STRANGE ATTRACTORS FOR PERIODICALLY FORCED PARABOLIC EQUATIONS

KENING LU, QIUDONG WANG, AND LAI-SANG YOUNG

Abstract. We prove that in systems undergoing Hopf bifurcations, the effects of periodic forcing can be amplified by the shearing in the system to create sustained chaotic behavior. Specifically, strange attractors with SRB measures are shown to exist. The analysis is carried out for infinite dimensional systems, and the results are applicable to partial differential equations. Application of the general results to a concrete equation, namely the Brusselator, is given.

CONTENTS

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INTRODUCTION

In this paper, we show, in the context of a system undergoing a Hopf bifurcation, how the effects of periodic forcing can be amplified by the shearing in the system to create sustained chaotic behavior. This is a general dynamical phenomenon, one that can occur in phase spaces of any dimension greater than or equal to two. In particular, it can occur in systems described by ordinary as well as partial differential equations. Moreover, this phenomenon is not specific to any particular class of equations, as long as it supports Hopf bifurcations. To stress the applicability of our results to evolutionary PDEs, we have elected to communicate this work in the context of infinite dimensional systems. This setting provides us with the opportunity to demonstrate how techniques from finite dimensional theory can be leveraged in infinite dimensions, and to address the implications of SRB measures in such systems.

Description of results.

Consider, for example, a 1-parameter family of semilinear parabolic equations

$$\begin{aligned} u_t &= D\Delta u + f_\mu(u), \qquad x \in \Omega,\ u \in \mathbb{R}^m, \\ u(x,t) &= 0, \qquad x \in \partial\Omega,\ t \geq 0. \end{aligned}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, μ is a parameter, D is a diagonal matrix with positive entries, and $f_\mu : \mathbb{R}^d \to \mathbb{R}^m$ is a polynomial with $f_\mu(0) = 0$ for all μ. We assume, in a sense to be made precise, that the solution $u(x,t) \equiv 0$ is stable for $\mu < 0$, and that it loses its stability at $\mu = 0$. To this equation, we add a forcing that is close to impulsive, i.e.,

$$u_t = D\Delta u + f_\mu(u) + \rho\varphi(x)p_T(t) \tag{0.1}$$

where $\rho \in \mathbb{R}$ is a constant, $\varphi : \Omega \to \mathbb{R}^m$ is a smooth function satisfying mild conditions, $p_T = \sum_{n=-\infty}^{\infty} \varepsilon^{-1} I_{[nT, nT+\varepsilon]}$ and I_A is the indicator function on A. As a *dynamical system* on $\left(H_0^1(\Omega)\right)^m$, (0.1) is a special case of a system generated by an equation of the form

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathbf{f}_\mu(\mathbf{u}) + \rho\mathbf{\Phi}(\mathbf{u})p_T(t). \tag{0.2}$$

Of interest to us is the effect of the forcing on solutions near $\mathbf{u} = \mathbf{0}$.

This paper is about evolutionary equations of the form (0.2); the exact PDE that gives rise to it is immaterial. We assume $-\mathcal{A}$ is a sectorial operator on a Hilbert space $\mathbb{H}$, $\mathcal{D}(\mathcal{A}) \subset \mathbb{H}^\sigma \subset \mathbb{H}$ (see Sect. 1.1), and $\mathbf{f}_\mu : \mathbb{H}^\sigma \to \mathbb{H}$ is smooth with $\mathbf{f}_\mu(\mathbf{0}) = \mathbf{0}$. The phase space of our dynamical system is $\mathbb{H}^\sigma$, and our main assumption is that the unforced equation undergoes a *generic supercritical Hopf bifurcation* at $\mu = 0$. By a well known result of Andronov and Hopf, when such a bifurcation occurs, a stable periodic solution emerges from the stationary point $\mathbf{u} = \mathbf{0}$ as it becomes unstable. We will refer to this stable periodic solution as the *Hopf limit cycle*. Given a forcing function $\mathbf{\Phi} : \mathbb{H}^\sigma \to \mathbb{H}^\sigma$, the question before us is: what is the effect of the forcing on this bifurcation?

It is not hard to see that for $\mu > 0$, if $T \geq C\mu^{-1}$ where T is the period of the forcing and C is large enough, then the time-T map F_T of the forced system has an attractor roughly where the Hopf limit cycle used to be. We call it the *Hopf attractor*. By an attractor, we refer to a compact, F_T-invariant set $\Lambda \subset \mathbb{H}^\sigma$ with the property that for all u_0 in a neighborhood of Λ, $F_T^n(u_0)$ converge to Λ as $n \to \infty$. These u_0 are said to be in the *basin of attraction* of Λ. We will show that under certain conditions on the unforced system and the forcing amplitude, the Hopf attractor is chaotic. The term "chaos" in this paper refers exclusively to *dynamical complexity* or complexity in the time evolution of the system (0.2). It is not to be confused with spatial chaos.

The following is an example of what we mean by dynamical complexity. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two open sets in $\mathbb{H}^\sigma$ representing two different kinds of, say, density profiles. Let us label the profiles in $\mathcal{U}_1$ as "H" (heads) and those in $\mathcal{U}_2$ as "T" (tails). Following Smale, we say the system (0.2) has a *horseshoe* if for every infinite sequence of heads and tails (e.g. HHHTHTT .....), there exists a solution $u(t)$ of (0.2) exhibiting the sequence of profiles in the order specified as it evolves with time. In Theorem 1, we give conditions that guarantee that the Hopf attractor contains horseshoes. This establishes the *existence* of many solutions with diverse time evolutions.

In Theorem 2, we prove a stronger form of chaos, namely the presence of a *strange attractor*. This result is proved under stronger assumptions on the unforced equation. It asserts that "almost every" solution in the basin of attraction of the Hopf attractor is unstable. Instability here is in the sense of positive Lyapunov exponents. We postpone explaining the meaning of "almost every" to Section 2; suffice it to say for now that it refers to a set in our function space whose complement is very small, and that our statements apply to "typical" solutions.

The results in the last paragraph are proved by constructing on the attractor a special invariant probability measure called an *SRB measure*. This measure is the law that describes the statistics of time evolutions of almost all solutions starting from the basin of attraction. The concept of SRB measures is well known in finite dimensional dynamics. There is, however, no direct generalization of the ideas surrounding SRB measures to infinite dimensions, due in part to the absence of a notion of Lebesgue measure in function spaces. In this paper, we propose an interpretation that we hope will be of interest for dissipative parabolic PDEs beyond the phenomenon considered here.

Leaving precise definitions to Section 1, we proceed to explain what lies behind the chaotic phenomena described above. Key to it all is a geometric invariant of the unforced system we call its *twist number*. This invariant is captured in the third order term of the normal form of the Hopf bifurcation at $\mu = 0$. It acts as a form of shear to amplify the deformations of the Hopf limit cycle due to the forcing, causing the cycle to "fold" as the system relaxes (see Sect. 2.4). If the twist is weak, the system is likely to remain non-chaotic; see Theorem 3. The larger the twist, the more pronounced the folds. This geometric mechanism is responsible for the creation of horseshoes. It is

necessary but not sufficient for the formation of strange attractors. Proving the existence of SRB measures is considerably more subtle; one has to fight against the system's tendency to form sinks.

We finish by demonstrating how the results above can be applied to a specific set of equations, namely the *Brusselator* in one physical dimension:

$$\begin{aligned} u_t &= d_1 u_{xx} + a - (b+1)u + u^2 v, \\ v_t &= d_2 v_{xx} + bu - u^2 v, \end{aligned} \tag{0.3}$$

where $x \in (0,1)$. In this (simplified) model of chemical reaction invented by Lefever and Prigogine [LP], u and v represent concentrations of two chemicals, d_1 and d_2 are their diffusion constants, and a and b are constants representing the concentrations of two other chemicals. Both Dirichlet and Neumann boundary conditions are considered. It is well known that a Hopf bifurcation occurs when a and b are varied. We show that the twist number can be made as high as one wishes by suitably adjusting a and b. Using a forcing that amounts to periodically altering by external means the concentration of one of the chemicals (namely u), we give conditions that lead to horseshoes and strange attractors in the periodically forced systems.

Relation to existing literature.

There is an extensive literature on the complex behavior of systems defined by ODEs that is clearly relevant to the present work. This literature is, however, too vast a subject for us to review here. We will limit ourselves to the two topics closest to this paper in terms of mathematical content, namely *(I) attractors and chaotic behavior for PDEs*, and *(II) attractors in finite dimensional hyperbolic theory.*

With regard to the first topic, the existence of absorbing sets and attractors has been established for large classes of dissipative PDEs, likewise for upper bounds on Lyapunov exponents and Hausdorff dimension (see [Ha, T, BV], also [KR]); these methods do not, in general, give information on the structure of the attracting sets. A number of other results in the literature discuss the manifestation of chaotic behavior for various PDEs, proving the existence of complicated behavior such as horseshoes and homoclinic solutions (which suggest chaos); see e.g. [C, G, HM, W, HL, HMO, LMSW, L, SZ, Z]. Unlike the papers just cited, the present paper is not about any specific PDE; it is about a generic dynamical phenomenon that gives rise to strange attractors and large sets of unstable solutions. We are not aware of other results exhibiting this strong form of chaos that apply to PDEs.

Moving to the second topic, in finite dimensions much progress has been made in hyperbolic theory, by which we include both uniform hyperbolic theory and its probabilistic or nonuniform version; see [Sm, S, B, P, R2], to cite only a few of the major advances. An important idea that grew out of this theory is that of an SRB measure; see [S, R1, LY]. These measures provide a good qualitative understanding of the dynamical picture, but *proving* that they exist for concrete

systems is often challenging. Outside of the Axiom A and piecewise uniformly hyperbolic category, SRB measures were first constructed for the Hénon attractors [BY]; this work is based on [BC], with some ideas going back to [J]. Borrowing techniques from [BC], the authors of [WY1, WY4] developed a general theory of *rank one* attractors, i.e., attractors with a single direction of instability and strong contraction in all other directions. The attractors in this paper are rank one, and we establish their existence by appealing to [WY1, WY4]. For other examples of rank one attractors, see [MV, DRV, WY2, WY3, WO]. The present paper is closely related to [WY3], which contains a prototypical version of a similar result in two dimensions.

Finally, it remains to connect the two topics in the first paragraph. Finite dimensional techniques are applicable through the use of center manifolds. SRB measures, once constructed, give information on all solutions of the PDE starting from "typical" initial conditions in an open set of a function space. The passage from center manifold to this open set is made possible by the regularity of a strong stable foliation. Much is known about center and stable manifolds (see e.g. [HPS, CLL]), though more refined versions of some results are needed.

1. Basic Definitions and Facts

In this section, we introduce the main objects that appear in our results. Background information is provided for the benefit of readers who may not be familiar with these aspects of dynamical systems theory. All of the material here can be found in the standard literature.

