on the geometry of the Hill's regions is obtained at the end.

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1. Introduction

The n-body problem admits ten first integrals of linear momentum, angular momentum and energy. These integrals define a $6n - 10$ dimensional algebraic variety in the $6n$ dimensional phase space $(\vec{q}, \vec{p})$, where $\vec{q}$ is the position vector and $\vec{p}$ is the linear momentum. This algebraic variety is the **integral manifold** of the n-body problem. By projecting the integral manifold into the $3n$-dimensional position space $\vec{q}$, we get the so called **Hill's region** of the n-body problem. It has been a long standing objective to understand the geometry of the integral manifold and the Hill's region in the history of the n-body problem.

According to the differences in the method used, all the existing literatures on the study of the integral manifold can be roughly put into two categories. Those in the first category follow Birkhoff's example. They try to locate possible bifurcations of the integral manifold in the spirit of Morse theory. Here the attention has been on the singularities of the gradient vector field generated by the angular momentum and the energy functions. Among the most important studies in this category are Birkhoff's discussion in his 1927 book [**2**], Smale's sequence of papers in early 1970's [**14, 15**] and Albouy's recent work [**1**]. Studies in this category usually avoid detailed analysis on individual integral manifold therefore do not provide concrete description on the geometric characteristics of the integral manifold and that of the Hill's regions.

The studies in the second category trace back to Hill's work on the geometry of the integral manifold of the restricted three-body problem [**8**]. His strategy was to first project the integral manifold from the phase space to the position space, then to draw pictures of the projections. The restricted three-body problem is a much simpler model, so Hill's analysis was not hard at all. Nevertheless, Hill's region has played an important role in the study of the restricted three-body problem, and his analysis has become a prototype for the studies of the integral manifold in this category.

There exist serious obstacles in applying Hill's strategy to the study of the integral manifold of the n-body problem. Even in the case of $n = 3$, we need to work on ten equations in 18 variables. This explains why similar analysis had not occurred until early 1970's when Easton studied the geometry of the integral manifold and the Hill's region of the planar three-body problem [**5**]. His study provided a complete topological classification of the integral manifold for the planar 3-body problem. Later, Easton tried to further use a projection scheme to confirm Birkhoff's conjecture on the bifurcation of the integral manifold of the spatial three-body problem [**6**]. He also tried to analyze the corresponding Hill's region. However, these are much harder goals and his progress was limited.

There are also many other studies on the Hill's region of the spatial three-body problem. The most important ones are due to Saari [**10, 11**] and Simo [**13**]. The study of the integral manifold of the spatial three-body problem is recently completed in [**9**], to which we refer for more details, including a full review of history. We remark that almost all the studies in this category have been only on the case of $n = 3$. As n increases the dimension of the manifold and the Hill's region, therefore the complexity of the problem, increases dramatically.

In this article we generalize and apply the analysis that was presented in [**9**] to study the integral manifold of the spatial four-body problem. It turns out that every single step of the analysis we made in the case of $n = 3$ can be generalized to the case of $n = 4$. Some generalizations are straightforward, others highly non-trivial. We obtain at the end, as the first step towards a complete resolution of the problem at hand, a preliminary description on the geometry of the Hill's region for the spatial four-body problem. Many things are left to be done.

We start section 2 with some basic concepts and formulations. Then we give a list of propositions on the geometry of the integral manifold. These propositions are straight forward generalizations of their correspondences in [**9**]. Section 3 contains a representation of configuration space and derivations of basic relationships. A preliminary description on the geometry of the integral manifold of the spatial four-body problem is obtained in section 4.

2. On the case of n bodies

2.1. Notations.

2.1.1. *Position vectors.* Consider the system of n gravitational particles $(m_1, \cdots, m_n)$ moving in the Euclidean space R^3. We use Jacobi's coordinates $(\vec{R}_1, \cdots, \vec{R}_{n-1})$ to write the equations of motion. Recall that $\vec{R}_i$ is a vector in R^3 pointing from the center of masses of $(m_1, \cdots, m_i)$ to m_{i+1}.

The equations of motion are

$$\mu_i \frac{d^2 \vec{R}_i}{dt^2} = \frac{\partial U}{\partial \vec{R}_i} \tag{1}$$

for $i = 1, \cdots, n-1$. Where

$$\mu_i = \frac{m_{i+1}(m_1 + \cdots + m_i)}{m_1 + \cdots + m_{i+1}},$$

$$U = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{r_{ij}},$$

and r_{ij} is the distance from m_i to m_j.

Equation (1) has first integrals

$$\sum_{i=1}^{n-1} \mu_i \vec{R}_i \times \vec{V}_i = c_0 \vec{k} \tag{2}$$

$$\frac{1}{2} \sum_{i=1}^{n-1} \mu_i \mid \vec{V}_i \mid^2 - U = h \tag{3}$$

where $\vec{V}_i = \frac{d\vec{R}_i}{dt}$, $c_0, \vec{k}$ are the magnitude and the direction of the angular momentum, and h is the constant of energy. Without loss of generality we set $\vec{k} = (0, 0, 1)$ and will only consider the non-trivial case of $h < 0$ in this article.

To remove the complexity caused by μ_i, we further introduce

$$\vec{r}_i = \sqrt{\mu_i} \vec{R}_i \qquad \vec{v}_i = \sqrt{\mu_i} \vec{V}_i$$

to write equations (2) and (3) as

$$\sum_{i=1}^{n-1} \vec{r}_i \times \vec{v}_i = c_0 \vec{k}, \tag{4}$$

$$\frac{1}{2}\sum_{i=1}^{n-1} \mid \vec{v}_i \mid^2 - U = h. \tag{5}$$

Throughout this article we let $\vec{r}_i = (x_i, y_i, z_i)$, $\vec{v}_i = (v_{ix}, v_{iy}, v_{iz})$, $\vec{r} = (\vec{r}_1, \cdots, \vec{r}_{n-1})$, $\vec{v} = (\vec{v}_1, \cdots, \vec{v}_{n-1})$. Further, for a 3×3 rotation matrix O, we denote

$$O\vec{r} = (O\vec{r}_1, \cdots, O\vec{r}_{n-1}), \quad O\vec{v} = (O\vec{v}_1, \cdots, O\vec{v}_{n-1}).$$

$\vec{r} \in R^{3(n-1)}$ is the **position vector** of the n-body system.

2.1.2. *Standard position vectors.* Let $\vec{r}, \vec{r}' \in S$ where

$$S = \{\vec{r} \in R^{3(n-1)}; \ \mid \vec{r} \mid = 1\}.$$

We say that $\vec{r} \sim \vec{r}'$ if there exists a 3×3 rotation matrix O, such that $O\vec{r} = \vec{r}'$. With this equivalence relation we defined the **space of configurations** of the n-body problem as

$$D = S/\sim .$$

The map of projection is denoted as $\pi : S \mapsto D$.

DEFINITION 2.1. *$\vec{r} \in S$ is a* **standard position vector** *if*

$$\sum_{i=1}^{n-1} x_i y_i = 0, \quad \sum_{i=1}^{n-1} x_i z_i = 0, \quad \sum_{i=1}^{n-1} y_i z_i = 0,$$

and $\rho_x \geq \rho_y \geq \rho_z$ where $\rho_x = \sum x_i^2$, $\rho_y = \sum y_i^2$ and $\rho_z = \sum z_i^2$.

We have

(i) For any given $\vec{r} \in S$, there is a rotation matrix O, such that $O\vec{r}$ is a standard position vector.

(ii) For any given $p \in D$ and $\vec{r}, \vec{r}' \in \pi^{-1}(p)$, if both $\vec{r}, \vec{r}'$ are standard position vectors, then

$$\rho_x(\vec{r}) = \rho_x(\vec{r}') \quad \rho_y(\vec{r}) = \rho_y(\vec{r}') \quad \rho_z(\vec{r}) = \rho_z(\vec{r}').$$

2.1.3. *Basic geometric objects.* We study the integral manifold by gradually projecting it into lower dimensional spaces. Here we list all the intermediate objects created by projections.

(i) **The integral manifold**

$$M(c_0, h) = \{(\vec{r}, \vec{v}) : \ \textit{such that } (4)\ (5)\ \textit{hold}\}.$$

(ii) **The Hill's region**

$$H(c_0, h) = \{\vec{r} : \ \textit{there exist } \vec{v}, \ \textit{such that } (\vec{r}, \vec{v}) \in M(c_0, h)\}.$$

(iii) **The un-scaled Hill's region**

$$K(c_0, h) = \{\vec{r}/\mid \vec{r} \mid : \ \vec{r} \in H(c_0, h)\}.$$

(iv) **The set of admissible configuration**

$$AC(c_0, h) = \{p \in D : \ \pi^{-1}(p) \cap K(c_0, h) \neq \emptyset\}.$$

2.2. Necessary and sufficient conditions. In this subsection we give necessary and sufficient conditions for a point being in one of the geometric objects defined in section 2.1.3. The proofs for all the propositions presented are identical to their corrspondences in the case of $n = 3$. We refer the reader to [**9**] for more details.

2.2.1. *Conditions for $AC(c_0, h)$.* For $p \in D$, let $\vec{r} \in \pi^{-1}(p)$ be a standard position vector representing p. We have

PROPOSITION 2.1. *$p \in AC(c_0, h)$ if and only if*

$$\frac{1}{\rho_x(\vec{r}) + \rho_y(\vec{r})} \leq \frac{U^2(\vec{r})}{2 \mid h \mid c_0^2}. \tag{6}$$

2.2.2. *Conditions for $K(c_0, h)$.* Again let $\vec{r}$ be a standard position vector in $\pi^{-1}(p)$ representing p. We have

PROPOSITION 2.2. *$K(c_0, h)$ is a singular fiber bundle over $AC(c_0, h)$. The fiber of $K(c_0, h)$ over $p \in AC(c_0, h)$ is determined as follows:*

(i) *Let c_1, c_2 be such that*

$$\begin{aligned} & c_1^2\left(\frac{1}{\rho_y + \rho_z} - \frac{1}{\rho_x + \rho_y}\right) \quad + \quad c_2^2\left(\frac{1}{\rho_x + \rho_z} - \frac{1}{\rho_x + \rho_y}\right) \\ \leq\ & \frac{U^2}{2 \mid h \mid c_0^2} - \frac{1}{\rho_x + \rho_y} \end{aligned} \tag{7}$$

and $c_1^2 + c_2^2 \leq 1$. Define V_p as the set of unit vectors $\vec{c} = (c_1, c_2, c_3) \in R^3$ with the indicated c_1, c_2 and $c_3 = \pm\sqrt{1 - c_1^2 - c_2^2}$.

