Hyperbolic Theory: Stable and Unstable Manifolds

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r mapping such that

$$\begin{aligned} x_1 &= \lambda x + \alpha(x,y) \\ y_1 &= \mu y + \beta(x,y). \end{aligned}$$

We assume that

(i) $0 < \mu < 1 < \lambda$; and

(ii) $\alpha(0,0) = \beta(0,0) = 0$ and $\|\alpha\|_{C^1}, \|\beta\|_{C^1} < \varepsilon$.

Theorem 1 Under the assumption that $\varepsilon > 0$ is sufficiently small, there exists $\phi : \mathbb{R} \to \mathbb{R}$ such that $(x, \phi(x))$ is a unique Lipschitz graph that is T-invariant.

– By a Lipschitz graph we mean that there exists $\gamma > 0$ such that, for all $x, x' \in \mathbb{R}$,

$$\frac{|\phi(x) - \phi(x')|}{|x - x'|} < \gamma.$$

– By T-invariant we mean that, for every $x \in \mathbb{R}$ there exists $x' \in \mathbb{R}$ such that

$$T(x, \phi(x)) = (x', \phi(x')).$$

– Let $W^u = \{(x, \phi(x)), x \in \mathbb{R}\}$. We call W^u as the unstable manifold for T.

Theorem 2 Let $(x, \phi(x))$ be as in the above. Then

(i) $\phi(x)$ is C^1.

(ii) Let u be a tangent vector of W^u at $z \in W^u$, then

$$\|DT_z^n u\| \geq K(\hat{\lambda})^n \|u\|$$

where $\hat{\lambda} > 1$ is smaller than λ.

Theorem 2 claims in particular that W^u is more than Lipschitz. It is C^1.

Theorem 3 Assume that α, β have bounded C^r-norms. Then $(x, \phi(x))$ in the above is a C^r-graph.

1. Proof of Theorem 1

Let

$$\mathcal{L}_\gamma = \{\phi; \frac{|\phi(x) - \phi(y)|}{|x-y|} \le \gamma \quad \forall x, y \in \mathbb{R}\}.$$

To make $\mathcal{L}_\gamma$ a norm-space we let

$$\|\phi\| = \sup_{x \in \mathbb{R}} \frac{|\phi(x)|}{|x|}.$$

For $\phi_1, \phi_2 \in \mathcal{L}_\gamma$, let

$$d(\phi_1, \phi_2) = \|\phi_1 - \phi_2\| = \sup_{x \in \mathbb{R}} \frac{|\phi_1(x) - \phi_2(x)|}{|x|}$$

$(\mathcal{L}_\gamma, d)$ is a complete metric space.

Step 1: First we prove that, for ε sufficiently small, the image of a Lipschitz graph in $\mathcal{L}_\gamma$ under T is also a Lipschitz graph in $\mathcal{L}_\gamma$.

Let $\phi \in \mathcal{L}_\gamma$.

Claim 1.1: $\forall x \in \mathbb{R}$, there exists a unique g such that

$$x = \lambda g + \alpha(g, \phi(g)),$$

which we denote as $g_\phi(x)$

Remark: Observe that

$$T(x, \phi(x)) = (\lambda x + \alpha(x, \phi(x)), \mu\phi(x) + \beta(x, \phi(x))$$

Claim 1.1 implies that the $\{T(x, \phi(x)); x \in \mathbb{R}\}$ is a graph of a function, which we denote as $T_*\phi$.

Note that $T_*\phi(x)$ is defined by

$$x \to g \to \mu\phi(g) + \beta(g, \phi(g))$$

where $g = g_\phi(x)$ is as in Claim 1.1.

Proof: For x given, let $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be defined as

$$\mathcal{F}(g) = \lambda^{-1}x - \lambda^{-1}\alpha(g, \phi(g))$$

It suffices to show that $\mathcal{F}(g)$ is a contraction:

$$
\begin{aligned}
|\mathcal{F}(g_1) - \mathcal{F}(g_2)| &< \lambda^{-1}\varepsilon(|g_1 - g_2| + |\phi(g_1) - \phi(g_2)|) \\
&\leq \lambda^{-1}(1+\gamma)\varepsilon|g_1 - g_2| \\
&< \rho|x_1 - x_2|
\end{aligned}
$$

for some $\rho < 1$. For the first inequality, we use $\|\alpha\|_{C^1} < \varepsilon$. For the second, we use $\phi \in \mathcal{L}_\gamma$ and the last is obtained by letting ε small.

Claim 1.2: $T_*\phi \in \mathcal{L}_\gamma$ provided that ε is sufficiently small.