1.1. Sectorial operators.

Let $\mathbb{H}$ be a Hilbert space with norm $\| \ \ \|$. An operator $\mathbf{A}$ on $\mathbb{H}$ is called a *sectorial operator* if it is closed, densely defined, and has the following properties: There exist constants $\alpha \in (0, \pi/2)$, $M \geq 1$, and $b \in \mathbb{R}$ such that

$$\Omega_{\alpha,b} := \{\lambda \in \mathbb{C} \mid \alpha \leq |\arg(\lambda - b)| \leq \pi, \ \lambda \neq b\} \subseteq \rho(\mathbf{A})$$

where $\rho(\mathbf{A})$ is the resolvent set of $\mathbf{A}$, and

$$\|(\lambda \mathrm{I} - \mathbf{A})^{-1}\| \leq \frac{M}{|\lambda - b|} \quad \text{for all } \lambda \in \Omega_{\alpha,b}.$$

Sectorial operators generate analytic semigroups $e^{-\mathbf{A}t}, t \geq 0$. See [H, K].

Let $\mathbf{A}$ be a sectorial operator on $\mathbb{H}$ with domain $\mathcal{D}(\mathbf{A})$. Associated with $\mathbf{A}$ are its *fractional power spaces* $(\mathbb{H}^\sigma, | \ \ |_\sigma)$, $0 \leq \sigma \leq 1$, defined as follows: Let $a \geq 0$ be an arbitrary (but fixed) number such that the real parts of the spectrum of $(\mathbf{A} + a\mathrm{I})$ are positive. Then

$$\mathbb{H}^\sigma = \mathcal{D}((\mathbf{A} + a\mathrm{I})^\sigma) \qquad \text{and} \qquad |u|_\sigma = \|(\mathbf{A} + a\mathrm{I})^\sigma u\|$$

where $(\mathbf{A} + a\mathrm{I})^\sigma$ is the inverse of

$$(\mathbf{A} + a\mathrm{I})^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-(\mathbf{A}+a\mathrm{I})t} dt$$

and Γ is the *gamma* function. The family $\mathbb{H}^\sigma, 0 \le \sigma \le 1$, interpolates between $\mathbb{H}$ and $\mathcal{D}(\mathbf{A})$. For $\sigma_1 > \sigma_2$, $\mathbb{H}^{\sigma_1} \subset \mathbb{H}^{\sigma_2}$ with continuous inclusion.

1.2. **Dynamical systems defined by evolutionary equations.**

Consider a nonlinear evolutionary equation of the form

$$\dot{u} = \mathbf{A}u + \mathbf{F}(u, \mu) \tag{1.1}$$

where $\mu \in (-\mu_1, \mu_1)$ is a parameter, $u \in \mathbb{H}$, and $-\mathbf{A}$ is a sectorial operator. We assume for some $0 \le \sigma < 1$ that

$$\mathbf{F} : \mathbb{H}^\sigma \times (-\mu_1, \mu_1) \to \mathbb{H}$$

is C^r for some $r \ge 1$. (Proofs of existence and uniqueness of solutions require $\sigma < 1$; see [H].) Let $S_t, t \ge 0$, denote the semi-group of time-t-maps defined by (1.1), i.e. $S_t(u_0) = u(t)$ where $t \mapsto u(t)$ is the mild solution of (1.1) satisfying $u(0) = u_0$. It is known that S_t is a C^r mapping of $(\mathbb{H}^\sigma, |\ \ |_\sigma)$ into itself (see e.g. [H]). Thus we may view $(\mathbb{H}^\sigma, |\ \ |_\sigma)$ as the phase space of a dynamical system generated by (1.1), and will write the norm on $\mathbb{H}^\sigma$ as $|\ \ |$ (omitting the subscript σ) once σ is fixed.

In applications, the choice of phase space is determined by the nonlinear term $\mathbf{F}$, which is often the Nemytski operator induced by a smooth scalar function such as $f(u)$ or $f(\bigtriangledown u, u)$. Such operators may not be smooth from $\mathbb{H}$ to itself. In the equation $\dot{u} = u_{xx} + \sin u$ with $u(0) = 0 = u(1)$, for example, $\sin u$ induces an $\mathbf{F}$ that is differentiable from $\mathbb{H}^{\frac{1}{2}} = H_0^1([0,1])$ to $L^2([0,1])$ but not from $L^2([0,1])$ to $L^2([0,1])$.

Suppose now a time-dependent forcing term $\mathbf{\Phi}(u,t)$ is added to (1.1), resulting in

$$\dot{u} = \mathbf{A}u + \mathbf{F}(u, \mu) + \mathbf{\Phi}(u, t). \tag{1.2}$$

Let S_{t_1,t_2} be the time-(t_1, t_2) map for this forced system, i.e. $S_{t_1,t_2}(\hat{u}) = u(t_2)$ where $t \mapsto u(t)$ is the solution of (1.2) with $u(t_1) = \hat{u}$. If $\mathbf{\Phi}$ is time-periodic, i.e. if there exists $T > 0$ such that $\mathbf{\Phi}(u, t + T) = \mathbf{\Phi}(u, t)$ for all $u \in \mathbb{H}^\sigma$ and $t \ge 0$, then the time evolution of (1.2) can be studied through the iteration of $S_{0,T}$. In this case we write $S_T = S_{0,T}$, and refer to it simply as the time-T map.

1.3. **Hopf bifurcations.**

Let the setting be as in Sect. 1.2, and assume additionally that $\mathbf{F}(0, \mu) = 0$ for all μ, so that $u(t) \equiv 0$ is a stationary solution. We rewrite equation (1.1) as

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u), \quad f_\mu(0) = 0, \quad \partial_u f_\mu(0) = 0, \tag{1.3}$$

by letting $\mathcal{A}_\mu = \mathbf{A} + \partial_u \mathbf{F}(0, \mu)$ and $f_\mu(u) = -\partial_u \mathbf{F}(0, \mu)u + \mathbf{F}(u, \mu)$. The mapping $\mathbf{F}$, and hence f_μ, are assumed to be C^5.

We will say the system defined by (1.3) undergoes a *generic supercritical Hopf Bifurcation* at $\mu = 0$ if conditions (H1) and (H2) below hold:

(H1) *The spectrum of* $\mathcal{A}_\mu$, $\Sigma(\mathcal{A}_\mu)$, *is decomposed into* $\Sigma^c(\mathcal{A}_\mu) \cup \Sigma^s(\mathcal{A}_\mu)$ *where* $\Sigma^c(\mathcal{A}_\mu) = \{a(\mu) \pm i\omega(\mu)\}$ *consists of a conjugating pair of complex numbers satisfying* $a(0) = 0, \omega(0) \neq 0$ *and* $a'(0) > 0$, *and* $\Sigma^s(\mathcal{A}_\mu) \subset \{\lambda \in \mathbb{C} : \ Re\ \lambda < -\beta^* < 0\}$ *for some* $\beta^* > 0$.

Corresponding to the eigenvalues $a(\mu) \pm i\omega(\mu)$, equation (1.3) has a 2-dimensional local center manifold W^c at $u = 0$ (see e.g. [CLL]). Via a change of coordinates, we may express the flow on W^c, i.e. the so-called *central flow*, in normal form as

$$\dot{z} = (a(\mu) + i\omega(\mu))z + k_1(\mu)z^2\bar{z} + k_2(\mu)z^3\bar{z}^2 + \cdots \tag{1.4}$$

where $z, k_1(\mu), k_2(\mu), \cdots \in \mathbb{C}$.

(H2) $Re(k_1(0)) < 0$.[1]

A well known result of Andronov and Hopf [A, Ho] asserts that under conditions (H1) and (H2), a stable periodic solution of diameter $\mathcal{O}(\mu^{\frac{1}{2}})$ bifurcates out of $u = 0$ as μ increases past 0. This periodic solution will be referred to as the **Hopf limit cycle** in this paper.

1.4. **A few ideas from dynamical systems.**

Let $(\mathbb{E}, |\cdot|)$ be a Hilbert space, and consider the dynamical system generated by iterating a C^1 map $F : \mathbb{E} \to \mathbb{E}$. We collect here a few ideas that are well known for finite dimensional systems and whose generalizations to infinite dimensions are straightforward.

For $u_0 \in \mathbb{E}$, we call $u_n = F^n(u_0)$, $n = 0, 1, 2, \cdots$, the **orbit** of u_0. We also refer to a bi-infinite sequence $\{u_n, n \in \mathbb{Z}\}$ as an orbit if for every n, $u_{n+1} = F(u_n)$.

A compact set $\Lambda \subset \mathbb{E}$ is called an **attractor** if (i) $F^{-1}(\Lambda) = \Lambda$; and (ii) there is an open set $\mathcal{U}$ in $\mathbb{E}$ such that $\Lambda \subset \mathcal{U}$, $F^N(\mathcal{U}) \subset \mathcal{U}$ for some $N \geq 1$, and for all $u \in \mathcal{U}$, $d(F^n(u), \Lambda) \to 0$ as $n \to \infty$. Here $d(F^n(u), \Lambda) = \min_{v\in\Lambda} |F^n(u) - v|$. The set $\mathcal{B}(\Lambda) := \{u \in \mathbb{E} : d(F^n(u), \Lambda) \to 0 \text{ as } n \to \infty\}$ is called the **basin of attraction** of Λ.

Attractors are important because they capture the asymptotic behavior of large sets of orbits. In general, Λ itself tends to be relatively small (compact and of finite Hausdorff dimension) while $\mathcal{B}(\Lambda)$, which, by definition contains an open set, is quite visible in the phase space. Notice that our attractors are not necessarily global attractors in the sense of [Ha, T].

A manifestation of dynamical complexity is the presence of a great variety of orbit types. The following is a topological version of Smale's horseshoe: Let $X := \Pi_{-\infty}^{\infty}\{1, 2, \cdots, r\}$, $r \geq 2$, be endowed with the product topology, and let $S : X \to X$ be the shift map, i.e. for $\mathbf{a} \in X$, the nth coordinate of $S(\mathbf{a})$ is the $(n+1)$st coordinate of $\mathbf{a}$. We say F has a **topological horseshoe** if there is an embedding $\Psi : X \to \mathbb{E}$ such that $\Psi \circ S = F \circ \Psi$. Less formally, this implies there are r pairwise disjoint open sets $U_1, \cdots U_r$ in $\mathbb{E}$ (representing r different types of "profiles") such that for

[1]Transforming $z \mapsto cz$, k_1 is changed to $|c|^2 k_1$. This is the only ambiguity for k_1. In particular, the sign of $Re(k_1(0))$ and $\arg(k_1(0))$ are uniquely determined.

every $\mathbf{a} = \{a_n\}_{n=-\infty}^{\infty} \in X$, there is an orbit $\{u_n\}_{n=-\infty}^{\infty}$ such that $u_n \in U_{a_n}$ for all n. In particular, if $\mathbf{a}$ is periodic, i.e. if for some p, $a_{n+p} = a_n$ for all n, then $u_{n+p} = u_n$ for all n. Since X contains periodic sequences of all periods, it follows that a map with a horseshoe has periodic orbits of all periods.