(ii) *For any $\vec{c} \in V_p$, let*

$$O(\vec{c}) = \{O \in SO(3) : \text{ such that } O\vec{c} = \vec{k}\}$$

where $SO(3)$ is the set rotation matrices in R^3.

(iii) *The fiber of $K(c_0, h)$ over $p \in AC(c_0, h)$ is*

$$F_p = \{O\vec{r} : \ O \in O(\vec{c}) \text{ and } \vec{c} \in V_p\}.$$

2.2.3. *Conditions for $H(c_0, h)$.*

PROPOSITION 2.3. *$H(c_0, h)$ is a singular bundle over $K(c_0, h)$. For any $\vec{r} \in K(c_0, h)$, the fiber over $\vec{r}$ in $H(c_0, h)$ is determined as follows:*

(i) *According to the description on $K(c_0, h)$, there exist O, a rotation matrix; $\vec{r}'$, a standard position vector; and $\vec{c}$, a unit vector in R^3; such that $\vec{r} = O\vec{r}'$ and $O\vec{c} = \vec{k}$.*

(ii) *Let $I_{\vec{r}}$ be the set of all positive real ρ satisfying*

$$\begin{aligned} & 2 \mid h \mid \rho^2 - 2\rho U(\vec{r}) \leq \\ & c_0^2\left(\frac{c_1^2}{\rho_y + \rho_z} + \frac{c_2^2}{\rho_x + \rho_z} + \frac{c_3^2}{\rho_x + \rho_y}\right), \end{aligned} \tag{8}$$

then the fiber in $H(c_0, h)$ over $\vec{r} \in K(c_0, h)$ is

$$H_{\vec{r}} = \{\rho\vec{r}' : \quad \rho \in I_{\vec{r}}\}.$$

Note that ρ_x, ρ_y and ρ_z in (8) is computed using the standard position vector $\vec{r}'$.

(iii) *It is easy to check that $I_{\vec{r}}$ is not empty for $\vec{r} \in K(c_0, h)$. Furthermore, $I_{\vec{r}}$ is a closed interval in general. This interval shrinks to one point over the boundaries of $K(c_0, h)$.*

2.2.4. *Conditions for $M(c_0, h)$.*

PROPOSITION 2.4. *$M(c_0, h)$ is given by equations (4) and (5). It is a singular fiber bundle over $H(c_0, h)$. For each $\vec{r} \in H(c_0, h)$, the fiber over $\vec{r}$ is a $3(n-1)-4$ dimensional sphere if $\vec{r}$ is not collinear. It is a $3(n-1)-3$ dimensional sphere if $\vec{r}$ is collinear. Each fiber shrinks to one point over the boundaries of $H(c_0, h)$.*

2.2.5. *Some remarks.*

(i) Propositions listed above and their proofs are almost identical to their correspondences in the case of $n = 3$. We refer the reader to [**9**] for details.
(ii) All the objects of projections, according to the statements above, are functions of one parameter $\mu = 2 \mid h \mid c_0^2$. Therefore we will denote these objects as M_μ, H_μ, K_μ and AC_μ respectively.
(iii) To further investigate the structure of the integral manifold, we now need to study
    (1) the geometry of AC_μ,
    (2) the fabrication of K_μ over AC_μ,
    (3) the fabrication of H_μ over K_μ and that of M_μ over H_μ.

The third item is rather simple according to the descriptions above. The hard part is the study of the first two items.

2.2.6. *Fabrication of K_μ over AC_μ.* The boundary of AC_μ is defined by

$$U^2(\vec{r})(\rho_x + \rho_y) = \mu.$$

This is the first key algebraic surface we need to study.

The fabrication of K_μ over AC_μ is a little more complicated. The geometry of the fibers depend on the intersection of the ellipse (7) and the unit circle $c_1^2 + c_2^2 = 1$. It changes when the long semi-axis or the short semi-axis of the ellipse (7) changes through one. So, to globally understand the fabrication, we need to study the algebraic surfaces in the configuration space D on which the values of these semi-axis equal to one. Set the short semi-axis to one we get

$$U^2(\vec{r})(\rho_y + \rho_z) = \mu,$$

and similarly set the long semi-axis to one we get

$$U^2(\vec{r})(\rho_x + \rho_z) = \mu.$$

Therefore to study of the geometry of the integral manifold and that of the Hill's region it suffices to study the three algebraic surfaces defined by

$$f_1 = (\rho_y + \rho_z)U^2(\vec{r}) = \mu \tag{9}$$

$$f_2 = (\rho_x + \rho_z)U^2(\vec{r}) = \mu \tag{10}$$

$$f_3 = (\rho_x + \rho_y)U^2(\vec{r}) = \mu. \tag{11}$$

These surfaces are the focus of our study. We will refer (9) as **the surface** Ω_1, (10) **the surface** Ω_2 and (11) **the surface** Ω_3.

3. On the case of four bodies

We now study the surfaces Ω_1, Ω_2 and Ω_3 assuming $n = 4$. We further assume $m_1 = m_2 = m_3 = m_4 = 1$. For the analysis that follows, the assumption on the number of particles involved is essential but the assumption on the masses is not.

$m_i = 1$ is introduced only to make the materials in this section and the next one easier to present.

3.1. Representing the configurations. In this subsection we introduce a set of variables $X = (X_1, X_2, X_3, X_4, X_5) \in R^5$ to represent the configurations of the four bodies.

3.1.1. *Position vectors.* Assume now $n = 4$, $m_i = 1$ for $1 \le i \le 4$ and that the physical space is R^3. The Jacobi coordinates for the 4-body system are $\vec{R}_1, \vec{R}_2$ and $\vec{R}_3$. We have by definition

$$\vec{r} = (\vec{r}_1, \vec{r}_2, \vec{r}_3),$$

where

$$\vec{r}_1 = \sqrt{\mu_1}\vec{R}_1, \quad \vec{r}_2 = \sqrt{\mu_2}\vec{R}_2, \quad \vec{r}_3 = \sqrt{\mu_3}\vec{R}_3,$$

and

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{2}{3}, \quad \mu_3 = \frac{3}{4}.$$

We further assume in this section that $\vec{r}$ is a standard position vector.

3.1.2. *Mutual distances.* Let r_{ij} be the distance from m_i to m_j. We have

$$r_{12}^2 = \vec{R}_1 \cdot \vec{R}_1 \tag{12}$$

$$r_{13}^2 = (\vec{R}_2 + \frac{1}{2}\vec{R}_1) \cdot (\vec{R}_2 + \frac{1}{2}\vec{R}_1) \tag{13}$$

$$r_{23}^2 = (\vec{R}_2 - \frac{1}{2}\vec{R}_1) \cdot (\vec{R}_2 - \frac{1}{2}\vec{R}_1) \tag{14}$$

$$r_{14}^2 = (\vec{R}_3 + \frac{1}{3}\vec{R}_2 + \frac{1}{2}\vec{R}_1) \cdot (\vec{R}_3 + \frac{1}{3}\vec{R}_2 + \frac{1}{2}\vec{R}_1) \tag{15}$$

$$r_{24}^2 = (\vec{R}_3 + \frac{1}{3}\vec{R}_2 - \frac{1}{2}\vec{R}_1) \cdot (\vec{R}_3 + \frac{1}{3}\vec{R}_2 - \frac{1}{2}\vec{R}_1) \tag{16}$$

$$r_{34}^2 = (\vec{R}_3 - \frac{2}{3}\vec{R}_2) \cdot (\vec{R}_3 - \frac{2}{3}\vec{R}_2). \tag{17}$$

3.1.3. *Position vectors and mutual distances.* From section 3.1.2 we have

$$\vec{r}_1 \cdot \vec{r}_1 = \frac{1}{2}r_{12}^2 \tag{18}$$

$$\vec{r}_2 \cdot \vec{r}_2 = \frac{1}{3}(r_{13}^2 + r_{23}^2 - \frac{1}{2}r_{12}^2) \tag{19}$$

$$\vec{r}_3 \cdot \vec{r}_3 = \frac{1}{4}(r_{14}^2 + r_{24}^2 + r_{34}^2) - \frac{1}{12}(r_{13}^2 + r_{23}^2 + r_{12}^2) \tag{20}$$

$$\vec{r}_1 \cdot \vec{r}_2 = \frac{1}{2\sqrt{3}}(r_{13}^2 - r_{23}^2) \tag{21}$$

$$\vec{r}_1 \cdot \vec{r}_3 = \frac{\sqrt{3}}{2\sqrt{2}}(\frac{1}{2}r_{14}^2 - \frac{1}{2}r_{24}^2 - \frac{1}{6}r_{13}^2 + \frac{1}{6}r_{23}^2) \tag{22}$$

$$\begin{aligned}\vec{r}_2 \cdot \vec{r}_3 &= \frac{1}{\sqrt{2}}(\frac{1}{4}r_{14}^2 + \frac{1}{4}r_{24}^2 - \frac{1}{2}r_{34}^2 + \frac{1}{12}r_{13}^2 \\ &\quad + \frac{1}{12}r_{23}^2 - \frac{1}{6}r_{12}^2).\end{aligned} \tag{23}$$

3.1.4. *Variables $X \in R^5$.* We set $r_{12} = 1$ to fix the size of a given configuration. Let $X = (X_1, X_2, X_3, X_4, X_5) \in R^5$ be defined as

$$X_1 = \vec{r}_1 \cdot \vec{r}_2 \tag{24}$$

$$X_2 = \vec{r}_1 \cdot \vec{r}_3 \tag{25}$$

$$X_3 = \vec{r}_2 \cdot \vec{r}_3 \tag{26}$$

$$X_4 = \vec{r}_2 \cdot \vec{r}_2 - \frac{1}{2} \tag{27}$$

$$X_5 = \vec{r}_3 \cdot \vec{r}_3 - \frac{1}{2}. \tag{28}$$

Remember that $\vec{r}_1 \cdot \vec{r}_1 = \frac{1}{2}$ since $r_{12} = 1$.