Proof: Let $T(x_1, \phi(x_1)) = (\widehat{x}_1, \widehat{y}_1)$, $T(x_2, \phi(x_2)) = (\widehat{x}_2, \widehat{y}_2)$. We need to prove that

$$
\frac{|\widehat{y}_1 - \widehat{y}_2|}{|\widehat{x}_1 - \widehat{x}_2|} < \gamma.
$$

This follows from

$$
\begin{aligned}
|\widehat{x}_1 - \widehat{x}_2| &\geq \lambda|x_1 - x_2| - |\alpha(x_1, \phi(x_1)) - \alpha(x_2, \phi(x_2))| \\
&\geq (\lambda - \varepsilon(1+\gamma))|x_1 - x_2| \\
|\widehat{y}_1 - \widehat{y}_2| &\leq \mu|\phi(x_1) - \phi(x_2)| + |\beta(x_1, \phi(x_1)) - \beta(x_2, \phi(x_2))| \\
&\leq (\mu + \varepsilon(1+\gamma^{-1}))\gamma|x_1 - x_2|.
\end{aligned}
$$

These two inequalities are combine to confirm Claim 2.

Step 2: We show that $\phi \to T_*\phi$ is a contraction.

By the way $\|\cdot\|$ is defined on $\mathcal{L}_\gamma$, we need to estimate

$$\frac{|T_*\phi_1(x) - T_*\phi_2(x)|}{|x|}$$

Note that $T_*\phi(x)$ are defined indirectly by

$$x \to g := g_\phi(x) \to \mu\phi(g) + \beta(g, \phi(g))$$

So need to first estimate $|g_{\phi_1}(x) - g_{\phi_2}(x)|$.

Estimate $|g_{\phi_1}(x) - g_{\phi_2}(x)|$**:**

- Let $g_1 = g_{\phi_1}(x)$, $g_2 = g_{\phi_2}(x)$. From

$$\begin{aligned} x &= \lambda g_1 + \alpha(g_1, \phi_1(g_1)) \\ x &= \lambda g_2 + \alpha(g_2, \phi_2(g_2)) \end{aligned}$$

we obtain

$$\lambda(g_1 - g_2) + \alpha(g_1, \phi_1(g_1)) - \alpha(g_2, \phi_2(g_2)) = 0.$$

It follows that

$$\lambda(g_1 - g_2) + \partial_x\alpha(g_1 - g_2) + \partial_y\alpha(\phi_1(g_1) - \phi_2(g_2)) = 0.$$

We write

$$\phi_1(g_1) - \phi_2(g_2) = (\phi_1(g_1) - \phi_1(g_2)) + (\phi_1(g_2) - \phi_2(g_2))$$

in the above to conclude

$$(\lambda + \partial_x\alpha + \partial_y\alpha\frac{\phi_1(g_1) - \phi_1(g_2)}{g_1 - g_2})(g_1 - g_2) = -\partial_y\alpha(\phi_1(g_2) - \phi_2(g_2)).$$

It follows that

$$|g_1 - g_2| \le \frac{\varepsilon}{\lambda - \varepsilon(1+\gamma)} |g_2| \|\phi_1 - \phi_2\|$$

- We also have

$$\begin{aligned}
&|T_*\phi_1(x) - T^*\phi_2(x)| \\
&\quad = |\mu(\phi_1(g_1) - \phi_2(g_2)) + \beta(g_1, \phi_1(g_1)) - \beta(g_2, \phi_2(g_2))| \\
&\quad = |(\mu + \partial_y \beta)(\phi_1(g_1) - \phi_2(g_2)) + \partial_x \beta(g_1 - g_2)| \\
&\quad \le (\mu + \varepsilon)|\phi_1(g_1) - \phi_1(g_2)| + (\mu + \varepsilon)|\phi_1(g_2) - \phi_2(g_2)| \\
&\qquad + |\partial_x \beta(g_1 - g_2)| \\
&\quad \le ((\mu + \varepsilon)\gamma + \varepsilon)|g_1 - g_2| + (\mu + \varepsilon)\|\phi_1 - \phi_2\||g_2| \\
&\quad < (\mu + K(\gamma)\varepsilon)|g_2|\|\phi_1 - \phi_2\|.
\end{aligned}$$

Note that in obtaining the last inequality we used our previous estimate on $|g_1 - g_2|$.

To finish this proof we also need

$$\begin{aligned}
|x| &= \lambda g_2 + \beta(g_2, \phi_2(g_2)) \\
&\ge (\lambda - \varepsilon(1+\gamma))|g_2|.
\end{aligned}$$

Combining the last two we finally conclude that

$$\|T_*\phi_1 - T_*\phi_2\| \le \frac{\mu + K\varepsilon}{\lambda - K\varepsilon} \|\phi_2 - \phi_1\|.$$

So $T_* : \mathcal{L}_\gamma \to \mathcal{L}_\gamma$ is a contraction. Theorem 1 is proved.

2. Proof of Theorem 2

Step 1 Preparatory remarks

First note that a Lipschitz function is not necessarily differentiable.

Ex: $f(x) = x \sin \frac{1}{x}$ is a Lipschitz function on $\mathbb{R}$ but it is not differentiable at $x = 0$:

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

is not well-defined at $x = 0$.