Local instability is often expressed in terms of Lyapunov exponents. For $u_0 \in \mathbb{E}$, we say F has a **positive Lyapunov exponent** at u_0 if

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\partial F^n(u_0)\| \ > \ 0.$$

Here $\partial F(u_0)$ is the Fréchet derivative of F at u_0. For a periodic orbit, having a positive Lyapunov exponent is synonymous with linear instability. This interpretation extends to arbitrary orbits.

Finally, we remark that if F is the time-T map of equation (1.2) with $\mathbf{\Phi}(u, t+T) = \mathbf{\Phi}(u, t)$, then the ideas above carry over to the solutions of (1.2) in the obvious way. For example, if F has a topological horseshoe, then (1.2) has infinitely many periodic solutions, at least one for each period nT, $n \in \mathbb{Z}^+$. Positive Lyapunov exponents translate into corresponding exponential growths for the variational equations associated with (1.2).

1.5. **SRB measures.**

There is no direct generalization of some of the ideas surrounding SRB measures to function spaces. We review first the situation in *finite* dimensions.

SRB measures were introduced by Sinai, Ruelle and Bowen in the context of uniformly hyperbolic attractors [B, R1, S]. The idea has since been made quite general: Let F be a C^2 diffeomorphism of a compact n-dimensional manifold, and let ν be an F-invariant Borel probability measure. We say ν is an *SRB measure* if (i) F has a positive Lyapunov exponent ν-a.e. and (ii) the conditional measures of ν on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these manifolds.

From (i), we see that SRB measures are associated with chaotic systems. We now state a result that explains why these measures are important: A point $x \in M$ is said to be *generic* with respect to a measure ν if for every continuous function $\varphi : M \to \mathbb{R}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(F^i(x)) \ \to \ \int \varphi d\nu \ .$$

A theorem says that if ν is an ergodic SRB measure with no zero Lyapunov exponents, then its generic points have positive Lebesgue measure. That is to say, *if one equates positive Lebesgue measure sets with observable events, then SRB measures describe patterns of behavior that can be observed.* In the case of a dissipative system, such as on an attractor, all invariant measures are singular with respect to Lebesgue, so this is not a statement of the Birkhoff Ergodic Theorem, and the existence of SRB measures cannot be taken for granted. For more information, see [Y2] and the references therein.

While the formal definition of SRB measures can be transported *verbatim* to infinite dimensional systems (when the relevant unstable manifolds are defined), it is less clear what the analog of the observability result above would be. In Section 2, we propose a natural idea for dissipative parabolic PDEs that we hope will be useful beyond the present paper.

2. Statement of Theorems

Two sets of results are stated in this section. The first describes the effect of periodic forcing on the dynamics of general evolutionary equations undergoing Hopf bifurcations. These results are stated in Sects. 2.1 and 2.2. A concrete application is given in Sect. 2.3.

2.1. **Setting and standing hypotheses.**

Let $\mathbb{H}$ be a Hilbert space.

Unforced system. We consider an equation of the form

$$\dot{u} = \mathbf{A}u + \mathbf{F}(u,\mu), \qquad u \in \mathbb{H}, \quad \mu \in (-\mu_1, \mu_1) \subset \mathbb{R}, \tag{2.1}$$

where $-\mathbf{A}$ is a sectorial operator and $\mathbf{F} : \mathbb{H}^\sigma \times (-\mu_1, \mu_1) \to \mathbb{H}$ is C^5 for some $\sigma \in [0,1)$ (definitions are given in Sects. 1.1 and 1.2). We assume $\mathbf{F}(0,\mu) = 0$ for all μ, and rewrite (2.1) to obtain

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u), \quad f_\mu(0) = 0, \quad \partial_u f_\mu(0) = 0. \tag{2.2}$$

The system defined by (2.2) is assumed to undergo a generic supercritical Hopf bifurcation at $u = 0$; more precisely, it is assumed to satisfy Conditions (H1) and (H2) in Sect. 1.3.

Periodically forced system. To the right side of (2.2) we now add a forcing term of the form $\rho\Phi(u)p_{T,\iota}(t)$, resulting in a new equation

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u) + \rho\Phi(u)p_{T,\iota}(t). \tag{2.3}$$

Here $\rho \geq 0$ and $0 < \iota << 1 << T$ are constants, $\Phi : \mathbb{H}^\sigma \to \mathbb{H}^\sigma$ is assumed to be C^5 with uniformly bounded C^5-norms, and

$$p_{T,\iota}(t) = \sum_{n=-\infty}^{\infty} p_\iota(t - nT) \quad \text{with} \quad p_\iota(t) = \begin{cases} \iota^{-1} & 0 \leq t < \iota, \\ 0 & \text{elsewhere.} \end{cases}$$

That is to say, the forcing has period T, and is close to an impulse followed by a long relaxation. Let E^s_μ and E^c_μ be the $\mathcal{A}_\mu$-invariant subspaces associated with $\Sigma^s(\mathcal{A}_\mu)$ and $\Sigma^c(\mathcal{A}_\mu)$ respectively (see Sect. 1.3). Other than its regularity, our only real assumption on Φ is $\Phi(0) \notin E^s_0$. Without loss of generality, let us set

(H3) $|P^c_0(\Phi(0))| = 1$ where P^c_0 is the projection of $\mathbb{H}^\sigma$ onto E^c_0.

This completes the description of the setting for our general results. Throughout this paper, the following 3 items, namely (i) the unforced equation, (ii) the form of the forcing term, and (iii)

the forcing function Φ, are regarded as given and fixed, while the 4 numbers, μ, ρ, T and ι, are treated as parameters that can be varied. We view (2.3) as defining a time-periodic dynamical system on $\mathbb{H}^\sigma$, and study its time evolution by iterating its time-T map $F_{\mu,\rho,T,\iota}$. When we wish to emphasize the dependence on a parameter, such as T, we will write F_T instead of $F_{\mu,\rho,T,\iota}$. Observe that $F_T = G_{T-\iota} \circ S_{0,\iota}$ where S_{t_1,t_2} is the time-(t_1, t_2) map of (2.3) and G_t is the time-t map of (2.2). Since both $S_{0,\iota}$ and $G_{T-\iota}$ are C^5, so is F_T.

Two geometric invariants. Crucial geometric information is contained in the following two numbers; they both appear in the statements of all of our results.

(i) The first is the **twist number** τ of the unforced system (2.2), defined to be

$$\tau := \frac{Im(k_1(0))}{-Re(k_1(0))}$$

where $k_1(0)$ is the coefficient of the third order term in the normal form of the central flow associated with (2.2) at $\mu = 0$ (see Sect. 1.3).

(ii) The second is the **kick ratio** γ, defined formally as

$$\gamma := \rho(d\mu^{\frac{1}{2}})^{-1}.$$

Here $d\mu^{\frac{1}{2}}$ is approximately the radius of the Hopf limit cycle in the unforced system in the direction of the forcing; the formal definition of d is given in Sect. 6.1. The geometry for $\gamma < 1$ and $\gamma > 1$ are a little different; see Sect. 2.4. With γ bounded away from 1, our analysis applies to both cases, and the proofs are similar (though parts have to be expressed differently). For definiteness, we will state and prove our results under the assumption

(H4) $\gamma < \frac{1}{2}$.

2.2. General results for periodically forced Hopf bifurcations.

Let $D^c(\varepsilon) = \{v \in E_0^c : |v| < \varepsilon\}$, $D^s(\varepsilon) = \{w \in E_0^s : |w| < \varepsilon\}$, and

$$\mathcal{N}(\varepsilon) = \{u = v + w : v \in D^c(\varepsilon), w \in D^s(\varepsilon)\}.$$

A priori bounds. A set of *a priori* bounds that limit our considerations to certain regions of $\mathbb{H}^\sigma$ and to certain parameter ranges will be assumed throughout. We explain the nature of these bounds without giving explicit values. Let $\varepsilon_0, \mu_0, \iota_0, \rho_0$ and M_0 be numbers depending only on $a'(0)$ ($a(\mu)$ is as in (H1)), the unforced system (2.2) at $\mu = 0$ and the forcing function Φ. The dynamics of interest will take place inside $\mathcal{N}(\varepsilon_0)$ for parameters

$$0 < \mu < \mu_0, \quad 0 < \iota < \iota_0, \quad 0 \le \rho < \rho_0 \quad \text{and} \quad T > M_0\mu^{-1}.$$

These constants are chosen with the following considerations:

– $\mathcal{N}(\varepsilon_0)$ is chosen so that structures associated with the linear part of the system at 0, namely center manifolds and strong stable foliations, are defined on $\mathcal{N}(\varepsilon_0)$;

- μ_0 is small enough that the Hopf limit cycles and attractors are well inside $\mathcal{N}(\varepsilon_0)$;
- ρ is primarily restricted through (H4);
- ι is small enough so that the net effect of the impulsive force is close to $u \mapsto u + \rho\Phi(0)$;
- $T > M_0\mu^{-1}$ is large enough so that the effect of the weak contraction of the Hopf limit cycle is played out sufficiently between kicks.

The numbers $\varepsilon_0, \mu_0, \iota_0, \rho_0$ and M_0 are determined by what is needed in our proofs and are revised a finite number of times ($\varepsilon_0, \mu_0, \iota_0, \rho_0$ downward and K_0 upward) as we go along.

Let $\mathcal{B}(\cdot)$ denote the basin of attraction of an attractor.

Proposition 2.1. (Trapping region and attractor) *Assuming M_0 is sufficiently large, there is, for each $\mu \in (0, \mu_0)$, an open set $\mathcal{U} = \mathcal{U}(\mu) \subset \mathcal{N}(\varepsilon_0)$ with the properties that*

(i) *for the* unforced *equation: $\Omega \subset \mathcal{U} \subset \mathcal{B}(\Omega)$ where Ω is the Hopf limit cycle;*

(ii) *for the* forced *equation: For all $T > M_0\mu^{-1}$, there exists $N \in \mathbb{Z}^+$ such that $F_T^N(\mathcal{U}) \subset \mathcal{U}$. It follows that*

$$\Lambda \ := \ \cap_{n\geq 0} F_T^n(\mathcal{U})$$

is an attractor of F_T with $\mathcal{U} \subset \mathcal{B}(\Lambda)$.

As we will see in Sect. 3.3, $\mathcal{U}$ can be taken to be a fairly large subset of $\mathcal{N}(\varepsilon_0)$. Under the conditions of Proposition 2.1, Λ is what becomes of the Hopf limit cycle when the forcing is applied. We call it the **Hopf attractor**.

Theorem 1. (Existence of horseshoes) *Assume*

$$|\tau| > 20\gamma^{-1}.$$

Then there exists $T_1(\mu) > M_0\mu^{-1}$ such that for all $T > T_1$, F_T has a topological horseshoe in Λ. This implies in particular that the system (2.3) has infinitely many periodic solutions with arbitrarily large periods.