3.1.5. *X in mutual distances.* From sections 3.1.3 and 3.1.4 we have

$$X_1 = \frac{1}{2\sqrt{3}}(r_{13}^2 - r_{23}^2) \tag{29}$$

$$X_2 = \frac{\sqrt{3}}{2\sqrt{2}}(\frac{1}{2}r_{14}^2 - \frac{1}{2}r_{24}^2 - \frac{1}{6}r_{13}^2 + \frac{1}{6}r_{23}^2) \tag{30}$$

$$X_3 = \frac{1}{\sqrt{2}}(\frac{1}{4}r_{14}^2 + \frac{1}{4}r_{24}^2 - \frac{1}{2}r_{34}^2 + \frac{1}{12}r_{13}^2 + \frac{1}{12}r_{23}^2 - \frac{1}{6}) \tag{31}$$

$$X_4 = \frac{1}{3}(r_{13}^2 + r_{23}^2 - 2) \tag{32}$$

$$X_5 = \frac{1}{4}(r_{14}^2 + r_{24}^2 + r_{34}^2) - \frac{1}{12}(r_{13}^2 + r_{23}^2 + 1) - \frac{1}{2}. \tag{33}$$

3.1.6. *Mutual distances in X.* Solve from section 3.1.5 for r_{ij}, we have

$$r_{12}^2 = 1 \tag{34}$$

$$r_{13}^2 = \frac{3}{2}X_4 + \sqrt{3}X_1 + 1 \tag{35}$$

$$r_{23}^2 = \frac{3}{2}X_4 - \sqrt{3}X_1 + 1 \tag{36}$$

$$r_{34}^2 = \frac{4}{3}X_5 - \frac{4\sqrt{2}}{3}X_3 + \frac{2}{3}X_4 + 1 \tag{37}$$

$$r_{14}^2 = \frac{4}{3}X_5 + \frac{1}{6}X_4 + \frac{2\sqrt{2}}{3}X_3 + \frac{2\sqrt{2}}{\sqrt{3}}X_2 + \frac{1}{\sqrt{3}}X_1 + 1 \tag{38}$$

$$r_{24}^2 = \frac{4}{3}X_5 + \frac{1}{6}X_4 + \frac{2\sqrt{2}}{3}X_3 - \frac{2\sqrt{2}}{\sqrt{3}}X_2 - \frac{1}{\sqrt{3}}X_1 + 1. \tag{39}$$

$X = (X_1, X_2, X_3, X_4, X_5)$ will be used to represent the configuration space D of the four bodies.

3.2. ρ_x, ρ_y, ρ_z in X. In this subsection we write ρ_x, ρ_y and ρ_z, therefore equations (9)-(11) in terms of the new variables $X = (X_1, X_2, X_3, X_4, X_5)$.

3.2.1. *A characteristic equation.* Assume that $\vec{r} = (\vec{r}_1, \vec{r}_2, \vec{r}_3)$ is a standard position vector. Let

$$\vec{r}_1 = (x_1, y_1, z_1), \quad \vec{r}_2 = (x_2, y_2, z_2), \quad \vec{r}_3 = (x_3, y_3, z_3)$$

and denote

$$\vec{a}_1 = (x_1, x_2, x_3) \quad \vec{a}_2 = (y_1, y_2, y_3) \quad \vec{a}_3 = (z_1, z_2, z_3).$$

By setting

$$A = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

we have

$$A^T = (\vec{r}_1^T, \vec{r}_2^T, \vec{r}_3^T) = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{pmatrix}$$

and

$$AA^T = \begin{pmatrix} \vec{r}_1\cdot\vec{r}_1 & \vec{r}_1\cdot\vec{r}_2 & \vec{r}_1\cdot\vec{r}_3 \\ \vec{r}_2\cdot\vec{r}_1 & \vec{r}_2\cdot\vec{r}_2 & \vec{r}_2\cdot\vec{r}_3 \\ \vec{r}_3\cdot\vec{r}_1 & \vec{r}_3\cdot\vec{r}_2 & \vec{r}_3\cdot\vec{r}_3 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} \vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 & \vec{a}_1\cdot\vec{a}_3 \\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 \\ \vec{a}_3\cdot\vec{a}_1 & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 \end{pmatrix}.$$

Since $\vec{r}$ is a standard position vector, we have $\vec{a}_i\cdot\vec{a}_j = 0$ when $i \neq j$ and $\rho_x = \vec{a}_1\cdot\vec{a}_1$, $\rho_y = \vec{a}_2\cdot\vec{a}_2$ and $\rho_z = \vec{a}_3\cdot\vec{a}_3$. So

$$A^TA = \begin{pmatrix} \rho_x & 0 & 0 \\ 0 & \rho_y & 0 \\ 0 & 0 & \rho_z \end{pmatrix}.$$

On the other hand

$$A^{-1}BA = A^{-1}(AA^T)A = A^TA.$$

Therefore we have

PROPOSITION 3.1. *ρ_x, ρ_y, ρ_z are the eigenvalues of the matrix*

$$B = \begin{pmatrix} 1/2 & X_1 & X_2 \\ X_1 & 1/2 + X_4 & X_3 \\ X_2 & X_3 & 1/2 + X_5 \end{pmatrix}. \tag{40}$$

Now let

$$\begin{vmatrix} -\lambda & X_1 & X_2 \\ X_1 & -\lambda + X_4 & X_3 \\ X_2 & X_3 & -\lambda + X_5 \end{vmatrix} = 0. \tag{41}$$

The eigenvalues of the matrix B are $\lambda - 1/2$, where λ are the solutions of the equation (41).

3.2.2. *Solution formulas.* Write (41) in λ explicitly we have

$$\lambda^3 - a\lambda^2 + b\lambda - c = 0 \tag{42}$$

where

$$\begin{aligned} a &= X_4 + X_5 \\ b &= X_4X_5 - X_1^2 - X_2^2 - X_3^2 \\ c &= 2X_1X_2X_3 - X_4X_2^2 - X_5X_1^2. \end{aligned}$$

To solve this equation we set $\lambda = y + \frac{a}{3}$ to obtain

$$(y + \frac{a}{3})^3 - a(y + \frac{a}{3})^2 + b(y + \frac{a}{3}) - c = 0.$$

So

$$y^3 + py + q = 0 \tag{43}$$

where

$$\begin{aligned} p &= -\frac{a^2}{3} + b \\ q &= -\frac{2a^3}{27} + \frac{ab}{3} - c \end{aligned}$$

and the solutions are

$$y_1 = (-\frac{q}{2} + \sqrt{D})^{1/3} + (-\frac{q}{2} - \sqrt{D})^{1/3} \tag{44}$$

$$y_2 = \omega(-\frac{q}{2} + \sqrt{D})^{1/3} + \omega^2(-\frac{q}{2} - \sqrt{D})^{1/3} \tag{45}$$

$$y_3 = \omega^2(-\frac{q}{2} + \sqrt{D})^{1/3} + (-\frac{q}{2} - \sqrt{D})^{1/3} \tag{46}$$

where $\omega = e^{2\pi i/3} = (-1 + \sqrt{3}i)/2$ and

$$D = (\frac{q}{2})^2 + (\frac{p}{3})^3.$$

Note that we also have $D \leq 0$ therefore (41) have three real solutions.

Write p, q directly in X, we have

$$p = -\frac{1}{3}(X_4^2 - X_4X_5 + X_5^2) - X_1^2 - X_2^2 - X_3^2; \tag{47}$$

$$\begin{aligned} q = & -\frac{2}{27}(X_4 + X_5)^3 - \frac{1}{3}(X_4 + X_5)(X_1^2 + X_2^2 + X_3^2 - X_4X_5) \\ & -2X_1X_2X_3 + X_4X_2^2 + X_5X_1^2. \end{aligned} \tag{48}$$

3.2.3. *Triangular form of the solutions.* Let

$$A = -\frac{q}{2}, \quad B = \sqrt{-D}.$$

and recall that $-D \geq 0$. We define θ by letting

$$\cos\theta = \frac{A}{r} = \frac{-q/2}{(-p/3)^{3/2}},$$

$$\sin\theta = \frac{B}{r} = \frac{\sqrt{-(q/2)^2 + (-p/3)^3}}{(-p/3)^{3/2}}.$$

where $r = A^2 + B^2$. We can now write the solutions as

$$y_1 = 2(-\frac{p}{3})^{1/2} \cos\frac{\theta}{3} \tag{49}$$

$$y_2 = 2(-\frac{p}{3})^{1/2} \cos(\frac{\theta}{3} + \frac{2\pi}{3}) \tag{50}$$

$$y_3 = 2(-\frac{p}{3})^{1/2} \cos(\frac{\theta}{3} + \frac{4\pi}{3}). \tag{51}$$

Denote the angles inside the cosine functions in (49) - (51) as $\theta_{min}, \theta_{mid}$ and θ_{max} respectively, and let

$$\cos\theta_{min} \leq \cos\theta_{mid} \leq \cos\theta_{max}. \tag{52}$$

We have $cos\theta_{min} < 0,\ \cos\theta_{max} > 0$. We further have

$$\theta_{mid} - \theta_{min} = 2\pi/3, \quad \theta_{max} - \theta_{mid} = 2\pi/3.$$

Writing these solutions in order as

$$y_{min} \leq y_{mid} \leq y_{max},$$

we have

$$\rho_z = \frac{1}{2} + \frac{1}{3}(X_4 + X_5) + y_{min} \tag{53}$$

$$\rho_y = \frac{1}{2} + \frac{1}{3}(X_4 + X_5) + y_{mid} \tag{54}$$

$$\rho_x = \frac{1}{2} + \frac{1}{3}(X_4 + X_4) + y_{max}. \tag{55}$$

Note that it is always true that

$$\cos\theta_{min} + \cos\theta_{mid} + \cos\theta_{max} = 0. \tag{56}$$

We remark that

(i) The same computation applies to the case of $n = 3$, in which we only need to deal with a 2×2 matrix therefore a quadratic equation. This is in fact a more direct way of getting some of the formulas in [**9**].