Definition: (a) Let $\phi : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and let x be fixed. We say that a unit vector $\tau \in \mathbb{R}^2$ is a tangent direction of ϕ at $(x, \phi(x))$ if there exists $x_n \to x$ such that

$$\lim_{n \to \infty} \frac{(x_n, \phi(x_n)) - (x, \phi(x))}{|(x_n, \phi(x_n)) - (x, \phi(x))|} = \tau.$$

(b) Let $\tau_{\phi_x} := \{k\tau; \quad k \in \mathbb{R}; \ \tau$ is a tangent direction of ϕ at $(x, \phi(x))\}$. τ_{ϕ_x} is the collection of all tangent vectors of ϕ at $(x, \phi(x))$.

Claim 2.1: $\phi(x)$ is differentiable if and only if τ_{ϕ_x} is a one dimensional subspace of $\mathbb{R}^2$.

Proof: If $\phi(x)$ is differentiable, then

$$\phi(x_n) = \phi(x) + \phi'(x)(x_n - x) + \mathbf{o}(|x_n - x|).$$

It follows that

$$\frac{(x_n, \phi(x_n)) - (x, \phi(x))}{|(x_n, \phi(x_n)) - (x, \phi(x))|} \to \frac{(1, \phi'(x))}{|(1, \phi'(x))|}.$$

So we have only one tangent direction and τ_{ϕ_x} is a one dimensional subspace.

If τ_{ϕ_x} is a 1-d subspace, then there exists a real number L such that $(u, v) \in \tau_{\pi_x}$ if and only if $v = Lu$. To prove that ϕ is differentiable at x, it suffices to prove that, for any $x_n \to x$,

$$\lim_{n\to\infty} \frac{|\phi(x_n) - \phi(x) - L(x_n - x)|}{|x_n - x|} \to 0.$$

To prove this we note that by assumption,

$$\frac{(x_n - x, \phi(x_n) - \phi(x))}{|(x_n - x, \phi(x_n) - \phi(x))|} \to (u, Lu),$$

which is written componentwise as

$$\frac{x_n - x}{|(x_n - x, \phi(x_n) - \phi(x))|} \to u$$

and

$$\frac{\phi(x_n) - \phi(x)}{|(x_n - x, \phi(x_n) - \phi(x))|} \to Lu.$$

It follows that

$$\begin{aligned} \phi(x_n) - \phi(x) &\to Lu|(x_n - x, \phi(x_n) - \phi(x))| \\ &\to L(x_n - x) \end{aligned}$$

Claim 2.2: Let $\phi \in \mathcal{L}_\gamma$ be the T-invariant graph in Theorem 1. Then τ_{ϕ_x} is T-invariant, i.e. $\forall \tau \in \tau_{\phi_x}$,

$$DT_{(x,\phi(x))}\tau \in \tau_{\phi_{T(x,\phi(x))}}.$$

Proof: This claim follows from

$$\frac{T(x_n, \phi(x_n)) - T(x, \phi(x))}{|T(x_n, \phi(x_n)) - T(x, \phi(x))|} \to \frac{DT_{(x,\phi(x))}\tau}{|DT_{(x,\phi(x))}\tau|}$$

provided that

$$\frac{(x_n, \phi(x_n)) - (x, \phi(x))}{|(x_n, \phi(x_n)) - (x, \phi(x))|} \to \tau.$$

In summary, to prove that ϕ in Theorem 1 is differentiable, it suffices to prove that associated with each $(x, \phi(x))$, there exists a 1D-subspace E_x such that

(a) $(u, v) \in \tau_{\phi_x}$ if and only if $(u, v) \in E_x$;

(b) E_x is DT-invariant.

Furthermore, if

(c) E_x is continuous with respect to x,

then ϕ is C^1, and Theorem 2(i) is proved.

Step 2: Proof of (a)-(c): an invariant cone argument

– Let $T_p\mathbb{R}^2$ be the space of tangent vectors of $\mathbb{R}^2$ at $p \in \mathbb{R}^2$. Then $T_p\mathbb{R}^2 \approx \mathbb{R}^2$.

Definition For $0 < \gamma < 1$ and $p \in \mathbb{R}^2$. We call

$$H_p = \{(u, v) \in T_p\mathbb{R}^2 : \quad |v| \leq \gamma |u|\}$$

the *horizontal cone of size* γ at $p \in \mathbb{R}^2$. Similarly,

$$V_p = \{(u,v) \in T_p\mathbb{R}^2 : \quad |u| \leq \gamma |v|\}$$

is the *vertical cone of size* γ.

In what follows we first chose $\gamma > 0$ small then $\varepsilon > 0$ smaller.