While Theorem 1 asserts the *existence* of many solutions with different time evolutions, it does not assert the dynamical complexity for solutions starting from "most" or "almost all" initial conditions. In finite dimensions, such a result is often deduced from the existence of an SRB measure together with the absolute continuity of stable foliations. The following proposition is key in bridging the gap between finite and infinite dimensions.

Proposition 2.2. (Strong stable foliation) *F_T has a codimension-2 stable foliation $\mathcal{W}^{ss}$ defined on an open set containing $\mathcal{N}(\varepsilon_0)$. This foliation is Lipschitz continuous, and its leaves are roughly parallel to E_0^s.*

A precise formulation of Proposition 2.2 is given in Sect. 3.2. Since $\mathcal{W}^{ss}$ is a foliation that is Lipschitz continuous, there is a well defined Lebesgue measure class transversal to its leaves. We say a property holds **almost everywhere transversal to** $\mathcal{W}^{ss}$ if for every embedded 2-dimensional surface S transversal to the leaves of $\mathcal{W}^{ss}$, it holds almost everywhere with respect to the Riemannian measure on S. Where no ambiguity arises, such as when referring to open subsets of $\mathcal{N}(\varepsilon_0)$, we will abbreviate the terminology above as "a.e."

The definition of an SRB measure is as in finite dimensions; it is given in Sect. 1.5.

Theorem 2. (Strange attractors with SRB measures) *There exists a constant $L_0 > 1$ (not depending on specifics of the unforced system or the forcing) such that if*

$$|\tau| > L_0\gamma^{-1},$$

then there is a positive measure set $\Delta \subset (T_2, \infty)$, $T_2(\mu) >> M_0\mu^{-1}$, with the property that for every $T \in \Delta$, the time evolution of (2.3) has the following description:

(i) *for a.e. $u_0 \in \mathcal{U}$, the solution $u(t)$ of (2.3) with $u(0) = u_0$ is unstable in the sense of having a positive Lyapunov exponent;*

(ii) *F_T has an ergodic SRB measure ν with respect to which a.e. $u_0 \in \mathcal{U}$ is generic.*

Every subinterval of (T_2, ∞) of length greater than the period of the Hopf limit cycle meets Δ in a set of positive Lebesgue measure.

In the rest of this paper, we will use the phrase "F_T has a **strange attractor**" as shorthand for (i) and (ii) in Theorem 2. Theorems 1 and 2 are the main results of this paper. To provide contrast, we finish with a result in the opposite direction.

Theorem 3. (Non-chaotic dynamics) *Assume*

$$|\tau| < (100\gamma)^{-1}.$$

Then there exists $T_0(\mu)$ such that for all $T > T_0$, the attractor Λ of F_T is diffeomorphic to a circle, and for every u_0 in $\mathcal{U}$, there exists $v_0 \in \Lambda$ such that $|F_T^n(u_0) - F_T^n(v_0)| \to 0$ as $n \to \infty$.

2.3. **An application: The Brusselator.**

The *Brusselator*, as described by the equations below, is a simplified model of an autocatalytic chemical reaction with diffusion [LP]:

$$\begin{aligned} u_t &= d_1\Delta u + a - (b+1)u + u^2 v, \\ v_t &= d_2\Delta v + bu - u^2 v. \end{aligned} \tag{2.4}$$

We consider this model in one physical dimension, i.e. $u = u(x,t)$ and $v = v(x,t)$ for $x \in [0,1]$, and let $\Delta = \partial_{xx}$. Here a and b are constants representing concentrations of certain initial substances, u and v are variables representing concentrations of two intermediates, and $d_1, d_2 > 0$ are their

respective diffusion coefficients. The term u^2v represents the autocatalytic step in the reaction. Two other chemicals are produced; they play no role in this reaction and are not represented.

One sees immediately that $(u(t), v(t)) \equiv (a, ba^{-1})$ is a stationary solution. It is well known that a Hopf bifurcation occurs as the parameters a and b are varied.

We claim that our results in Sect. 2.2 apply when the system (2.4) is periodically forced. Letting $U = u - a$, $V = v - ba^{-1}$, and $\mathbf{u} = (U, V)$, we write (2.4) as an evolutionary equation

$$\dot{\mathbf{u}} = \mathcal{A}_{a,b}\mathbf{u} + f_{a,b}(\mathbf{u}), \tag{2.5}$$

where

$$\mathcal{A}_{a,b} = \begin{pmatrix} d_1\frac{\partial^2}{\partial x^2} + (b-1) & a^2 \\ -b & \theta d_1\frac{\partial^2}{\partial x^2} - a^2 \end{pmatrix}, \qquad \theta = \frac{d_2}{d_1}, \quad d_1 \neq 0;$$

and

$$f_{a,b}(\mathbf{u}) = \begin{pmatrix} UV^2 + ba^{-1}U^2 + 2aUV \\ -UV^2 - ba^{-1}U^2 - 2aUV \end{pmatrix}.$$

Two types of boundary conditions are considered, with forcings chosen to respect them:

Neumann boundary conditions. We consider

$$\begin{aligned} u_t &= d_1\Delta u + a - (b+1)u + u^2v + \rho(1+\cos\pi x)p_{T,\iota}(t), \\ v_t &= d_2\Delta v + bu - u^2v\ ; \\ \partial_x u(0,t) = \partial_x u(1,t) &= 0, \qquad \partial_x v(0,t) = \partial_x v(1,t) = 0\ . \end{aligned} \tag{2.6}$$

In (U, V)-coordinates, $-\mathcal{A}_{a,b}$ is a sectorial operator with compact resolvent in $\mathbb{H} = L^2([0,1]) \times L^2([0,1])$, $\mathcal{D}(\mathcal{A}_{a,b}) = (H^2([0,1]) \cap \{\partial_x = 0 \text{ at } x = 0,1\})^2$, and $f_{a,b} : \mathbb{H}^\sigma \to \mathbb{H}$ is smooth for $\sigma \in (\frac{1}{2}, 1)$. Using a standard cut-off procedure to modify $f_{a,b}$ outside of a small neighborhood of 0, we may assume it is globally Lipschitz continuous.

Recall that the twist number τ of the unforced equation is an important factor in determining the type of dynamics that ensue when the system is forced.

Proposition 2.3. (Hopf bifurcation) *For each $d_1 > 0$ and $\theta < 1$, the following hold for the unforced equation, i.e. system (2.6) without the forcing term:*

(i) *for each fixed $a > 10$, as b is increased, a supercritical Hopf bifurcation satisfying Conditions (H1) and (H2) occurs at $b = a^2 + 1$;*

(ii) *$|\tau|$, which depends only on a, tends to ∞ as $a \to \infty$.*

Using Proposition 2.3, we verify that there are regions of parameters for which Theorems 1–3 apply. In particular, we have

Theorem 4. (Effects of forcing) *Let $d_1 > 0$, $\theta < 1$ be fixed. Then for a sufficiently large, there is an open set of ρ, ι and b (depending on d_1, θ and a), $b \approx a^2 + 1$, for which*

(i) *the time-T map F_T has a horseshoe for all large T;*

(ii) *F_T has a strange attractor for a positive measure set of large T.*

Dirichelet boundary conditions: We consider

$$\begin{aligned} u_t &= d_1\Delta u + a - (b+1)u + u^2 v + \rho \sin \pi x \; p_{T,\iota}(t), \\ v_t &= d_2 \Delta v + bu - u^2 v \; ; \\ u(0,t) = u(1,t) &= a, \qquad v(0,t) = v(1,t) = ba^{-1}. \end{aligned} \tag{2.7}$$

As before, we write (2.7) in (U, V)-coordinates. Here, $-\mathcal{A}_{a,b}$ is a sectorial operator with compact resolvent in $\mathbb{H} = L^2([0,1]) \times L^2([0,1])$ and $\mathcal{D}(\mathcal{A}_{a,b}) = (H^2([0,1]) \cap H_0^1([0,1]))^2$. Moreover, $f_{a,b} : \mathbb{H}^{\frac{1}{2}} = H_0^1([0,1]) \times H_0^1([0,1]) \to \mathbb{H}$ is smooth and globally Lipschitz continuous with the usual cutoff.

The computation of τ in this case is considerably more involved than for Neumann boundary conditions. We limit ourselves in this paper to the special case $d_1 = \pi^{-2}$.

Proposition 2.4. (Hopf bifurcation) *Let $d_1 = \pi^{-2}$. Then for each θ with $0 < \theta << 1$, the following hold for the unforced equation, i.e. system (2.7) without the forcing term:*

(i) *there exists $A_0 = A_0(\theta) > 1$ such that for each $a \in (1, A_0)$, as b is increased, a supercritical Hopf bifurcation satisfying Conditions (H1) and (H2) occurs at $b = 2 + a^2 + \theta$;*

(ii) *$|\tau|$, which depends on θ and a but not on b, tends to ∞ as $a \to A_0$.*

Using Proposition 2.4, we verify that there are regions of parameters for which Theorems 1–3 apply. In particular, we have

Theorem 5. (Effects of forcing) *Let $d_1 = \pi^{-2}$, $0 < \theta << 1$. Then there are open sets of a, b, ρ and ι, with $a \approx A_0$ and $b \approx a^2 + 2 + \theta$, for which*

(i) *the time-T map F_T has a horseshoe for all large T;*

(ii) *F_T has a strange attractor for a positive measure set of large T.*

2.4. **Discussion of results.**

A. *Geometric mechanism in production of chaos.*

To make transparent the underlying geometry, we consider the following simplified situation: Assume (i) the phase space is 2-dimensional, (ii) the Hopf limit cycle for parameter μ is the circle centered at 0 of radius $\sqrt{\mu}$, and (iii) the forcing is impulsive, and is a rigid translation of the entire phase plane by a distance ρ. Assuming the limit cycle is positively oriented, the implication of $\tau > 0$ is that the angular velocities of orbits revolving around 0 increase with distance from 0.