(ii) For the n-body problem in R^3 with $n > 4$, this computation does not apply. However, it works if we put the n bodies in R^{n-1}. Most of the results obtained in this article also apply to such high dimensional setting of the n-body problem.

3.3. On the configuration space.

3.3.1. *Set of orientations.* Let $X = (X_1, X_2, X_3, X_4, X_5) \in R^5$ and D be the set of X representing a 4-body configuration in R^3. Also let S be the set of unit vectors in R^5 (the **set of orientations** in R^5). $int(D)$ represents all the true 3-dimensional configurations and $bd(D)$ a collection of all the planar, collinear as well as collision configurations of the four bodies. For any fixed unit vector $\vec{u} \in S$, the half ray

$$X_{\vec{u}} = R \cdot \vec{u}, \qquad R \in (0, +\infty)$$

is **the set of configurations of orientation** $\vec{u}$. Since $X = 0$ is in D, $X_{\vec{u}} \cap int(D) \neq \emptyset$ for any given $\vec{u}$ in S.

According to (47) and (48), p, q are both homogeneous functions in X. p is degree two and q is degree three. As a direct consequence, $\theta_{min}, \theta_{mid}$ and θ_{max} are all independent of R. In fact, for a given orientation $\vec{u}$,

$$\rho_z(R\vec{u}) = \frac{1}{2} + k_{min}(\vec{u})R \tag{57}$$

$$\rho_y(R\vec{u}) = \frac{1}{2} + k_{mid}(\vec{u})R \tag{58}$$

$$\rho_x(R\vec{u}) = \frac{1}{2} + k_{max}(\vec{u})R \tag{59}$$

where

$$k_{min} = \frac{1}{3}(u_4 + u_5) + 2(-\frac{p(\vec{u})}{3})^{1/2}\cos\theta_{min}(\vec{u}) \tag{60}$$

$$k_{mid} = \frac{1}{3}(u_4 + u_5) + 2(-\frac{p(\vec{u})}{3})^{1/2}\cos\theta_{mid}(\vec{u}) \tag{61}$$

$$k_{max} = \frac{1}{3}(u_4 + u_5) + 2(-\frac{p(\vec{u})}{3})^{1/2}\cos\theta_{max}(\vec{u}). \tag{62}$$

3.3.2. *ρ_z and $X_{\vec{u}} \cap int(D)$.* For a given orientation $\vec{u} \in S$, ρ_x, ρ_y, ρ_z are linear functions of R. Using R as an independent variable and V as a dependent variable, we now study the graphs of all these functions in the (R, V)-plane. We are only interested in $R \geq 0$. By $\rho_x \geq \rho_y \geq \rho_z$, $R\vec{u} \in D$ if and only if $\rho_z \geq 0$. Equality holds on $bd(D)$.

We have two cases depending on the sign of k_{min} for ρ_z. In the case of $k_{min} \geq 0$, the part of the line $V = \rho_z$ over $R > 0$ does not intersect the R axis therefore $X_{\vec{u}}$ is entirely in D. In the case of $k_{min} < 0$, this line intersects R-axis at $R = R_{min}$ therefore $R\vec{u} \in int(D)$ if and only if $R \in [0, R_{min})$. $X_{\vec{u}}$ hit $bd(D)$ at $R = R_{min}$. Any point $X \in R^5$ with $R > R_{min}$ represents **no real four-body configuration**. The configuration space D is homeomorphic to a 5-dimensional ball, and if we set $R_{min} = +\infty$ for the orientations of $k_{min} \geq 0$, then

$$bd(D) = \{R_{min}\vec{u}; \quad \vec{u} \in S^4\}.$$

3.3.3. *Degenerate orientations.*

Definition 3.1. *An orientation $\vec{u}$ is* **degenerate** *if either $k_{min} \geq 0$ or there exist an index pair (i,j) such that $r_{ij} = 0$ for the configuration $R_{min}\vec{u}$.*

Denote

$$\Delta_{12} = \{\vec{u} \in S^4; \quad k_{min} \geq 0\},$$

$$\Delta_{ij} = \{\vec{u} \in S^4; \quad r_{ij} = 0 \ for \ R_{min}\vec{u}\},$$

and let $\Delta = \cup \Delta_{ij}$ be the set of all degenerate orientations. $\vec{u}$ is in Δ if and only if the half ray $R\vec{u}$ hits the boundary of D at a configuration of collisions.

3.3.4. *The set Δ_{12}.* By definition Δ_{12} is the set of orientations $\vec{u}$ such that $X_{\vec{u}}$ is completely contained in D. Denote $\vec{u} = (u_1, u_2, u_3, u_4, u_5)$. We have

Proposition 3.2. *$\vec{u} \in \Delta_{12}$ if and only if*

$$u_1 = u_2 = 0; \tag{63}$$

$$u_4 \geq 0; \ u_5 \geq 0; \tag{64}$$

$$u_4^2 + u_5^2 + u_4 u_5 \geq 1; \tag{65}$$

$$u_3^2 + u_4^2 + u_5^2 = 1. \tag{66}$$

Proof: First we prove that (63)-(66) are necessary conditions for $\vec{u}$ being in Δ_{12}. Let $\vec{r} = R\vec{u}$ for some $\vec{u} \in \Delta_{12}$. By (27) and (28) we have

$$\begin{aligned} |\vec{r}_2|^2 &= u_4 R + \frac{1}{2}, \\ |\vec{r}_3|^2 &= u_5 R + \frac{1}{2} \end{aligned}$$

for all $R \geq 0$. So (64) is neccesary.

Similarly we have

$$|\vec{r}_1| = \sqrt{\frac{1}{2}}, \quad |\vec{r}_2| = \sqrt{u_4 R + \frac{1}{2}}, \quad |\vec{r}_3| = \sqrt{u_5 R + \frac{1}{2}}$$

therefore

$$(X_1 =)u_1 R = \sqrt{\frac{1}{2}}\sqrt{u_4 R + \frac{1}{2}}\cos\theta_{12} (= \vec{r_1}\cdot\vec{r_2}) \tag{67}$$

$$(X_2 =)u_2 R = \sqrt{\frac{1}{2}}\sqrt{u_5 R + \frac{1}{2}}\cos\theta_{13} (= \vec{r_1}\cdot\vec{r_2}) \tag{68}$$

$$(X_3 =)u_3 R = \sqrt{u_4 R + \frac{1}{2}}\sqrt{u_5 R + \frac{1}{2}}\cos\theta_{23} (= \vec{r_2}\cdot\vec{r_3}) \tag{69}$$

where θ_{ij} are the angles between $\vec{r}_i$ and $\vec{r}_j$.

In order to make the first two equalities hold for all R it is neccesary to have $u_1 = u_2 = 0$. We further have from (69) that

$$\cos\theta_{23} = \frac{u_3 R}{\sqrt{(u_4 R + \frac{1}{2})(u_5 R + \frac{1}{2})}}$$

so

$$\mid u_3 \mid \leq \sqrt{(u_4 + \frac{1}{2R})(u_5 + \frac{1}{2R})}$$

for all R. By setting $R = \infty$ we conclude that $u_3^2 \leq u_4 u_5$, which implies, when combined with (66), that

$$u_4^2 + u_5^2 + u_4 u_5 \geq 1.$$

To proof that (63)-(66) are sufficient for $\vec{u} \in \Delta_{12}$, it suffices to prove that (63)-(66) implies $k_{min} = 0$. Observe that $k_{min} = 0$ if and only if y_{min}, the smallest solution for (43), equals to $-\frac{1}{3}(u_4 + u_5)$. By setting $u_1 = u_2 = 0$ in (43), we see that $y = -\frac{1}{3}(u_4 + u_5)$ is a solution. We also have $-\frac{1}{3}(u_4 + u_5) \leq 0$ by (64). To prove that this is the smallest solution we need to further check

$$\frac{d}{dy}(y^3 + py + q) \mid_{y=-\frac{1}{3}(u_4+u_5)} \geq 0,$$

which is precisely (65). □

See Figure 1(a) for a picture of Δ_{12} and Figure 1(b) for its projection on the (u_4, u_5)-plane.

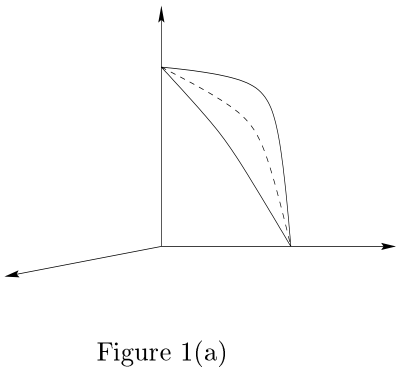

Figure 1(a)

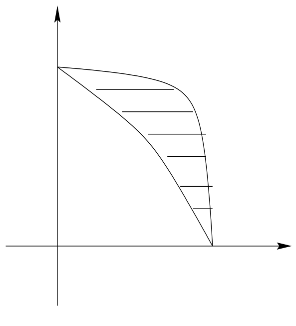

Figure 1(b)

3.3.5. *The set* Δ. Δ_{ij} and Δ_{12} have the same structure by symmetry. To understand the geometry of the set Δ, we need to further compute $\Delta_{12} \cap \Delta_{ij}$.