Claim 2.3: $\{H_p,\ p \in \mathbb{R}^2$ is invariant under DT, i.e.,

$$DT_p(H_p) \subset H_{T(p)}.$$

Similarly, $\{V_p : \quad p \in \mathbb{R}^2\}$ is invariant under DT^{-1}, i.e.,

$$DT_p^{-1} V_p \subset V_{T^{-1}p}$$

Claim 2.4: Vectors in H_p are **uniformly expanding** under the iterations of DT: there exists $\widehat{\lambda} > 1$, such that for $\tau \in H_p$,

$$|DT_p \tau| \geq \widehat{\lambda} |\tau|$$

Similarly, vectors in V_p are **uniformly expanding under the iterations of** DT^{-1}: We have for $\tau \in V_p$,

$$|DT_p^{-1}\tau| > \hat{\lambda}|\tau|$$

Proof: Note that

$$DT = \begin{pmatrix} \lambda + \partial_x\alpha & \partial_y\alpha \\ \partial_x\beta & \mu + \partial_y\beta \end{pmatrix}.$$

For $\tau := (u, v) \in H_p$, let $s(\tau) = \frac{|v|}{|u|}$. Then

$$\begin{aligned} s(DT(\tau)) &= \frac{|\partial_x\beta u + \mu v + \partial_y\beta v|}{|\lambda u + \partial_x\alpha u + \partial_y\alpha v|} \\ &\leq \frac{\mu s(\tau) - \varepsilon(1 + s(\tau))}{\lambda + \varepsilon(1 + s(\tau))} \\ &< \gamma. \end{aligned}$$

proving Claim 2.3. For the claim on uniform expanding we have

$$\begin{aligned} |DT(\tau)|^2 &= (\partial_x \beta u + \mu v + \partial_y \beta v)^2 + (\lambda u + \partial_x \alpha u + \partial_y \alpha v)^2 \\ &\geq u^2(\lambda^2 + \mu^2 s^2(\tau) - \mathcal{O}(\varepsilon)) \\ &\geq \frac{\lambda^2 - \mathcal{O}(\varepsilon)}{1+\gamma^2}|\tau|^2. \end{aligned}$$

Note that in obtaining the last inequality we used

$$|\tau|^2 = u^2 + v^2 = (1 + s^2(\tau))u^2 < (1+\gamma^2)u^2$$

since $\tau \in H_p$. It follow that

$$|DT(\tau)| \geq \frac{\lambda - \mathcal{O}(\varepsilon)}{\sqrt{1+\gamma^2}}|\tau|.$$

We are now ready to construct the 1D invariant spaces on $W^u = \{(x, \phi(x)\}$.

Claim 2.5: For $p \in W^u$, let

$$E_p = \cap_{n=1}^{\infty} DT^n_{p-n}(H_{p-n}).$$

Then $\{E_p, p \in W^u\}$ is a collection of DT-invariant 1D subspaces. Furthermore, $\{E_p\}$ is continuous in p.

Proof: Note that in the above $p_{-n} = T^{-n}p$. Let $E_p(n) := DT^n_{p_{-n}} H_{p_{-n}}$. It follows from Observation 1 that $E_p(n+1) \subset E_p(n)$. So E_p is non-empty, therefore at least a 1D-subspace of $\mathbb{R}^2$.

• We show that E_p is DT-invariant: this is because $E_{T(p)}(n+1) = DT \circ E_p(n)$; and it follows that

$$E_{T(p)} = \cap E_{T(p)}(n+1) = DT(\cap E_p(n)) = DTE_p.$$

• We now prove that E_p is not more than a 1D subspace: Our proof goes as follows.

(1) It follows from Claim 2.4 that, for $\tau \in E_p$,

$$|DT_p^{-n}\tau| < (\hat{\lambda})^{-n}|\tau|.$$

Reason: Let $\tau' = DT_p^{-n}\tau$, then we have

$$|DT^n_{p_{-n}}\tau'| > (\hat{\lambda})^n|\tau'|.$$

This is

$$|DT_p^{-n}\tau| < (\hat{\lambda})^n|\tau|.$$

(2) Similarly, let

$$S_p = \cap_{n=1}^{\infty} DT_{p_n}^{-n}(V_{p_n}).$$

is DT-invariant, and for any $\tau \in S_p$,

$$DT_p^{-n}\tau > (\hat{\lambda})^n |\tau|.$$

(3) Let e_1 be a unit vector in E_p, and e_2 be a unit vector in S_p. Assume that $\tau \in E_p$ is not in the subspace span by e_1. Then there exists $K_1, K_2 \neq 0$ such that

$$\tau = K_1 e_1 + K_2 e_2.$$

Applying DT^{-n}, we have

$$DT^{-n}\tau = K_1 DT^{-n} e_1 + K_2 DT^{-n} e_2.$$

This can not hold for all $n > 0$ since as $n \to \infty$, $|DT^{-n}\tau|$ and $|DT^{-n}e_1|$ decreasing exponentially towards zero but $|DT^{-n}e_2| \to \infty$ exponentially.

- Finally we show that E_p is continuous on p:

(a) First we note that E_p is the only set of vectors in H_p such that

$$|DT_p^{-n}\tau| < (\hat{\lambda})^{-n} |\tau|.$$

This is because any other vector in H_p can be written as $K_1 e_1 + K_2 e_2$ with $K_2 \neq 0$. But the images of e_2 grow exponentially under DT^{-1}.