In Figure 1, the circle shown in grey is the Hopf limit cycle; its images at the various times are shown in black. This figure shows how folds are created: The kick sends different points on the Hopf limit cycle to locations with unequal distances from 0. In the presence of a strong enough twist, the different angular velocities of these points after the kick then cause the circle to become deformed. During the period of relaxation, the "tail" is further elongated as the folded image of

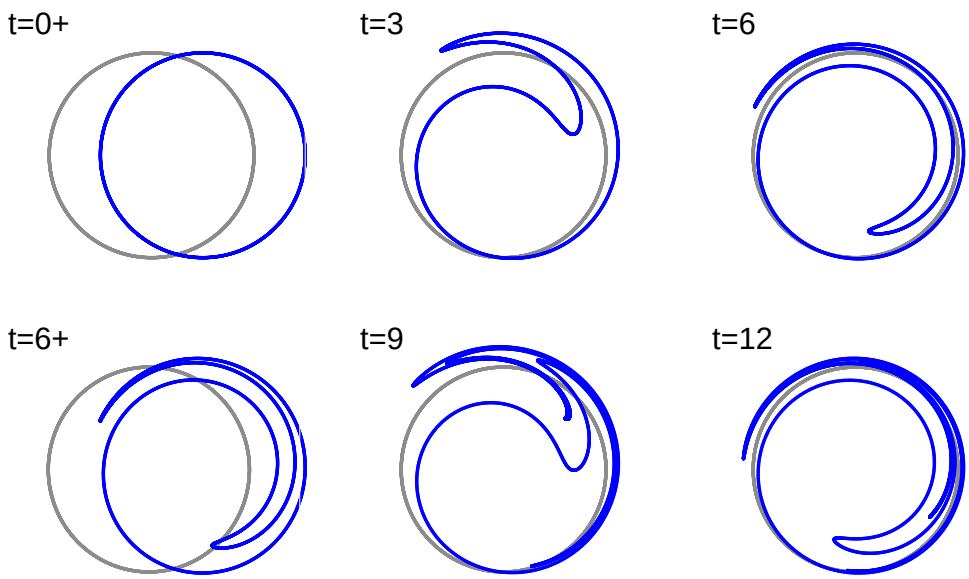

FIGURE 1. Hopf attractor with $\gamma = \rho\mu^{-\frac{1}{2}} = \frac{1}{2}$, $\tau = 10$ and $T = 6$

the kicked cycle returns to a neighborhood of the unkicked cycle. The process is repeated. Folds created this way are shown to give rise to horseshoes. They also provide the geometry behind the formation of strange attractors; see Part B.

It is not hard to see that if the kick is weak, or if the twist is not strong enough, then the fold seen in Figure 1 will not materialize, resulting in non-chaotic dynamics.

In the present simplified setting, $\gamma > 1$ corresponds to a kick that sends the Hopf cycle to an image disjoint from itself; see Figure 2. The mechanism for producing chaos in this case is as before, but the geometry is a little different: upon relaxation the cycle wraps around itself from the outside, i.e. the action of the time-T map on the Hopf cycle when projected back to itself has degree 0, while the corresponding action for $\gamma < 1$ has degree 1.

B. *Horseshoes* versus *strange attractors.*

We stress again that in this paper, chaos refers to *complexity in time evolutions*, not spatial chaos. Two different types of chaotic behaviors are asserted in Theorems 1 and 2:

The presence of horseshoes implies the existence of a Cantor set of initial conditions with complicated time evolutions. By continuous dependence on initial conditions, a solution starting near this set will appear chaotic initially. As time goes on, it may or may not remain chaotic; for example, it may tend to a stable equilibrium. The phenomenon in which the latter occurs is known as *transient chaos.*

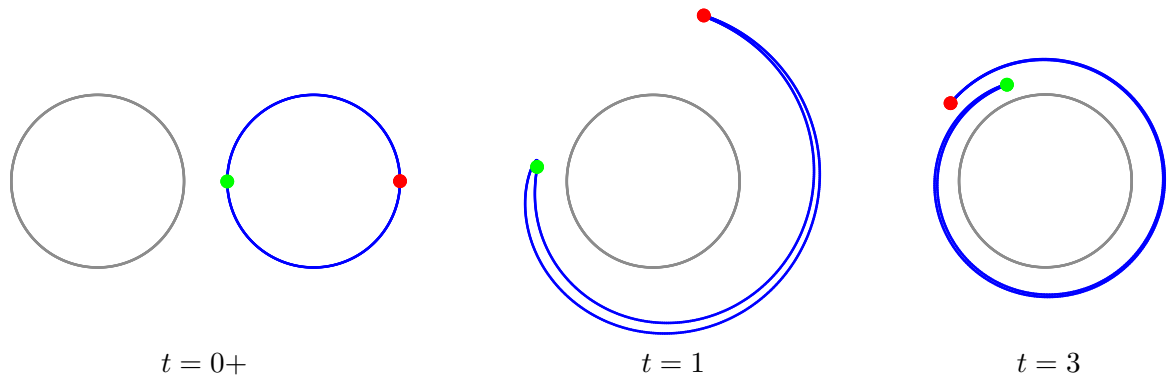

FIGURE 2. Hopf attractor with kick ratio $\gamma > 1$

Strange attractors represent a considerably stronger form of chaos, a kind that is both *sustained in time* and *ubiquitous in the phase space*, meaning it is seen in solutions starting from a large set of initial conditions and the chaotic behavior last indefinitely. The formation of strange attractors requires a subtle balance not required for horseshoes. As a result, proving the existence of SRB measures also presents a much greater challenge.

Notation used in the rest of this paper:

(1) The letter K is used as a generic constant that may depend on the unforced equation and the forcing function Φ but is independent of the parameters μ, ρ, ι or T. Its value may vary from line to line. When there is a need to distinguish between two constants in the same statement, or if a specific value is used in more than one place, $K_1, K_2, \cdots$ may be used instead of K.

(2) Let $\mathbb{H}$ and $\mathbb{H}'$ be Hilbert spaces, and let $g_\mu : U \to \mathbb{H}'$ be a family of C^r maps on $U \subset \mathbb{H}$ parametrized by μ. The notation "$g_\mu = \mathcal{O}_r(\mu^k)$" is our shorthand for $\|g_\mu\|_{C^r} < K\mu^k$ for some $K > 0$. By $\|g_\mu\|_{C^r}$, we mean the C^r-norm of the mapping g_μ with respect to phase variables; smoothness with respect to the parameter μ is not assumed.

(3) *Partial derivatives are denoted by ∂, not subscripts.* For example, if f is a mapping from (x, y)-space into (u, v)-space, then f_u and f_v are the u- and v-components of f, while $\partial_x f_u$ is the partial derivative of f_u with respect to x.

3. Invariant Manifolds

A dynamical picture associated with the linear parts of F_T near 0 is presented in this section. The main objects are a center manifold and a strong stable foliation. These structures are parameter-dependent but quite robust. The more delicate dynamics occur on the center manifold; they are treated in later sections.

3.1. **Standardizing the linear part of the unforced equation.**

First we introduce μ-dependent changes of coordinates that enable us to work in a space $\mathbb{E}$ with the following properties:

(i) there is a single splitting $\mathbb{E} = E^c \oplus E^s$ left invariant by $\mathcal{A}_\mu$ for all μ;
(ii) $\mathcal{A}_\mu$ restricted to E^c is in canonical form.

Let $\mathbb{E} := \mathbb{R}^2 \oplus E_0^s$ be endowed with the following inner product: The metric on $\mathbb{R}^2$ is Euclidean, E_0^s has the inner product inherited from $\mathbb{H}^\sigma$, and the two subspaces are orthogonal. For each μ, we define a linear transformation $L_\mu : \mathbb{H}^\sigma \to \mathbb{E}$ as follows: Let $e(\mu) = v_1(\mu) + iv_2(\mu)$ be a continuous family of eigenfunctions of $\mathcal{A}_\mu$ of norm 1 associated with the eigenvalue $a(\mu) - i\omega(\mu)$. We denote the coordinates on $\mathbb{E}$ by (x, y, w) where $(x, y) \in \mathbb{R}^2$ and $w \in E_0^s$, and define $L_\mu(u) = (x, y, w)$ where

$$P_\mu^c u = xv_1(\mu) + yv_2(\mu) \qquad \text{and} \qquad w = \left(P_\mu^s|_{E_0^s}\right)^{-1} P_\mu^s u.$$

Clearly, $L_\mu(E_\mu^c) = \mathbb{R}^2$ and $L_\mu(E_\mu^s) = E_0^s$. Note also that both L_μ and L_μ^{-1} are continuous and their operator norms are uniformly bounded. This is because we may assume $\|P_\mu^s - P_0^s\| \le \eta < \sqrt{2} - 1$ for all $\mu \in (-\mu_0, \mu_0)$; consequently, $P_\mu^s|_{E_0^s} : E_0^s \to E_\mu^s$ is an isomorphism with $\|P_\mu^s\| \le 1 + \eta$ and $\| \left(P_\mu^s|_{E_0^s}\right)^{-1} \| \le \frac{1}{1-\eta}$. Also, since P_μ^c and P_μ^s depend smoothly on μ, we have

$$\|P_\mu^c - P_0^c\| = \mathcal{O}(\mu), \qquad \|P_\mu^s - P_0^s\| = \mathcal{O}(\mu).$$

Furthermore, $\left(P_\mu^s|_{E_0^s}\right)^{-1} P_\mu^s$ is continuous in μ.

We write $\mathbb{E} = E^c \oplus E^s$ where $E^c = \mathbb{R}^2$ and $E^s = E_0^s$, and use $|\cdot|$ to denote the norm on $\mathbb{E}$. Let $D^c(\varepsilon) = \{u \in E^c : |u| < \varepsilon\}$, $D^s(\varepsilon) = \{u \in E^s : |u| < \varepsilon\}$ and $\mathcal{N}(\varepsilon) = D^c(\varepsilon) \times D^s(\varepsilon)$. (Under L_μ^{-1}, the image of $\mathcal{N}(\varepsilon)$ as defined here is not the set in $\mathbb{H}^\sigma$ with the same name but it contains a neighborhood in $\mathbb{H}^\sigma$ of the same kind for a smaller ε.)

In like manner, we introduce a space $\check{\mathbb{E}} := \mathbb{R}^2 \oplus \check{E}_0^s$ where $E_\mu^c \oplus \check{E}_\mu^s$ is the splitting in $\mathbb{H}$ corresponding to Σ_μ^c and Σ_μ^s. We put on $\check{\mathbb{E}}$ a weak norm $\|\cdot\|$ defined as follows: on $\mathbb{R}^2$ it is Euclidean, on $\check{E}_0^s$ it is the norm $\|\cdot\|$ inherited from $\mathbb{H}$, and the two subspaces are orthogonal. A linear change of coordinates $\check{L}_\mu : \mathbb{H} \to \check{\mathbb{E}}$ analogous to L_μ is defined. In particular, $\check{L}_\mu(u) = (x, y, \check{w})$ where $\check{w} = (\check{P}_\mu^s|_{\check{E}_0^s})^{-1} \check{P}_\mu^s u$ and $\check{P}_\mu^s : \mathbb{H} \to \check{E}_\mu^s$ is the projection. Since $E_\mu^s = \check{E}_\mu^s \cap \mathbb{H}^\sigma$, we may view $\mathbb{E}$ as a subspace of $\check{\mathbb{E}}$ with $L_\mu = \check{L}_\mu|_{\mathbb{H}^\sigma}$.