PROPOSITION 3.3. *We have*

(1) $\Delta_{12} \cap \Delta_{13} = \Delta_{12} \cap \Delta_{23} = (0, 0, 0, 0, 1)$.
(2) $\Delta_{12} \cap \Delta_{24} = \Delta_{12} \cap \Delta_{14} = (0, 0, -\frac{2\sqrt{2}}{\sqrt{73}}, \frac{8}{\sqrt{73}}, \frac{1}{\sqrt{73}})$.
(3) $\Delta_{12} \cap \Delta_{34} = (0, 0, \frac{\sqrt{2}}{\sqrt{7}}, \frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}})$.

Proof: For $\vec{u} \in \Delta_{12} \cap \Delta_{13}$ we have $r_{13} = r_{12} = 1$. We also have $r_{23} = 1$ because $u_1 = u_2 = 0$. So $\Delta_{12} \cap \Delta_{13} = \Delta_{12} \cap \Delta_{23}$. Now we set $r_{13} = 1$ to get a solution $\vec{u} = (0, 0, 0, 0, 1)$. Similarly we have $\Delta_{12} \cap \Delta_{24} = \Delta_{12} \cap \Delta_{14}$ and to find the intersection is to solve the equation

$$\frac{4}{3}u_5 + \frac{1}{6}u_4 + \frac{2\sqrt{2}}{3}u_3 = 0.$$

From the last equation we have

$$u_3 = -\sqrt{2}u_5 - \frac{1}{4\sqrt{2}}u_4$$

so

$$u_4^2 + u_5^2 + (-\sqrt{2}u_5 - \frac{1}{4\sqrt{2}}u_4)^2 = 1.$$

Therefore

$$(\sqrt{2}u_5 - \frac{1}{4\sqrt{2}}u_4)^2 = 1 - u_4^2 - u_5^2 - u_4 u_5.$$

The right hand-side is non-positive if $\vec{u} \in \Delta_{12}$ so the formulas on both sides of the last equation must be zero, producing the second solution. The third item in this proposition is proved along the same line. $\square$

The picture around Δ_{12} is as in Figure 2 according to proposition 3.3. We have similar pictures around every Δ_{ij} by symmetry. The set Δ is obtained by identifying disks of the same label in these pictures. It is embedded in S^4.

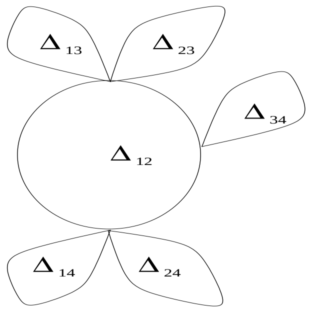

Figure 2

4. On the surfaces $\Omega_1 - \Omega_3$

In this section we study the geometry of the surfaces $\Omega_1 - \Omega_3$. Recall that Ω_3 is the boundary of the set of admissible configurations. Ω_1 and Ω_2 are the set of configurations on which the fabrication of K_μ over AC_μ bifurcates.

4.1. Graph of intersections.

4.1.1. *$f_i(R)$ and $g(R)$.* For a given $\vec{u}$ let ρ_x, ρ_y and ρ_z be defined by $X = R\vec{u}$, and let

$$f_1(R) = \rho_y + \rho_z; \quad f_2(R) = \rho_x + \rho_z; \quad f_3(R) = \rho_x + \rho_y.$$

We have,

$$f_i(R) = 1 + s_i(\vec{u})R$$

by (57)-(59) where

$$s_1 = \frac{2}{3}(u_4 + u_5) - y_{max}(\vec{u}) \tag{70}$$

$$s_2 = \frac{2}{3}(u_4 + u_5) - y_{mid}(\vec{u}) \tag{71}$$

$$s_3 = \frac{2}{3}(u_4 + u_5) - y_{min}(\vec{u}), \tag{72}$$

and $y_{max} \geq y_{mid} \geq y_{min}$ are the solutions for (43) with $p = p(\vec{u}), q = q(\vec{u})$.

We also let

$$g(R) = \mu U^{-2},$$

and

$$U = \sum_{1 \leq i < j \leq 4} \frac{1}{r_{ij}} = \sum_{1 \leq i < j \leq 4} (u_{ij}R + 1)^{-1/2} \tag{73}$$

where

$$u_{12} = 0 \tag{74}$$

$$u_{13} = \frac{3}{2}u_4 + \sqrt{3}u_1 \tag{75}$$

$$u_{23} = \frac{3}{2}u_4 - \sqrt{3}u_1 \tag{76}$$

$$u_{34} = \frac{4}{3}u_5 - \frac{4\sqrt{2}}{3}u_3 + \frac{2}{3}u_4 \tag{77}$$

$$u_{14} = \frac{4}{3}u_5 + \frac{1}{6}u_4 + \frac{2\sqrt{2}}{3}u_3 + \frac{2\sqrt{2}}{\sqrt{3}}u_2 + \frac{1}{\sqrt{3}}u_1 \tag{78}$$

$$u_{24} = \frac{4}{3}u_5 + \frac{1}{6}u_4 + \frac{2\sqrt{2}}{3}u_3 - \frac{2\sqrt{2}}{\sqrt{3}}u_2 - \frac{1}{\sqrt{3}}u_1. \tag{79}$$

Ω_i are defined by

$$f_i(R) = g(R).$$

We will draw the graphs of $V = f_i(R)$ and $V = g(R)$ on the plane (R, V) and $\Omega_i \cap X_{\vec{u}}$ is the set of intersections of these graphs.

4.1.2. *$V = g(R)$ and $V = f_i(R)$.* First we have

Lemma 4.1. *$V = g(R)$ is a concave function in R.*

Proof: We need to show that $g'' \leq 0$, a consequence of the following computation: By definition,

$$U' = -\frac{1}{2}\sum(u_{ij} + 1)^{-3/2}u_{ij} = \frac{1}{2}\sum r_{ij}^{-3}u_{ij},$$

$$U'' = \frac{3}{4}\sum(u_{ij}R + 1)^{-5/2}u_{ij}^2 = \frac{3}{4}\sum r_{ij}^{-5}u_{ij}^2.$$

So

$$\begin{aligned} g'(R) &= -2\mu U^{-3}U'; \\ g''(R) &= 6\mu U^{-4}(U')^2 - 2\mu U^{-3}U'' \\ &= 2\mu U^{-4}\sigma(R) \end{aligned}$$

where

$$\begin{aligned} \sigma(R) &= 3(U')^2 - UU'' \\ &= \frac{3}{4}\left[(\sum_{i<j} r_{ij}^{-3}u_{ij})^2 - (\sum_{i<j} r_{ij}^{-1})(\sum_{i<j} u_{ij}^2 r_{ij}^{-5}) \right] \\ &= \frac{3}{4}\sum (r_{ij}^{-3}r_{i'j'}^{-3}u_{ij}u_{i'j'} - r_{ij}^{-5}r_{i'j'}^{-1}u_{ij}^2). \end{aligned}$$

The last sum is over all the indices (i, j, i', j') such that $i < j, i' < j'$ and $(i, j) \neq (i', j')$. Now we combine the terms of indices (i, j, i', j') with that of indices (i', j', i, j) to obtain

$$\sigma(R) = \frac{3}{4}\sum r_{ij}^{-5}r_{i'j'}^{-5}(-1)(u_{ij}r_{ij}^2 - u_{i'j'}r_{i'j'}^2)^2 \leq 0.$$

This proves the lemma. □

Solving for R from $r_{ij} = u_{ij}R + 1 = 0$ we get

$$R(i, j) = -\frac{1}{u_{ij}}.$$

Let $(\hat{\imath}, \hat{\jmath})$ be the index pair such that

$$R(\hat{\imath}, \hat{\jmath}) > 0 \quad and \quad R(\hat{\imath}, \hat{\jmath}) = \min_{(i,j)}\{R(i, j) : \quad R(i, j) > 0\}.$$

The only part of $g(R)$ that is of interest to us is the segment over the interval $[0, R(\hat{\imath}, \hat{\jmath}))$. We denote this index pair as $(\hat{\imath}, \hat{\jmath})(\vec{u})$. If the indicated index pair does not exist, then we set $(\hat{\imath}, \hat{\jmath})(\vec{u}) = (1, 2)$.

The graph for $V = g(R)$ in the case of $R(\hat{\imath}, \hat{\jmath}) < \infty$ is given in Figure 3(a). For $\vec{u}$ fixed,

(i) $R_g^{\vec{u}}$ is the R-intercept of the graph $V = g(R)$ and we see that

$$R_g^{\vec{u}} = R(\hat{\imath}, \hat{\jmath}) = (R_g^{\vec{u}}, 0). \tag{80}$$

(ii) Similarly $T_g^{\vec{u}}$ is the V-intercept of the graph $V = g(R)$ and we have

$$T_g^{\vec{u}} = g(0) = (0, T_g^{\vec{u}}) \tag{81}$$

where

$$g(0) = \frac{\mu}{36}; \tag{82}$$

$$g'(0) = \frac{2}{3}(u_4 + u_5)g(0). \tag{83}$$

(iii) We drop the sup-script $\vec{u}$ in these conventions without causing any confusion.

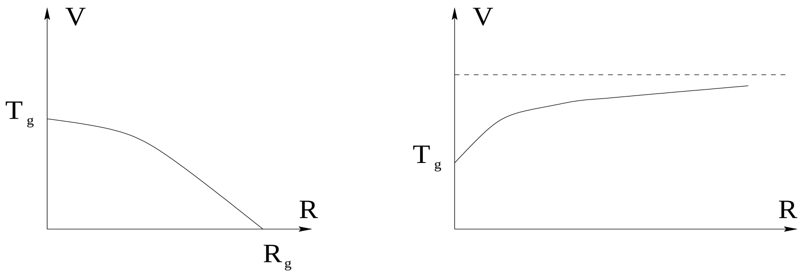

Figure 3(a) Figure 3(b)

When $(\hat{\imath}, \hat{\jmath}) = (1, 2)$, $V = g(R)$ is as in Figure 3(b). $g(0)$ and $g'(0)$ are again as in (82) and (83). This graph has a horizontal asymptotic line

$$V = g(\infty) = \mu/\tau^2(\vec{u}) \tag{84}$$

where $\tau(\vec{u})$ is the number of index pairs $(i, j), i < j$ such that $u_{ij} = 0$. Note that we need to count $(1, 2)$ as one of these pairs.