(b) For $p_n \to p$, let τ_n be the unit tangent vector in E_{p_n} and $\{\tau_{n_k}\}$ is any given subsequence such that $\tau_{n_k} \to \tau$. It suffices to argue that $\tau \in E_p$, which implies $E_{p_n} \to E_p$.

$\tau \in E_p$ because for any $m > 0$ fixed we have

$$|DT_{p_{n_k}}^{-m} \tau_{n_k}| < (\hat{\lambda})^{-m} |\tau_{n_k}|.$$

Let $k \to \infty$ we obtain, for all $m > 0$ that

$$|DT_p^{-m} \tau| < (\hat{\lambda})^{-m} |\tau|.$$

Therefore τ spans E_p by (a).

Remarks: (1) Theorem 2 follows from combining this claim and the conclusions of Step 1.

(2) Applying the same argument to T^{-1}, we obtain a stable manifold $W^s := \{(\psi(y), y)\}$ where $\psi(y)$ is C^1.

3. Proof of Theorem 3

Step 1 Formal computation of derivatives

Let $T^{-1} : (x, y) \to (x_{-1}, y_{-1})$ be written as

$$x_{-1} = \lambda^{-1}x + f(x, y), \quad y_{-1} = \mu^{-1}y + g(x, y).$$

We have $\|f\|_{C^r}, \|g\|_{C^r} < K\varepsilon$. We now regard x_{-1}, y_{-1} as functions of x for $y = \phi(x)$. In what follows $f = f(x, \phi(x)), g = g(x, \phi(x))$. We have

$$\begin{aligned} (x_{-1})' &= \lambda^{-1} + f_x + f_y\phi'(x) \\ (y_{-1})' &= \mu^{-1}\phi'(x) + g_x + g_y\phi'(x). \end{aligned}$$

Since $(x, \phi(x))$ is T^{-1}-invariant,

$$y_{-1} = \phi(x_{-1}).$$

Differentiate on both sides, we obtain

$$(\mu^{-1} + g_y)\phi'(x) + g_x = \phi'(x_{-1})(\lambda^{-1} + f_x + f_y\phi'(x)).$$

We now compute higher derivatives. Let $\phi^{(n)}(x)$ be the n-th derivative of $\phi(x)$ at x. From the last equality we have

$$(\mu^{-1}+g_y+\phi'(x_{-1})f_y)\phi^{(n)}(x) = \phi^{(n)}(x_{-1})(\lambda^{-1}+f_x+f_y\phi'(x))+\mathcal{P}$$

where $\mathcal{P}$ is a polynomial of the partial derivatives of f, g and ϕ of order $\leq n$. It then follows that

$$\begin{aligned} \phi^{(n)}(x) &= \frac{(\lambda^{-1} + f_x + f_y \phi'(x))}{(\mu^{-1} + g_y + \phi'(x_{-1}) f_y)} \phi^{(n)}(x_{-1}) \\ &+ \frac{\mathcal{P}}{(\mu^{-1} + g_y + \phi'(x_{-1}) f_y)} \end{aligned}$$

We now use this equality **inductively**, i.e. we first let x be x_{-1} and x_{-1} be x_{-2} respectively in the above to obtain

$$\begin{aligned} \phi^{(n)}(x_{-1}) &= \frac{(\lambda^{-1} + f_x + f_y \phi'(x))}{(\mu^{-1} + g_y + \phi'(x_{-2}) f_y)} \phi^{(n)}(x_{-2}) \\ &+ \frac{\mathcal{P}}{(\mu^{-1} + g_y + \phi'(x_{-2}) f_y)}. \end{aligned}$$

Note that here we evaluate f_x, f_y and P at $(x_{-1}, \phi(x_{-1}))$. We then substitute $\phi^{(n)}(x_{-1})$ in the right hand of the previous equality buy the right hand of the last equality. We write $\phi^{(n)}(x)$ in terms of $\phi^{(n)}(x_{-2})$ and all other derivatives of f, g and $\phi(x)$ of order $\leq n-1$ at x, x_{-1} and x_{-2} (We could write this formula explicitly if so desired).

We then repeat the same process to write $\phi^{(n)}(x)$ in terms of $\phi^{(n)}(x_{-3})$ and the other derivatives of lower order at x, x_{-1}, x_{-2} and x_{-3}. Clearly there is no obstacle for us to go from x_{-3} to $x_{-4}, x_{-5}, \cdots$. We at the end obtain a way to compute $\phi^{(n)}(x)$ by using the the lower derivatives of f, g and ϕ at $x, x_{-1}, x_{-2}, \cdots$.

Step 2 Let us assume that $n < r$ and $\phi^{(i)}(x)$ exists for all $i < n$. Further assume that there exist $K_n > 0$ such that $|\phi^{(n)}(x)| < K_n$.