From this point on, we view the phase space of our dynamical system as $\mathbb{E}$ (with further coordinate changes to follow). Results on $\mathbb{E}$ are easily translated back to $\mathbb{H}^\sigma$. For simplicity, we will use the same symbols as before to denote corresponding objects. For example, we will write $\mathcal{A}_\mu$ and f_μ instead of $\check{L}_\mu \mathcal{A}_\mu (L_\mu|\mathcal{D}(\mathbf{A}))^{-1}$ and $\check{L}_\mu f_\mu L_\mu^{-1}$, call the time-$T$ map F_T instead of $L_\mu F_T L_\mu^{-1}$, and so on. Let $\mathcal{A}_\mu^c$ and $\mathcal{A}_\mu^s$ denote the restriction of $\mathcal{A}_\mu$ on E^c and E^s respectively. Then on $\mathbb{E}$, equation

(2.3) has the form

$$\begin{aligned} \dot{x} &= ax - \omega y + f_x + \rho \Phi_x p_{T,\iota}(t), \\ \dot{y} &= \omega x + ay + f_y + \rho \Phi_y p_{T,\iota}(t), \\ \dot{w} &= \mathcal{A}^s w + f_{\check{w}} + \rho \Phi_w p_{T,\iota}(t), \end{aligned} \tag{3.1}$$

where $f = (f_x, f_y, f_{\check{w}})$ and $\Phi = (\Phi_x, \Phi_y, \Phi_w)$ are the component functions of f and Φ. All quantities depend on μ, although this dependence has been suppressed.

3.2. **Invariant manifolds.**

The results in this subsection are consequences of the following exponential dichotomy: For $v \in E^c$, we have

$$|e^{\mathcal{A}^c t} v| \le 2e^{a(\mu)|t|}|v| \quad \text{for } t \in \mathbb{R}, \tag{3.2}$$

and there exists β_0 with $0 < \beta_0 < \beta^*$ (see Sect. 1.3) such that for all $w \in E^s$,

$$|e^{A^s t} w| < Ce^{-\beta_0 t}|w| \quad \text{and} \quad |e^{A^s t} w| < C\frac{1}{t^\sigma} e^{-\beta_0 t}\|w\| \quad \text{for } t > 0. \tag{3.3}$$

We may assume $a(\mu) < a_0 << \beta_0$ for all $\mu < \mu_0$.

In Propositions 3.1–3.3 below, the assertions are to be understood to be preceded by the statement "There exist $\varepsilon_0, \rho_0, \mu_0, \iota_0$ and M_0 such that the following hold". (See Sect. 2.1, paragraph on *a priori* bounds.) Because the structures discussed in this section are robust, the conditions imposed on these constants are relatively mild compared to the ones imposed in connection with the more delicate dynamics in later sections.

Proposition 3.1. (Center manifolds) *For each $T > M_0\mu^{-1}$, there is a C^1 mapping $h^c : D^c(\varepsilon_0) \to E^s$ with the property that the 2-dimensional manifold*

$$\mathcal{W}^c = \{v + h^c(v) \mid v \in D^c(\varepsilon_0)\}$$

is F_T-invariant, i.e. $F_T(\mathcal{W}^c) \subset \mathcal{W}^c$, and $\|h^c\|_{C^1} < K(\rho + \varepsilon_0)$ for some $K > 0$.

Let us denote h^c above as h^c_ρ to indicate the dependence on ρ. The center manifold of the unforced equation is therefore represented by $\{v + h^c_0(v), v \in D^c(\varepsilon_0)\}$. Our next proposition compares h^c_ρ and h^c_0.

Proposition 3.2. $\|h^c_\rho - h^c_0\|_{C^1} < e^{-\frac{\beta_0}{2}T}$.

We now give precise definitions of the objects surrounding Proposition 2.2 in Sect. 2.2. Let $G : \mathcal{N}(\varepsilon_0) \times D^s(\varepsilon_0) \to E^c$ be a C^1 map, and for each $u \in \mathcal{N}(\varepsilon_0)$, let Λ_u be the graph of the map $w \mapsto G(u, w)$. We assume that Λ_u passes through u, i.e. if $u = u^c + u^s$ where $u^c \in E^c$ and $u^s \in E^s$, then $G(u, u^s) = u^c$.

(i) We say that $\{\Lambda_u, u \in \mathcal{N}(\varepsilon_0)\}$ defines a *codimension two foliation* if the following hold: for all $u, \bar{u} \in \mathcal{N}(\varepsilon_0)$, either $\Lambda_u = \Lambda_{\bar{u}}$ or $\Lambda_u \cap \Lambda_{\bar{u}} = \emptyset$;

(ii) the co-dimensional two foliation $\{\Lambda_u, u \in \mathcal{N}(\varepsilon_0)\}$ is F_T*-invariant* if, for all $u \in \mathcal{N}(\varepsilon_0) \cap F_T^{-1}(\mathcal{N}(\varepsilon_0))$, $F_T(\Lambda_u) \subset \Lambda_{F_T(u)}$;

(iii) Λ_u is a *codimension 2 strong stable manifold* through u, written $\mathcal{W}_u^{ss} = \Lambda_u$, if the following hold: for every $u \in \mathcal{N}(\varepsilon_0)$ and $\bar{u} \in \Lambda_u$,

$$|F_T^n(\bar{u}) - F_T^n(u)| < e^{-a_0 n}|\bar{u} - u| \qquad \text{for all } n > 0$$

where a_0 is as in the beginning of Sect. 3.1.

Notice that (i) implies that $\mathcal{W}_u^{ss}$ is uniquely characterized by the property in (iii). If all the Λ_u are strong stable manifolds, then we say G *defines a codimension 2 strong stable foliation* $\mathcal{W}^{ss}$ on $\mathcal{N}(\varepsilon_0)$ for F_T.

We need one more definition. For $g : D^c(2\varepsilon_0) \to E^s(\varepsilon_0)$, let $\Sigma_g = \{u + g(u) : u \in D^c(2\varepsilon_0)\}$ be the graph of g. We say $\mathcal{W}^{ss}$ is *Lipschitz continuous* if there exists $K > 0$ such that the following holds for every C^1 map g as above with $\|\partial g\| \le 1$: for each $u \in \mathcal{N}(\varepsilon_0)$, $\mathcal{W}_u^{ss}$ meets Σ_g at a unique point η, and the mapping $u \mapsto \eta(u)$ satisfies $|\eta(u_1) - \eta(u_2)| < K|u_1 - u_2|$.

The following is a more precise formulation of Proposition 2.2.

Proposition 3.3. (Strong stable foliations) *There is a codimension 2 strong stable foliation $\mathcal{W}^{ss}$ on $\mathcal{N}(\varepsilon_0)$ for F_T. Moreover, if $G : \mathcal{N}(\varepsilon_0) \times D^s(\varepsilon_0) \to E^c$ is the mapping defining $\mathcal{W}^{ss}$, then*

(a) for all $u \in \mathcal{N}(\varepsilon_0)$, $\|\partial_w G(u, w)\|_{L(E^s,E^c)} < K(\rho + \varepsilon_0)$; and

(b) the strong stable foliation $\mathcal{W}^{ss}$ is Lipschitz continuous.

To prove the assertions in Theorem 2 regarding properties that hold "almost everywhere transversal to $\mathcal{W}^{ss}$", we will prove the following: For every C^1 mapping $g : D^c(2\varepsilon_0) \to E^s(\varepsilon_0)$ with $\|\partial g\| \le 1$, if Σ_g is the graph of g and m_g is the Riemannian measure on Σ_g, then the properties in question holds m_g-a.e. It can be shown that this is equivalent to the formulation in Sect. 2.1, technical details are left to the reader. One may, if one so chooses, take the above as the definition of "a.e."

The arguments used in the proofs of Propositions 3.1–3.3 are quite standard, though we know of no references that treat exactly the present setting. (Most proofs of center manifolds assume *a priori* knowledge of a fixed point through which the manifold passes; we do not have that.) We will omit the proof of Proposition 3.1, referring the reader to e.g. [CLL]. Sketches of Propositions 3.2 and 3.3 are given in Appendix A.

3.3. **Trapping regions.**

The following are the two goals of this subsection: (a) We will identify a trapping region $\mathcal{U} \subset \mathbb{E}$ which occupies much of $\mathcal{N}(\varepsilon_0)$, and a small annular region $A \subset \mathcal{W}^c$ containing the Hopf attractor. (b) Using invariant manifolds, we will show how to pass from $\mathcal{U}$ to A; in particular, we will explain how "*a.e.*"-point in $\mathcal{U}$ is related to a point in A typical with respect to Lebesgue measure. Once this passage is carried out, one can focus on the dynamics on A alone. At this time, however, we

do not have all the technical estimates needed for a complete proof. We will proceed nonetheless assuming two estimates (that are quite reasonable based on what we already know), and provide technical verifications later on.

Continuing to work in the space $\mathbb{E}$, we denote phase points by $u = (v, w), v \in E^c, w \in E^s$. For each μ, ρ, ι and T, we let $\mathcal{W}^c = \mathcal{W}^c(\mu, \rho, \iota, T)$ denote the center manifold of the system with these parameters, often omitting the parameters when they are implicitly understood. Strong stable manifolds are denoted similarly. The special notation W^c (respectively W^{ss}) is reserved for the *unforced* equation.

In the coordinates of $\mathbb{E}$, it is easy to see that for each small $\mu > 0$, the limit cycle Ω of the unforced equation is approximately a round circle of radius $\tilde{d}\mu^{\frac{1}{2}}$ on W^c with $\tilde{d} := (\frac{a'(0)}{-Re(k_1(0))})^{\frac{1}{2}}$ (see Sect. 1.3). We will show in later sections that the Hopf attractor for the forced system lies in an $\mathcal{O}(\sqrt{\mu})$-neighborhood of this limit cycle. Let

$$A(\delta, \mathcal{W}^c) \;=\; \{|v - \tilde{d}\mu^{\frac{1}{2}}| < \delta\} \cap \mathcal{W}^c \;.$$

Recall that $F_T = G_{T-\iota} \circ S_{0,\iota}$ where S_{t_1,t_2} is the time-(t_1, t_2)-map of (2.3) and G_t is the time-t-map of the unforced equation. We will refer to $\kappa := S_{0,\iota}$ as the kick map. Recall also that the sizes of admissible kicks are regulated by (H4).

Provisional assumptions:

(1) For the unforced equation, $\Omega \subset A(\delta, W^c)$ for some $\delta = \mathcal{O}(\mu)$.

(2) For all $u = (v, w) \in \mathcal{N}(\varepsilon_0)$, if $\kappa(u) = (v', w')$, then $|v - v'| < \frac{51}{100}\tilde{d}\mu^{\frac{1}{2}}$.

(1) is verified in Sect. 4.3; (2) is verified in Sect. 6.1. We now proceed assuming (1) and (2).

Lemma 3.1. *For $u_0 = (v_0, w_0) \in \mathcal{W}^c = \mathcal{W}^c(\mu, \rho, \iota, T)$, let $F_T(u_0) = (v_0', w_0')$. The following hold for all admissible μ, ρ, ι, and $T > M_0\mu^{-1}$, M_0 sufficiently large:*

(i) If $3\tilde{d}\mu^{\frac{1}{2}} \le |v_0| \le \varepsilon_0$, then $|v_0'| < \frac{1}{2}|v|$.