The graphs for $f_i(x)$ are straight lines passing through $(0, 1)$ with slopes s_i.

4.1.3. *Intersections of $V = f_i(R)$ and $V = g(R)$.* Let

$$S_{kj} = \{\vec{u} \in S^4 \mid (\hat{k}, \hat{\jmath})(\vec{u}) = (k, j)\}.$$

We see that $\cup_{1 \le k < j \le 4} S_{kj} = S^4$. It suffices to only study the part of Ω_i that is over S_{12} by symmetry.

In the rest of this section we assume that $\vec{u} \in S_{12}$. $V = g(R)$ is as in Figure 3(b). The way in which $V = f_i(R)$ and $V = g(R)$ intersect depends on the slopes s_i. We have the following possibilities:

(a) $s_i < 0$. See Figure 4. For $0 < \mu < 36$ there is one point of intersection. This point moves toward $(0, 1)$ as μ approaches to 36. It disappears after $\mu = 36$.

Figure 4(a) Figure 4(b)

(b) $s_i = 0$. Recall that $\tau(\vec{u})$ is the number of index pairs (i, j) so that $u_{ij} = 0$. For $0 < \mu < \tau^2(\vec{u})$ there is no point of intersection. For $\tau^2(\vec{u}) \le \mu \le 36$, There is one and this point moves toward $(0, 1)$ first then disappears after $\mu = 36$ as μ increases. See Figure 5.

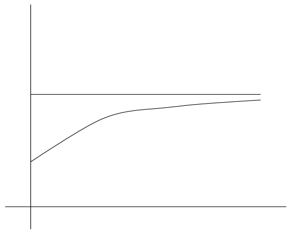

Figure 5(a)

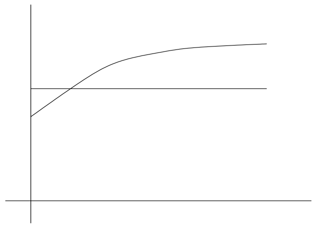

Figure 5(b)

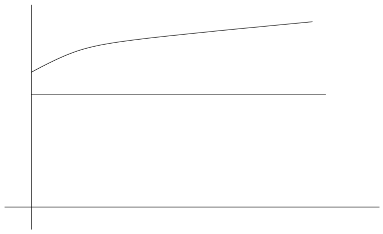

Figure 5(c)

(c) $0 < s_i < \frac{2}{3}(u_4 + u_5)$. For every $\vec{u}$ in this case there exist a $\mu(\vec{u}) > 0$, such that for $0 < \mu < \mu(\vec{u})$, there is no point of intersection. At $\mu = \mu(\vec{u})$, $V = f_i(R)$ and $V = g(R)$ are in tangential position. For $\mu(\vec{u}) < \mu < 36$, There are two points of intersection. One moves toward and the other moves away from $(0, 1)$. After $\mu = 36$, the first point disappears but the second one stays. It moves toward infinity as μ increases. See Figure 6.

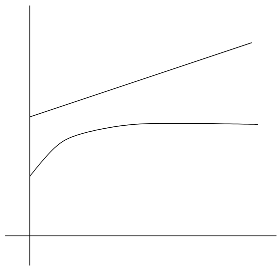

Figure 6(a)

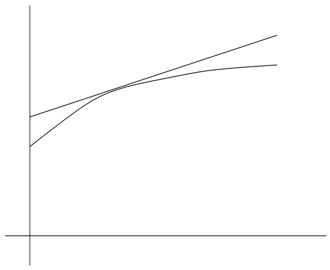

Figure 6(b)

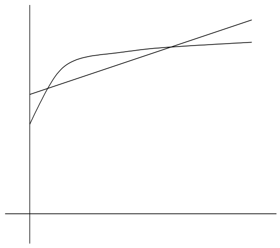

Figure 6(c)

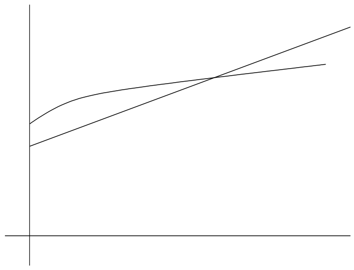

Figure 6(d)

Critical value $\mu(\vec{u})$ is determined by

$$g'(R,\mu) = f_i'(R)$$
$$g(R,\mu) = f_i(R)$$

with $\vec{u}$ fixed. Here we have two equations for two unknowns μ and R.

(d) $s_i > \frac{2}{3}(u_4 + u_5)$. Recall that $g'(0) = \frac{2}{3}(u_4 + u_5)$ and $g(R)$ is a concave function. So for $0 < \mu < 36$, there is no point of intersection. For $\mu > 36$ we have one. This point occurs at $(0,1)$ for $\mu = 36$. It then moves towards infinity as μ increases. See Figure 7.

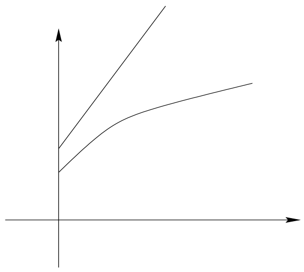

Figure 7(a)

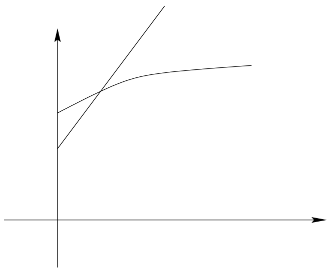

Figure 7(b)

We now further divide S_{12} into four subsets according to the listed possibilities (a)-(d). Denote for Ω_i;

$$S_{12} = S_{12}^a(i) \cup S_{12}^b(i) \cup S_{12}^c(i) \cup S_{12}^d(i).$$

4.2. A geometric description.

4.2.1. *On AC_μ.* In this subsection we study Ω_3, the boundary of the set of admissible configurations. We have

PROPOSITION 4.1. *For Ω_3, $S_{12}^a(3) = S_{12}^b(3) = S_{12}^c(3) = \emptyset$. $S_{12}^d(3) = S_{12}$.*

Proof: Let $u \in S_{12}$. Recall that

$$s_3 = \frac{2}{3}(u_4 + u_5) - y_{min}(\vec{u})$$

where y_{min} is the smallest solution of (43) with $p = p(\vec{u})$, $q = q(\vec{u})$. We have

$$s_3 > \frac{2}{3}(u_4 + u_5)$$

since $\frac{2}{3}(u_4 + u_5) > 0$ and $y_{min} < 0$. This proves the proposition. □

The following description on Ω_3 is based on proposition 4.1.

THEOREM 4.1. *For $\mu < 36$, every configuration in D is admissible. As μ passes through* 36, *A simply connected set of non-admissible configurations occurs. The boundary of this non-admissible set expands in all direction as μ increases.*

Remember that the boundary of AC_μ is the intersection of the configuration space D and Ω_3. As μ increases, Ω_3 will eventually touch the boundary of D. The intersection of Ω_3 and ∂D defines the Hill's region of the planar four-body problem.

Figure 8(a)-(d) provide an oversimplified picture on the evolution of the set AC_μ. In these pictures, we use the inside of a triangle to represent the configuration space and the triangle itself represents the boundary of D. We also regard the center of the triangle as $X = \vec{0}$ and the directions of vertices as Δ. Figure 8 only serves as a rough demonstration on what we know. The details on the evolution of $\Omega_3 \cap \partial D$ are not there. Ω_3 is empty for $\mu < 36$. It comes out of $X = \vec{0}$ after $\mu = 36$, touches ∂D at central configurations and grows bigger. However, as shown in Figure 8(d), Ω_3 stays inside of D forever over Δ.

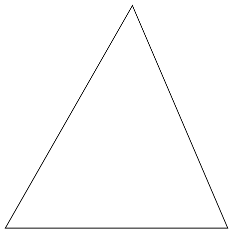

Figure 8(a)

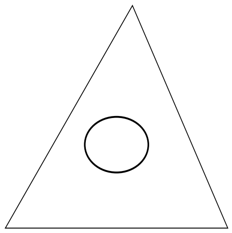

Figure 8(b)

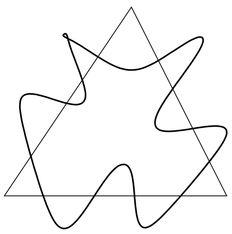

Figure 8(a)

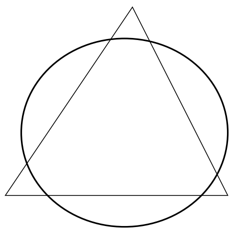

Figure 8(b)

4.2.2. *The surface* Ω_1. Ω_1 and Ω_2 are much more complicated surfaces that work together to determine the nature of the fiberations of K_μ over AC_μ. We have

PROPOSITION 4.2. *For* Ω_1, $S_{12}^a(1), S_{12}^b(1), S_{12}^c(1)$ *are all non-empty. However,* $S_{12}^d(1)$ *is empty.*

Proof: For the emptiness of $S_{12}^d(1)$, recall

$$s_1 = \frac{2}{3}(u_4 + u_5) - y_{max}(\vec{u})$$

and the fact that $y_{max} > 0$. So $s_1 < \frac{2}{3}(u_4 + u_5)$ for all $\vec{u} \in S_{12}$ and $S_{12}^d(1) = \emptyset$. Observe that the boundary of Δ_{12} is in $S_{12}^b(1)$, the interior of Δ_{12} is in $S_{12}^c(1)$ and S_{lin} is in $S_{12}^a(1)$ where S_{lin} is the set of all the orientations that hit the boundary of D at collinear four-body configurations. See section 4.3 for further discussions on S_{lin}. □

Ω_1 is simple over $S_{12}^a(1)$. . At $\mu = 0$, this surface is at $(\vec{u}, \frac{-1}{s_1(\vec{u})})$. It is mostly out of D but touches the boundary of D on S_{lin}. As μ increases, Ω_1 moves toward $X = \vec{0} \in R^5$ for all $\vec{u} \in S_{12}^a(1)$. When μ reaches 36, this component from Ω_1 reaches $X = \vec{0}$, then completely disappears afterwards.