According to Step 1, we have an iteration process to determine $\phi^{(n+1)}(x)$ based on the lower derivatives. So for $\phi^{(n+1)}(x)$ to exist *it suffices for us to show that the formal computational process detailed in Step 1 converges*.

Recall that

$$\begin{aligned}\phi^{(n)}(x) &= \frac{(\lambda^{-1} + f_x + f_y\phi'(x))}{(\mu^{-1} + g_y + \phi'(x_{-1})f_y)}\phi^{(n)}(x_{-1}) \\ &+ \frac{\mathcal{P}}{(\mu^{-1} + g_y + \phi'(x_{-1})f_y)},\end{aligned}$$

from which we obtain

$$|\phi^{(n)}(x)| \leq a|\phi^{(n)}(x_{-1})| + b$$

where $a < 1$ is a constant and b is also a constant that is a dependent of K_n and n. Use this inequality inductively, we obtain

$$|\phi^{(n)}(x)| \leq (\sum a^i)b < (1-a)^{-1}b.$$

This estimate implies the convergence of the indicated iteration process therefor the existence of $\phi^{(n)}(x)$. We have finished the proof of Theorem 3.

4. Extensions of Theorem 1-3

A. *Maps of higher dimensions*

In our discussion we restricted ourselves to the two dimensional case. This restriction is not necessary. The argument we made can be generalized in a straight forward fashion to higher dimensional maps (using a little more of the terms and conclusions of matrix analysis in linear algebra).

Let us state formally the higher dimensional version of theorems 1-3:

For $k < n$ let $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$ and $(x, y) \in \mathbb{R}^n$. Let $T : \mathbb{R}^n$ be such that

$$\begin{aligned} x_1 &= A^u x + \alpha(x, y) \\ y_1 &= A^s y + \beta(x, y) \end{aligned}$$

where A^u is a $k \times k$, and A^s is a $(n-k) \times (n-k)$ constant matrix. Assume that

(i) The magnitude of all eigenvalues of A^u is large than 1, and the magnitude of all eigenvalues of A^s is smaller than 1;

(ii) $\|\alpha\|_{C^1}, \|\beta\|_{C^1} < \varepsilon$ and the C^r-norms of α and β are uniformly bounded on $\mathbb{R}^n$.

Theorem Let T be as in the above satisfying (i) and (ii), and assume that ε is appropriately small. The there exists a unique k-dimensional manifolds $W^u = \{(x, \phi(x))\}$ such that

(a) W^u is T-invariant, and $\phi : x \to \mathbb{R}^k$ is a C^r-mappings of uniformly bounded C^r-norms.

(b) There exist $K, \lambda > 1$, such that $\forall x, y \in W^u$,

$$d(T^n(x), T^n(y)) \geq K^{-1}\lambda^n d(x, y).$$

For all $n > 0$.

Similarly, there exists a unique $W^s = \{(\psi(y), y)\}$, $(n-k)$-dimensional, such that

(a)' W^u is T-invariant, and $\phi : x \to \mathbb{R}^k$ is a C^r-mappings of uniformly bounded C^r-norms.

(b)' There exist K, $\mu < 1$, such that $\forall x, y \in W^s$,

$$d(T^n(x), T^n(y)) \leq K\mu^n d(x, y).$$

This Theorem is a simple generalization of Theorem 1-3 above.

B. *Local stable and unstable manifold theorem for hyperbolic fixed points*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be C^r-diff and $x_0 \in \mathbb{R}^n$ be such that $T(x_0) = x_0$. Let $A = DT_{x_0}$ be the tangent map of T at x_0.

Definition: We say that x_0 is a hyperbolic fixed point if DT_{x_0} has no eigenvalues of magnitude 1.

Let us assume that T is C^r-diff and x_0 is a hyperbolic fixed point. Let us write the eigenvalues of DT_{x_0} of magnitude > 1 as $\Lambda^u = \{\lambda_1, \cdots \lambda_k\}$, and the rest as $\Lambda^s = \{\mu_1, \cdots \mu_{n-k}\}$. Let E^u be the eigen-space for Λ^u and E^s be the eigen-space for Λ^s. E^u is a k-dimensional subspace, E^s is $(n-k)$-dimensional and $E^s + E^u = \mathbb{R}^n$.

Theorem In a sufficiently small neighborhood of x_0, there exists a k-dimensional C^r-embedded disk, which we denote as $W^u(x_0)$, such that

(i) $T^{-1}(W^u(x_0)) \subset W^u(x_0)$.

(ii) There exists $K > 0$ and $\lambda > 1$ such that $\forall x, y \in W^u(x_0)$, we have

$$d(T^{-n}(x), T^{-n}(y)) < K(\lambda)^{-n} d(x, y)$$

for all $n > 0$.

(iii) $W^u(x_0)$ is tangent to E^u at x_0.