(ii) If $\frac{3}{5}\tilde{d}\mu^{\frac{1}{2}} < |v_0| < 3\tilde{d}\mu^{\frac{1}{2}}$, then $F_T(u_0) \in A(\frac{1}{100}\tilde{d}\mu^{\frac{1}{2}}, \mathcal{W}^c)$.

Proof: We give the proof of (ii); the proof of (i) is similar and uses only cruder estimates.

Consider $u_0 \in \mathcal{W}^c$. We will estimate the location of $F_T(u_0)$ via the following sequence of points: $u_1 = \kappa(u_0)$, i.e., u_1 is the image of u_0 under the kick map; u_2 is the unique point in $W^c \cap W^{ss}_{u_1}$, $u_3 = G_{T-\iota}(u_2)$, and u_4 is the unique point in $W^{ss}_{u_3} \cap \mathcal{W}^c$. By the quasi-invariance of the W^{ss}-foliation in the unforced dynamics, $F_T(u_0) \in W^{ss}_{u_3}$. By the invariance of $\mathcal{W}^c$ under F_T, $F_T(u_0) \in \mathcal{W}^c$. Thus $u_4 = F_T(u_0)$.

Let us write $u_i = (v_i, w_i)$. By Assumption (2), $|v_1 - v_0| < \frac{51}{100}\tilde{d}\mu^{\frac{1}{2}}$, and $|w_1 - w_0| < K\sqrt{\mu}$. By Proposition 3.2, the C^1 distance between $\mathcal{W}^c$ and W^c is $< e^{-\frac{1}{2}\beta_0 T} << \frac{1}{T} = \mathcal{O}(\mu)$. These facts together with the Lipschitz bound on the functions defining individual W^{ss}-leaves (Proposition 3.3(a)) imply that $|v_2 - v_1| < K'\varepsilon_0\sqrt{\mu}$. Given that $\frac{3}{5}\tilde{d}\mu^{\frac{1}{2}} < |v_0| < 3\tilde{d}\mu^{\frac{1}{2}}$, we have $\frac{1}{20}\tilde{d}\mu^{\frac{1}{2}} < |v_2| <$

$4\tilde{d}\mu^{\frac{1}{2}}$ assuming ε_0 is small enough. From Assumption (1), we can arrange to have $\frac{199}{200}\tilde{d}\mu^{\frac{1}{2}} < |v_3| < \frac{201}{200}\tilde{d}\mu^{\frac{1}{2}}$ by taking M_0 large enough and $T \geq M_0\mu^{-1}$. This is because the rate of attraction to Ω in $\{\frac{1}{20}\tilde{d}\mu^{\frac{1}{2}} < |v| < 4\tilde{d}\mu^{\frac{1}{2}}\} \cap W^c$ is of order μ, a known fact about Hopf limit cycles (see also (4.16) in Sect. 4.3). In the notation above, then, we have shown that $u_3 \in A(\frac{1}{200}\tilde{d}\mu^{\frac{1}{2}}, W^c)$. Passing back to $\mathcal{W}^c$ by sliding along W^{ss}, we obtain the desired conclusion for u_4. □

Next we name an open set $\mathcal{U}(\mu)$ that is a common trapping region for $F_{\mu,\rho,\iota,T}$ for all admissible ρ, ι and T. The advantage of having a common trapping region is that one can imagine how, as ρ increases, the limit cycle is "broken" and turned into more complicated attractors. There are many choices of $\mathcal{U}$ with the properties in Proposition 2.1. For example, we may take

$$\mathcal{U}(\mu) \;=\; \mathcal{N}(\varepsilon_0) \setminus \{(v,w) : |v| \leq \frac{2}{3}\tilde{d}\mu^{\frac{1}{2}} + K\varepsilon_0|w|\}$$

where $K\varepsilon_0$ is a Lipschitz constant for the functions defining individual $\mathcal{W}^{ss}$-leaves.

Proof of Proposition 2.1: Let $\mathcal{U}$ be as above.

(i) For the unforced equation, since Ω has radius $\approx \tilde{d}\mu^{\frac{1}{2}}$ and W^c is tangent to E^c at $(0,0)$ so that $|w| = \mathcal{O}(\mu)$ on Ω, we have $\Omega \subset \mathcal{U}$. For ε_0 small enough, all points in $\mathcal{N}(\varepsilon_0)$ not in W_0^{ss} are contained in $\mathcal{B}(\Omega)$, the basin of attraction of Ω. The $K\varepsilon_0|w|$ term in the definition of $\mathcal{U}$ ensures that $W_0^{ss} \cap \mathcal{U} = \emptyset$. It follows that $\mathcal{U} \subset \mathcal{B}(\Omega)$.

(ii) For the forced equation, let $\Gamma = \cup\{\mathcal{W}_u^{ss} : u = (v,w) \in \mathcal{W}^c, \frac{3}{5}\tilde{d}\mu^{\frac{1}{2}} < |v| < \varepsilon_0\}$. By Lemma 3.1, $F_T(\Gamma \cap \mathcal{W}^c) \subset \{\frac{99}{100}\tilde{d}\mu^{\frac{1}{2}} < |v| < \varepsilon_0\} \cap \mathcal{W}^c$, which one checks easily is well within $\mathcal{U}$. Since $\mathcal{W}^{ss}$-leaves are contracted, it follows that $F_T^N(\Gamma) \subset \mathcal{U}$ for some N. Now an argument similar to that in the last paragraph gives $\mathcal{W}_u^{ss} \cap \mathcal{U} = \emptyset$ for every $u = (v,w) \in \mathcal{W}^c$ with $|v| \leq \frac{3}{5}\tilde{d}\mu^{\frac{1}{2}}$. Thus $\Gamma \supset \mathcal{U}$. This together with $F_T^N(\Gamma) \subset \mathcal{U}$ implies $F_T^N(\mathcal{U}) \subset \mathcal{U}$. □

The following proposition, which for us is the culmination of the ideas in Sects. 3.2 and 3.3, contains information we will need later on. Let $A = A(\frac{1}{100}\tilde{d}\mu^{\frac{1}{2}}, \mathcal{W}^c)$.

Proposition 3.4. *The following hold for all admissible μ, ρ, ι and $T > M_0\mu^{-1}$:*

(a) For every $u \in \mathcal{U}$, there exists $n \in \mathbb{Z}^+$ and $\hat{u} \in A$ such that $F_T^n(u) \in \mathcal{W}^{ss}(\hat{u})$.

(b) Let $\hat{A} \subset A$ be any Borel measurable subset having full Lebesgue measure. Then for a.e. $u \in \mathcal{U}$ transversal to $\mathcal{W}^{ss}$, one may take $\hat{u} \in \hat{A}$.

Proof: By Proposition 3.3(a), the $\mathcal{W}^{ss}$-leaf through u meets $\mathcal{W}^c$ in a point $\tilde{u}$, and by Lemma 3.1, $F_T^n(\tilde{u}) \in A$ for some n. This together with the quasi-invariance of $\mathcal{W}^{ss}$-leaves proves (a). The assertion in (b) uses in addition the Lipshitz property of the $\mathcal{W}^{ss}$-foliation (Proposition 3.3(b)), together with the fact that restricted to $\mathcal{W}^c$, F_T is a smooth map and therefore preserves the Lebesgue measure class. □

4. Canonical Form of Equations Around the Limit Cycle

This section contains a sequence of μ-dependent coordinate changes that will, in the end, render the Hopf limit cycle (of the unforced system) as a unit-size circle in a transformed space. Accompanying changes of the relevant equations are computed for future use.

4.1. **Normal form of Hopf bifurcation.**

Our starting point is equation (3.1) in Sect. 3.1. Setting $\rho = 0$, we obtain

$$\dot{x} = ax - \omega y + f_x, \quad \dot{y} = \omega x + ay + f_y, \quad \dot{w} = \mathcal{A}^s w + f_{\check{w}}.$$

Let W^c be the center manifold of the unforced equation on $D^c(\varepsilon_0)$. Then W^c is the graph of a C^5 function $W \equiv h_0^c : D^c(\varepsilon) \to E^s$ with $W(0,0) = 0$, $\partial_x W(0,0) = \partial_y W(0,0) = 0$, and $(x, y, W) \in W^c$ satisfies

$$\mathcal{A}^s W + f_{\check{w}} = (ax - \omega y + f_x)\partial_x W + (\omega x + ay + f_y)\partial_y W.$$

Coordinate change #1: "flattening" W^c. Let

$$\bar{w} = w - W(x, y). \tag{4.1}$$

(Each time a coordinate change is made on $\mathbb{E}$, there is an accompanying one on $\check{\mathbb{E}}$ which we leave implicit.) In $(x, y, \bar{w})$-coordinates, the center manifold is $\bar{w} = 0$, and equation (3.1) is transformed to

$$\begin{aligned} \dot{x} &= ax - \omega y + f_x(x, y, W) + \Delta f_x + \rho \Phi_x p_{T,\iota}(t), \\ \dot{y} &= \omega x + ay + f_y(x, y, W) + \Delta f_y + \rho \Phi_y p_{T,\iota}(t), \\ \dot{\bar{w}} &= \mathcal{A}^s \bar{w} + \Big[\Delta f_{\check{w}} - \Delta f_x \partial_x W - \Delta f_y \partial_y W\Big] + \rho \Big[\Phi_w - \Phi_x \partial_x W - \Phi_y \partial_y W\Big] p_{T,\iota}(t), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \Delta f_x(x, y, \bar{w}) &= f_x(x, y, \bar{w} + W(x, y)) - f_x(x, y, W(x, y)), \\ \Delta f_y(x, y, \bar{w}) &= f_y(x, y, \bar{w} + W(x, y)) - f_y(x, y, W(x, y)), \\ \Delta f_{\check{w}}(x, y, \bar{w}) &= f_{\check{w}}(x, y, \bar{w} + W(x, y)) - f_{\check{w}}(x, y, W(x, y)). \end{aligned}$$

These terms have very simple forms, which we make precise in the next lemma. Let $L(X, Y)$ be the space of linear maps from X to Y.