Things are a little more complicated over $S_{12}^b(1)$. We again divide $\Omega_{12}^b(1)$ into five subsets according to $\tau(\vec{u})$ and define

$$S_{12}^b(1)(j) = \{\vec{u} \in S_{12}^b(1) : \ \tau(\vec{u}) = j\}$$

where $j = 1, \cdots, 5$. Over $S^b_{12}(1)(j)$, Ω_1 first occurs at $\mu = j^2$. Then it behaves the same way as the part over $S^a_{12}(1)$ (It moves toward $X = \vec{0}$, reaches $X = \vec{0}$ at $\mu = 36$ then disappears).

It is harder to analyse the part of the surfaces that is over $S^c_{12}(1)$ where the intersection pattern is $(0, 1, 2, 1)$. For any given $\vec{u} \in S^d_{12}(1)$ there exists a critical value $\mu(\vec{u})$, such that Ω_1 occurs over the orientation $\vec{u}$ at $\mu = \mu(\vec{u})$. For $\mu(\vec{u}) < \mu < 36$, there are two points of Ω_1 over $\vec{u}$. The inside point moves toward, but the outside one moves away from, the critical configuration $X = \vec{0}$. At $\mu = 36$ the inside point reaches i $X = \vec{0}$ then disappears but the outside one stays forever.

To get a complete geometric description on Ω_1 we need to have

(1) A detailed understanding on the geometry of $S^b_{12}(1)$. ($S^a_{12}(1)$ and $S^c_{12}(1)$ as well)
(2) A description on the level set defined by $\mu(\vec{u}) = c$ in $S^c_{12}(1)$.
(3) A description on $\Omega_1 \cap \partial D$.

All these are non-trivial questions deserving further considerations. We strongly believe that there exist no bifurcation for the level surface $\mu(\vec{u}) = c$ other than the trivial ones (corresponding to the possible changes introduced by different values of τ).

We conclude this subsection again by a set of pictures. Again, these pictures only serve a purpose of demonstration, and we remind our reader on the inevitable drawbacks of trying to describe 4-d objects using 2-d pictures.

This time we use an octagon to represent the configuration space D. The center of the octagon represents the critical configuration $X = \vec{0}$. The shadowed part is for $S^c_{ij}(1)$. The direction of vertices is for $S^b_{ij}(1)$ and the blank part for $S^a_{ij}(1)$. We use different vertices to represent different τ values in $S^b_{ij}(1)$. See Figure 9(a).

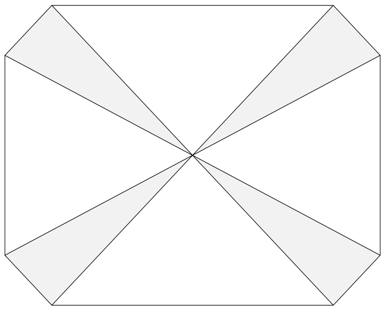

Figure 9(a)

All the boundary lines are dotted in Figure 9(b)-(f). The solid curves represent Ω_1. The initial position (for $\mu = 0$) is as in Figure 9(b). The part where the solid curve coincide the dotted boundary of the octagon represents S_{lin}.

As μ moves up, Ω_1 moves into D gradually. This is reflected in Figure 9(c). Then, after μ reaches $1, 2, 3, 4, 5$, Ω_1 is extended into the shadowed set of orientations. (Figure 9(d)). Gradually, Ω_1 evolves to become what it is in Figure 9(e). We get an inner part that is separated from the rest of the surface. As μ goes up, the inner part moves toward $X = \vec{0}$ then disappears after $\mu = 36$. The rest of the surface moves away from $X = \vec{0}$. (See Figure 9(f)). The majority of the surface will grow out of D. But it stays inside of D forever over Δ.

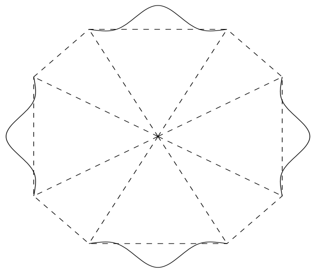

Figure 9(b)

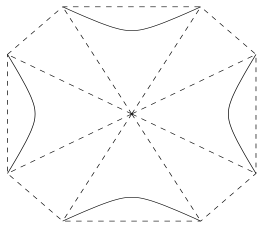

Figure 9(c)

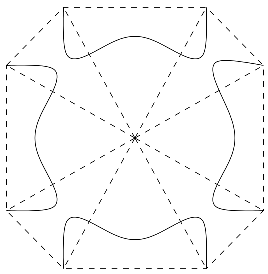

Figure 9(d)

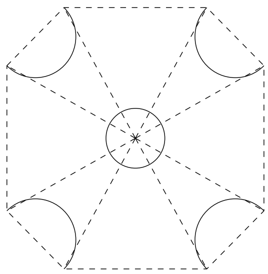

Figure 9(e)

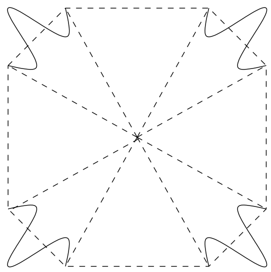

Figure 9(f)

4.2.3. *The surface* Ω_2. This is another surface on which we do not have complete control. First let us state what we know about Ω_2.

PROPOSITION 4.3. *For* Ω_2, $S^a_{12}(2) = S^b_{12}(2) = \emptyset$. *However,* $S^c_{12}(2)$ *and* $S^d_{12}(2)$ *are non-empty.*

Proof: The proof on the emptiness of $S^a_{12}(2)$ and $S^b_{12}(2)$ can be transferred to a problem of finding the maximum value of a function in $\vec{u}$ over S_{12}. This is a tedious calculus problem involving long computations, on which we chose to skip here.

For the rest of this proposition, we consider the set of orientations with $u_1 = u_2 = 0$. For these orientations we know that $\frac{-1}{3}(u_4 + u_5)$ is a solution for (43). We can now easily find the other two solutions in quadratic form. One solution leads to things in $S^c_{12}(2)$ and the other in $S^d_{12}(2)$. $\square$

Critical value $\mu(\vec{u})$ exist for every $\vec{u}$ in $S^c_{12}(2)$. The intersection pattern is $(0,1,2,1)$ according to $\mu < \mu(\vec{u}), \mu = \mu(\vec{u}), \mu(\vec{u}) < \mu < 36$ and $\mu > 36$. Over $S^d_{12}(2)$, there is no point of Ω_2 before $\mu = 36$. For $\mu > 36$, Ω_2 is a surface that is homeomorphic to S^4. It grows as μ increases.

In Figure 10 we demonstrate what we know about Ω_2. We again use an octagon as in Figure 9(a) to represent the configuration space. However, the shadowed part now represents those in the orientation set $S^d_{12}(2)$, and the rest, $S^c_{12}(2)$. μ increases as the pictures change from Figure 10(a) to Figure 10(d). Ω_2 grows to touch ∂D. It grows out of D in all directions except Δ, in which it stays inside of D forever.

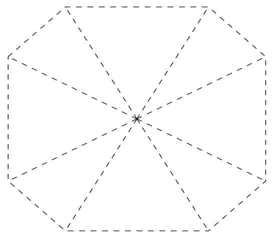

Figure 10(a)

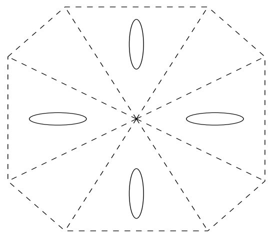

Figure 10(b)

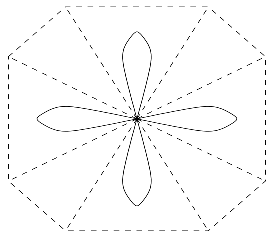

Figure 10(c)

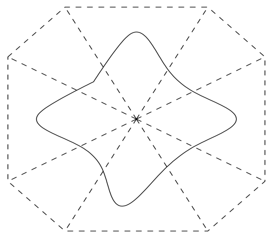

Figure 10(d)

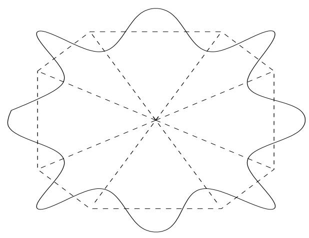

Figure 10(e)

4.2.4. *Fiberation over AC_μ: the Hill's regions.* We are finally in a position to discuss the fabrications of K_μ over AC_μ. Similar to what was obtained in [**9**] for the three-body problem, a fiber can be the entire sphere S^2, a band or two caps. They respectively correspond to the cases of (i) $a, b \geq 1$, (ii) $a < 1, b \geq 1$, and (iii) $a < 1, b < 1$ where a and b are the short and long axis of the ellipse (7). Recall that Ω_1 is the surface of $a = 1$ and Ω_2 is that of $b = 1$. See Figure 11(a)-(c).