Similarly, there is a $n-k$-dimensional C^r-embedded disk through x_0, which we denote as $W^s(x_0)$, such that

(i)' $T(W^s(x_0)) \subset W^s(x_0)$.

(ii)' There exists K' and $\mu < 1$ such that $\forall x, y \in W^s(x_0)$,

$$d(T^n(x), T^n(y)) < K\mu^n d(x, y)$$

for all $n > 0$.

(iii) W^s is tangent to E^s at x_0.

An outline of proof: (a) Make coordinate change $x \to x - x_0$ to move the fixed point to $x = 0$ in $\mathbb{R}^n$.

(b) Make a linear coordinate change to transfer DT_{x_0} into

$$A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix}$$

such that the eigenvalues of A_u is Λ^u and the eigenvalues of A_s is Λ^s,

(c) In a sufficiently small neighborhood of $(x, y) = 0$ where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$, T is written as

$$\begin{aligned} x_1 &= A_u x + \alpha(x, y) \\ y_1 &= A_s y + \beta(x, y) \end{aligned}$$

(d) Note that $\alpha(x, y)$ and $\beta(x, y)$ might be large when we are away from $(x, y) = (0, 0)$. However, we can always define as new map $\widehat{T}$ such that (i) $\hat{T} = T$ in a sufficiently small neighborhood of $(0, 0)$, and (ii) C^1 norm of $\hat{T}$ are as small as we want.

(e) We now apply previous proposition to $\hat{T}$.

C. *Local stable and unstable manifold theorem for periodic orbits*

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be C^r-diff and x_0 is such that there exists $p > 1$ such that $T^p x_0 = x_0$. We say that x_0 is a periodic point of period p.

Definition: We say that x_0 is a hyperbolic periodic point if it is a hyperbolic fix point for T^p.

Let us assume that T is C^r-diff and x_0 is a hyperbolic periodic point. Let $x_i = T^i x_0$. Then we have

Theorem For every $x_i, 0 \le i < p$, there exists a small neighborhood $B(x_i)$, such that in $B(x_i)$, there exists a k-dimensional C^r-embedded disk, which we denote as $W^u(x_i)$, such that

(i) $T^{-1}(W^u(x_i)) \subset W^u(x_{i-1})$.

(ii) There exists $K, \lambda > 1$ such that $\forall x, y \in W^u(x_i)$, we have

$$d(T^{-n}(x), T^{-n}(y)) < K^{-1}(\lambda)^{-n} d(x, y)$$

for all $n > 0$.

Similarly, there is a $(n-k)$-dimensional C^r-embedded disk through x_i, which we denote as $W^s(x_i)$, such that

(i)' $T(W^s(x_i)) \subset W^s(x_{i+1})$.

(ii)' There exists K' and $\mu < 1$ such that $\forall x, y \in W^s(x_0)$,

$$d(T^n(x), T^n(y)) < K \mu^n d(x, y)$$

for all $n > 0$.

This theorem follows from applying the previous Theorem to T^p.

D. *Local stable and unstable manifold theorem for hyperbolic orbit*

We now extend the stable and unstable manifold theorem to a 'hyperbolic orbit', which is not necessarily periodic. We start with the definition of a hyperbolic orbit.

Definition Let $\{x_i\}_{i=-\infty}^{\infty}$ be a given orbit of a C^r-diff $T : \mathbb{R}^n \to \mathbb{R}^n$. We say that $\{x_i\}$ is a hyperbolic orbit if there exists E_i^u and E_i^s, subspaces of dimensions k and $n-k$, $E_i^u + E_i^s = \mathbb{R}^n$, such that

(i) $DT_{x_i}(E_i^u) = E_{i+1}^u$, $DT_{x_i}(E_i^s) = E_{i+1}^s$;

(ii) There exists $\lambda > 1$ such that $\|DT_{x_i}\tau\| > \lambda\|\tau\|$, $\forall \tau \in E_i^u$. We also have $\mu < 1$ such that $\|DT_{x_i}\tau\| < \mu\|\tau\|$ $\forall \tau \in E_i^s$.

Remark: In practice, it is hard to identify E_i^u and E_i^s explicitly for a hyperbolic orbit. We usually replace it by the so called **invariant cone** condition.

First let $E \in \mathbb{R}^n$ be a k-dimensional subspace, and $E^\perp$ be the orthogonal complement. For every $\tau \in \mathbb{R}^n$, we write

$$\tau = x + y$$

where $x \in E$ and $y \in E^\perp$. We call

$$\mathcal{C} = \{\tau = x + y, \quad \frac{\|y\|}{\|x\|} < \gamma\}$$

a **cone of size γ centered around E**.

Definition We say that $\{x_i\}$, an orbit of T, satisfies *hyperbolic cone condition*. If there exists two cone families $\{\mathcal{C}_i^u\}$, $\{\mathcal{C}_i^s\}$ of size γ around $\{E_i^u\}$, $\{E_i^s\}$ respectively for some $\gamma > 0$, such that

(i) $DT_{x_i}\mathcal{C}_i^u \subset \mathcal{C}_{i+1}^u$; $DT_{x_i}^{-1}\mathcal{C}_i^s \subset \mathcal{C}_{i+1}^s$.