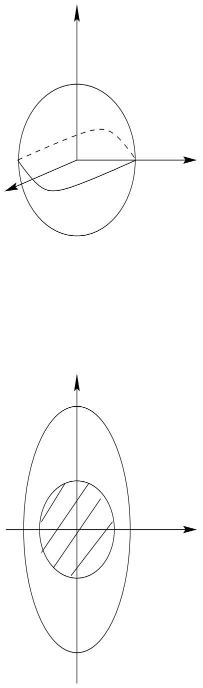

Figure 11(a)

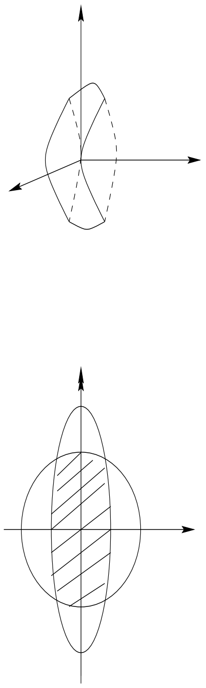

Figure 11(b)

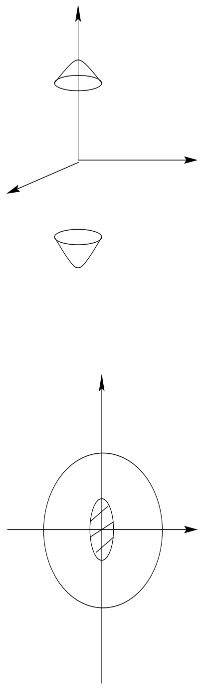

Figure 11(c)

Since no rigorous description is available, we will continue with our previously introduced octagon to convey what we know about this fibration. Again these pictures are not precise statements on the geometry of the Hill's region. They only serve a purpose of demonstration.

The octagon in Figure 9(a) is used to represent the configuration space. A region is shadowed if the pointwise fiber is a band. It is double shadowed if the fiber is two caps and is left blank if the fiber is the entire S^2. The set of configurations that are not admissible are blacked-out. These regions come and go and constantly change their shapes as μ increases from zero to infinity. See Figure 14(a)-(f).

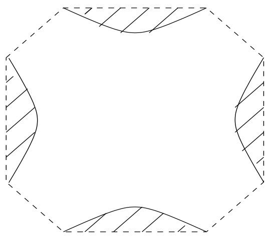

Figure 12(a)

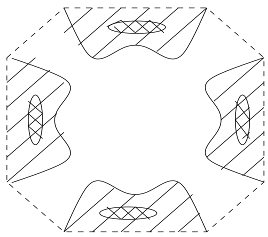

Figure 12(b)

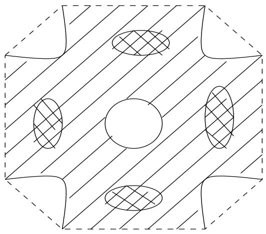

Figure 12(c)

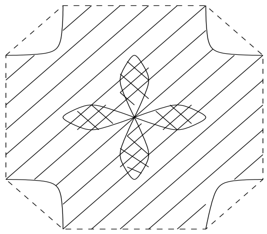

Figure 12(d)

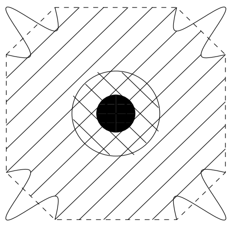

Figure 12(e)

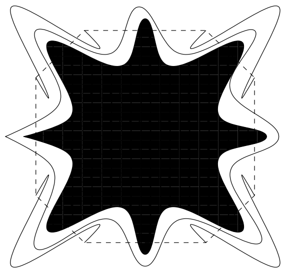

Figure 12(f)

4.3. Fiberations over S_{lin}. This subsection serves two purposes. In the case of $n = 3$, every orientation in the configuration space is in fact in S_{lin}. So by providing complete geometric description of the Hill's region over S_{lin}, we sustain our claim that everything we did in [**9**] has been generalized and applied to the four-body case. Secondly, objects studied in this subsection are the part of K_μ on which we have complete control.

DEFINITION 4.1. *Let*

$$S_{lin} = \{\vec{u} \in S^4 : \quad \rho_y(X_{\vec{u}} \cap bd(D)) = 0\}.$$

Since for $\vec{u} \in S_{lin}$

$$\rho_z(X_{\vec{u}} \cap bd(D)) = 0,$$

$\vec{u} \in S_{lin}$ if and only if $X_{\vec{u}} \cap \partial D$ is collinear. We have

LEMMA 4.2. *S_{lin} is equivalent to the projective plane RP^2.*

Proof: S_{lin} is equivalent to the set of four-body collinear configurations. Let us now use r_{12}, r_{23} and r_{34} as variables to represent a collinear configuration. To get all the index orders of the four bodies, we allow these variables to be either positive or negative. Taking away the size of the configurations we set

$$r_{12}^2 + r_{23}^2 + r_{34}^2 = 1. \tag{85}$$

We observe that (r_{12}, r_{23}, r_{34}) and $(-r_{12}, -r_{23}, -r_{34})$ represent the same point in the space of configurations. Therefore S_{lin} is indeed equivalent to RP^2.

Next we study $S_{lin} \cap \Delta$. $\Delta_{ij} \cap S_{lin}$ form big circles on the unit sphere (85). $\Delta_{12} \cap S_{lin}$ is defined by setting $r_{12} = 0$. Others are similar. We have in total six such circles to represent six different kinds of binary collisions. The intersections of all these circles correspond to multiple collisions. We again classify orientations in Δ according to their τ value. So

$$\Delta = \Delta(1) \cup \Delta(2) \cup \Delta(3)$$

where

$$\Delta(j) = \{\vec{u} \in \Delta : \quad \tau(\vec{u}) = j\}.$$

The great circles in Δ divide the 2-sphere (85) into triangular regions. See Figure 13. The sides of these triangles are in $\Delta(1)$. The vertices represent multiple collisions. two of them is in $\Delta(3)$ (triple collision) and one is in $\Delta(2)$ (simultaneous binary collision). $\square$

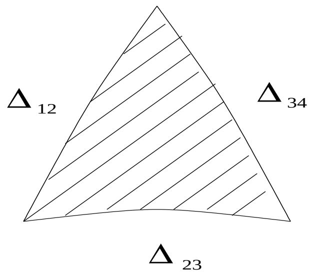

Figure 13

We also need the next lemma

LEMMA 4.3. *Following the conventions given in section 3.2.2. If* $\vec{u} \in S_{lin} \cap S_{12}$, *then*

$$b(\vec{u}) = u_4u_5 - u_1^2 - u_2^2 - u_3^2 \leq 0.$$

Proof: Under the assumption that $b(\vec{u}) > 0$ we have $u_3^2 < u_4u_5$. Therefore

$$c(\vec{u}) = 2u_1u_2u_3 - u_4u_2^2 - u_5u_1^2 < 0.$$

Since $\vec{u} \in S_{lin}$, we have $y_{min} = y_{mid}$. So (43) has one positive solution and two negative solutions that are equal. This implies

$$q(\vec{u}) = -\frac{2a^3}{27} + \frac{ab}{3} - c < 0$$

and

$$(\frac{q}{2})^2 + (\frac{p}{3})^3 = 0.$$

Remember that $p(\vec{u}) = -\frac{a^2}{3} + b$ is always negative and $c < 0$. We have

$$\frac{1}{3^3}(\frac{a^2}{3} - b)^3 < \frac{1}{2^2}(-\frac{2a^3}{27} + \frac{ab}{3})^2.$$

Therefore

$$b > \frac{a^2}{4}$$

which will make

$$q(\vec{u}) = -\frac{2a^3}{27} + \frac{ab}{3} - c > 0.$$

This is a contradiction since we have $q(\vec{u}) < 0$. The lemma is proved. $\square$

Now turn to the solutions for (43). We have

$$cos\theta_{min} = cos\theta_{mid} = -\frac{1}{2}, \quad cos\theta_{max} = 1.$$

So

$$f_2(R) = f_3(R) = 1 + \left(\frac{2}{3}(u_4+u_5) + \frac{1}{3}\sqrt{(u_4+u_5)^2 - 3b(\vec{u})}\right) R$$

and

$$f_1(R) = 1 + \frac{2}{3}\left((u_4+u_5) - \sqrt{(u_4+u_5)^2 - 3b(\vec{u})}\right) R$$

According to Lemma 4.3, $b \leq 0$ so $s_1 \leq 0$. We further have $s_1 = 0$ if and only if $b = 0$. For $\vec{u} \in S_{lin} \cap S_{12}$, $b = 0$ and $(\frac{p}{3})^3 + (\frac{q}{2})^2 = 0$ together imply $c = 0$ so $u_1 = u_2 = 0$. Therefore $s_1 = 0$ if and only if $\vec{u} \in S_{lin} \cap \Delta_{12}$.

Denote the part of the surface Ω_i that is over S_{lin} as $\hat{\Omega}_i$. We also denote the values of μ at which Ω_3 touches collinear central configurations as μ_{lin}. We have the following description on these surfaces:

(i) $\hat{\Omega}_2$, $\hat{\Omega}_3$ are identical. For $0 < \mu < 36$, it is empty. It occurs as one point ($X = \vec{0}$) at $\mu = 36$. For $36 < \mu \leq \mu_{lin}$, it is homeomorphic to $S_{lin} = RP^2$. For $\mu > \mu_{lin}$, part of the surface is out of D. In this case, $\hat{\Omega}_2 \cap D$ is homotopic to $\Delta \cap S_{lin}$.

(ii) For $\hat{\Omega}_1$. It starts from the boundary of D at $\mu = 0$ and shrinks in as μ increases. For $0 < \mu \leq 1$, this surface is homeomorphic to $S_{lin} \setminus \Delta$. For $1 < \mu \leq 4$, it is homeomorphic to $S_{lin} \setminus (\Delta(2) \cup \Delta(3))$. For $4 < \mu \leq 9$, it is homeomorphic to $S_{lin} \setminus \Delta(3)$. For $9 < \mu < 36$, it is homeomorphic to S_{lin}. At $\mu = 36$, it is at the critical configuration $X = \vec{0}$. For $\mu > 36$, $\hat{\Omega}_1$ is homeomorphic to the set $S_{lin} \cap \Delta$. As μ increases, it moves toward the set of collinear degenerate configurations on the boundary of D. It will stay inside of D forever.

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