(ii) There exists $\lambda > 1$ such that $|DT_{x_i}\tau| > \lambda|\tau|$ for all $\tau \in \mathcal{C}_i^u$. Similarly, we have $|DT_{x_i}^{-1}\tau| > \lambda|\tau|$ for all $\tau \in \mathcal{C}_i^s$.

Claim: If $\{x_i\}$ satisfies the hyperbolic cone condition, then it is a hyperbolic orbit.

Proof: The proof of this claim is a straight forward generalization of Theorem 2. Note that for the mapping T studies in Theorem 1-3, every orbit satisfies the hyperbolic cone condition therefore all orbits are hyperbolic. The second step of the proof of Theorems 2 consists of (a) check the cone condition, then (b) to prove that the hyperbolic cone condition implies the existence of DT-invariant stable and unstable subspaces.

Let $\{x_i\}$ be a hyperbolic orbit for a C^r-diff T.

Theorem For all x_i, $i \in \mathbb{Z}$, there exists a small neighborhood $B(x_i)$, such that in $B(x_i)$, there exists a k-dimensional C^r-embedded disk, which we denote as $W^u(x_i)$, such that

(i) $T^{-1}(W^u(x_i)) \subset W^u(x_{i-1})$.

(ii) There exists $K, \lambda > 1$ such that $\forall x, y \in W^u(x_i)$, we have

$$d(T^{-n}(x), T^{-n}(y)) < K^{-1}(\lambda)^{-n} d(x, y)$$

for all $n > 0$.

Similarly, there is a $n-k$-dimensional C^r-embedded disk through x_i, which we denote as $W^s(x_i)$, such that

(i)' $T(W^s(x_i)) \subset W^s(x_{i+1})$.

(ii)' There exists K' and $\mu < 1$ such that $\forall x, y \in W^s(x_0)$,

$$d(T^n(x), T^n(y)) < K\mu^n d(x, y)$$

for all $n > 0$.

We also have $W^u(x_i)$, $W^s(x_i)$ are tangent to E_i^u and E_i^s respectively at x_i for all i.

Remarks: (1) To prove this theorem, we need to prove a more general version of Theorem 1-3. However, the essence of the proof are the same.

(2) This theorem now implies that through every point in $\mathbb{R}^n$, there exists a local stable and unstable manifold for T studied in Theorem 1-3.

E. *Conjugate to linear mappings*

Theorem Assume that x_0 is a hyperbolic fixed point of a C^1-diff T. Then in a sufficiently small neighborhood of x_0, T is conjugate topologically to the tangent map $u \to DT_{x_0}u$.

An outline of proof: (a) We conjugate T restricted on $W^u(x_0)$ to DT_{x_0} restricted to $E^u(x_0)$ and $W^s(x_0)$ to $E^s(x_0)$. (This is not hard since we already have proved the existence and the smoothness of $W^u(x_0)$ and $W^s(x_0)$. Also, the restricted flow are basically uniform contractions).

(b) (See remark (2) in the above) There exist stable and unstable manifold for every $p \in$

$W^u(x_0)$ and $q \in W^s(x_0)$. We need to further prove that these manifolds are actually continuously depending on the base points. We also need to prove that for $\forall p \in W^u(x_0) \setminus \{x_0\}, q \in W^s(x_0) \setminus \{x_0\}$,

$$W^s(p) \cap W^u(q)$$

contains only one point. Denote this intersection as $(p + q)$. We also need to prove that $(p, q) \to (p + q)$ is continuous.

(c) Let $h_1 : E^u \to W^u(x_0)$ and $h_2 : E^s \to W^s(x_0)$ be the conjugacies claimed in (a). Then

$$h : p + q \to (h_1(p) + h_2(q))$$

is the desired conjugacy.

F. *Hartman's Theorem*

Theorem:

Proof: All we need is to conjugate two linear mappings with the same stable and unstable dimensions.

Remark The conjugating mapping h constructed in Hartman's theorem is a homeomorphism. Let us make the observation that h n general is *not* differentiable.

Let T and S be two differentiable mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(0) = 0$, $S(0) = 0$. Hartman's theorem claims that T is topologically conjugate to S if, around $z = 0$, T and S have the same stable and unstable dimensions, i.e. DT and DS have the same number of eigenvalues with magnitude < 1.

If h is differentiable, then from

$$T \circ h = h \circ S,$$

we have

$$DT \cdot Dh = Dh \cdot DS \quad \to \quad DS = (Dh)^{-1} DT(Dh).$$

This will require that DT and DS possess the *identical set of eigenvalues*. A much stronger requirement.

Most of the times we regard two systems *as the same* if there exists a conjugating function h that is a homeomorphism. Differentiable conjugates usually are regarded as **too much** to ask.