Lemma 4.1. *There are C^4 maps $B_x, B_y : (x, y, \bar{w}) \mapsto L(E^s, \mathbb{R})$ and $B_{\check{w}} : (x, y, \bar{w}) \mapsto L(E^s, \check{E}^s)$ with $B_x(0,0,0) = B_y(0,0,0) = B_{\check{w}}(0,0,0) = 0$ such that*

$$\Delta f_x(x, y, \bar{w}) = B_x(x, y, \bar{w})\bar{w}\ , \qquad \Delta f_y(x, y, \bar{w}) = B_y(x, y, \bar{w})\bar{w}\ ,$$

$$(\Delta f_{\check{w}} - \Delta f_x \partial_x W - \Delta f_y \partial_y W)(x, y, \bar{w}) \ = \ B_{\check{w}}(x, y, \bar{w})\bar{w}\ .$$

Proof: We write, for example,

$$\Delta f_x(x,y,\bar{w}) = \int_0^1 \partial_w f_x(x,y,\tau\bar{w}+W(x,y))\bar{w}\ d\tau = B_x(x,y,\bar{w})\bar{w},$$

and note that $B_x(0,0,0)=0$ because $f_x(x,y,\bar{w}) = O(|(x,y,\bar{w})|^2)$. Other terms are treated similarly. □

Coordinate change #2: normal form. For the unforced system, the flow on the center manifold $\bar{w}=0$ is defined by

$$\dot{x} = ax - \omega y + f_x(x,y,W(x,y)), \quad \dot{y} = \omega x + ay + f_y(x,y,W(x,y)). \tag{4.3}$$

Standard normal form theory [GH] assures us that there exist $h_x(x,y)$ and $h_y(x,y)$ satisfying $h_x(0,0)=h_y(0,0)=0$ and $\partial_x h_x(0,0)=\partial_y h_x(0,0)=\partial_x h_y(0,0)=\partial_y h_y(0,0)=0$ such that the change of variables

$$\bar{x} = x + h_x(x,y), \quad \bar{y} = y + h_y(x,y) \tag{4.4}$$

transforms equation (4.3) into

$$\begin{aligned} \dot{\bar{x}} &= a\bar{x} - \omega\bar{y} - \alpha(\bar{x}^2+\bar{y}^2)\bar{x} - \beta(\bar{x}^2+\bar{y}^2)\bar{y} + \bar{f}_x(\bar{x},\bar{y}), \\ \dot{\bar{y}} &= \omega\bar{x} + a\bar{y} + \beta(\bar{x}^2+\bar{y}^2)\bar{x} - \alpha(\bar{x}^2+\bar{y}^2)\bar{y} + \bar{f}_y(\bar{x},\bar{y}), \end{aligned} \tag{4.5}$$

where $\bar{f}_x, \bar{f}_y$ are of order ≥ 5 in $\bar{x}$ and $\bar{y}$. Comparing (4.5) for the unforced equation and (1.4) in Sect. 1.3, we see that the twist number $\tau = \frac{\beta}{\alpha}$.

Using $(\bar{x},\bar{y},\bar{w})$ as new variables, equation (4.2) is transformed into

$$\begin{aligned} \dot{\bar{x}} &= a\bar{x} - \omega\bar{y} - \alpha(\bar{x}^2+\bar{y}^2)\bar{x} - \beta(\bar{x}^2+\bar{y}^2)\bar{y} + \bar{f}_x + \bar{B}_x\bar{w} + \rho\hat{\Phi}_x p_{T,\iota}(t), \\ \dot{\bar{y}} &= \omega x + ay + \beta(\bar{x}^2+\bar{y}^2)\bar{x} - \alpha(\bar{x}^2+\bar{y}^2)\bar{y} + \bar{f}_y + \bar{B}_y\bar{w} + \rho\hat{\Phi}_y p_{T,\iota}(t), \\ \dot{\bar{w}} &= \mathcal{A}^s\bar{w} + \bar{B}_{\check{w}}\bar{w} + \rho\hat{\Phi}_w p_{T,\iota}(t), \end{aligned} \tag{4.6}$$

where $\bar{f}_x, \bar{f}_y$ are as above, $\bar{B}_x, \bar{B}_y$ and $\bar{B}_{\check{w}}$ are operator-valued functions satisfying $\bar{B}_x(0,0,0) = \bar{B}_y(0,0,0) = \bar{B}_{\check{w}}(0,0,0) = 0$; and

$$\begin{aligned} \hat{\Phi}_x &= (1+\partial_x h_x(x,y))\Phi_x + \partial_y h_x(x,y)\Phi_y, \\ \hat{\Phi}_y &= \partial_x h_y(x,y)\Phi_x + (1+\partial_y h_y(x,y))\Phi_y, \\ \hat{\Phi}_w &= \Phi_w - \partial_x W(x,y)\Phi_x - \partial_y W(x,y)\Phi_y. \end{aligned} \tag{4.7}$$

Coordinate change #3: polar coordinates. Let $\bar{x} = r\cos\theta,\ \bar{y} = r\sin\theta$. In $(r,\theta,\bar{w})$-coordinates with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and $r>0$, equation (4.6) is transformed to

$$\begin{aligned} \dot{r} &= (a-\alpha r^2)r + r^5 f_r(r,\theta) + B_r(r,\theta,\bar{w})\bar{w} + \rho\hat{\Phi}_r(r,\theta,\bar{w})p_{T,\iota}(t), \\ \dot{\theta} &= \omega + \beta r^2 + r^4 f_\theta(r,\theta) + \frac{1}{r}B_\theta(r,\theta,\bar{w})\bar{w} + \rho\frac{1}{r}\hat{\Phi}_\theta(r,\theta,\bar{w})p_{T,\iota}(t), \\ \dot{\bar{w}} &= \mathcal{A}^s\bar{w} + B_{\check{w}}(r,\theta,\bar{w})\bar{w} + \rho\hat{\Phi}_w(r,\theta,\bar{w})p_{T,\iota}(t), \end{aligned} \tag{4.8}$$

where $B_r, B_\theta, B_{\bar{w}}$ are operator-valued functions of order at least one in r and $\bar{w}$, f_r and f_θ are smooth functions of r and θ, and

$$\hat{\Phi}_r = \cos\theta\hat{\Phi}_x + \sin\theta\hat{\Phi}_y, \qquad \hat{\Phi}_\theta = \cos\theta\hat{\Phi}_y - \sin\theta\hat{\Phi}_x. \tag{4.9}$$

4.2. **Blow-ups.**

Coordinate change #4: blow-up by $\sim \mu^{-\frac{1}{2}}$. It follows from (4.8) that for $\mu > 0$, a stable periodic solution of radius $\approx \sqrt{\frac{a(\mu)}{\alpha}} \approx$ constant$\cdot\sqrt{\mu}$ emerges on the center manifold $\bar{w} = 0$. To normalize the radius of this limit cycle, we introduce new variables

$$\eta = \sqrt{\frac{\alpha}{a(\mu)}}\, r, \quad \text{and} \quad \mathbf{w} = \sqrt{\frac{\alpha}{a(\mu)}}\, \bar{w}. \tag{4.10}$$

We now rewrite equation (4.8) in these new variables paying attention to the orders of magnitudes of the various terms in relation to μ:

(4.11)

$$\begin{aligned}
\dot{\eta} &= a(\mu)(1-\eta^2)\eta + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_r(\eta,\theta) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_r\Big(\sqrt{\frac{a(\mu)}{\alpha}}\eta, \theta, \sqrt{\frac{a(\mu)}{\alpha}}\mathbf{w}\Big)p_{T,\iota}(t),\\
\dot{\theta} &= \omega + \frac{\beta}{\alpha}a(\mu)\eta^2 + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_\theta(\eta,\theta) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\frac{1}{\eta}\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_\theta\Big(\sqrt{\frac{a(\mu)}{\alpha}}\eta, \theta, \sqrt{\frac{a(\mu)}{\alpha}}\mathbf{w}\Big)p_{T,\iota}(t),\\
\dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_w\Big(\sqrt{\frac{a(\mu)}{\alpha}}\eta, \theta, \sqrt{\frac{a(\mu)}{\alpha}}\mathbf{w}\Big)p_{T,\iota}(t),
\end{aligned}$$

where

$$\hat{f}_r(\eta,\theta) = \eta^5 f_r(\sqrt{\frac{a(\mu)}{\alpha}}\eta, \theta), \qquad \hat{f}_\theta(\eta,\theta) = \eta^4 f_\theta(\sqrt{\frac{a(\mu)}{\alpha}}\eta, \theta),$$

and "$\mathcal{O}_4(\sqrt{\mu})\mathbf{w}$" above means $B\mathbf{w}$ where as an operator-valued function, B satisfies $\|B\|_{C^4(\hat{V})} < K\sqrt{\mu}$ on a bounded domain $\hat{V}$ of the form

$$\hat{V} := \{(\eta,\theta,\mathbf{w}) \ : \ K_1^{-1} \le \eta \le K_1, \quad |\mathbf{w}| \le K_1\}.$$

See Sect. 4.4 for a suitable choice of K_1. We note that in order for a term in equation (4.11) to be $\mathcal{O}_4(\sqrt{\mu})$, all it takes is for it to have a factor $\sqrt{\mu}$ in front after the rescaling (4.10) is performed on the previous equation (and for the rest to have bounded C^4 norm). Derivatives with respect to parameters including μ are not relevant here.

4.3. **Final adjustments.**

We finish with 3 adjustments that involve only the 2-dimensional part.

(i) After coordinate change #4, the Hopf limit cycle is close to $\eta = 1$. First we let $\bar{\eta} = \eta - 1$. Equation (4.11) is transformed to

$$\begin{aligned}
\dot{\bar{\eta}} &= -a(\mu)(\bar{\eta}+1)(\bar{\eta}+2)\bar{\eta} + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_r(\bar{\eta}+1,\theta) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_r p_{T,\iota}(t),\\
\dot{\theta} &= \hat{\omega} + \frac{\beta}{\alpha}a(\mu)(\bar{\eta}+2)\bar{\eta} + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_\theta(\bar{\eta}+1,\theta) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\frac{1}{\bar{\eta}+1}\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_\theta p_{T,\iota}(t),\\
\dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\hat{\Phi}_w p_{T,\iota}(t),
\end{aligned} \tag{4.12}$$

where $\hat{\omega} = \omega + \frac{\beta}{\alpha}a(\mu)$.

Lemma 4.2. *Let $\bar{\eta} = \phi(\theta)$ be the Hopf limit cycle. Then $\|\phi\|_{C^4} < K\mu$.*

Proof: Note that the Hopf limit cycle is on the center manifold $\mathbf{w} = 0$, and $\phi(\theta)$ satisfies

$$\begin{aligned}
&- a(\mu)(\phi(\theta)+1)(\phi(\theta)+2)\phi(\theta) + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_r(\phi(\theta)+1,\theta)\\
&= \phi'(\theta)\left[\hat{\omega} + \frac{\beta}{\alpha}a(\mu)(\phi(\theta)+2)\phi(\theta) + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_\theta(\phi(\theta)+1,\theta)\right].
\end{aligned}$$

First, $|\phi'(\theta)| < K\mu$ for all θ since the left side of the equality above is $\mathcal{O}(\mu)$. Since $\phi(\theta)$ is periodic, there exists θ_0 such that $\phi'(\theta_0) = 0$. For this value of θ, we obtain $|\phi(\theta_0)| < K\mu$. Putting these facts together, we conclude that $|\phi(\theta)| < K\mu$ for all θ. Estimates on higher derivatives are obtained by differentiating both sides of the equality above. □

Note that provisional assumption (1) in Sect. 3.3 follows from Lemma 4.2.

(ii) We set

$$\xi = \bar{\eta} - \phi(\theta), \tag{4.13}$$

so that the Hopf limit cycle is given by $\xi = 0$. In the new variables $(\xi, \theta, \mathbf{w})$, equation (4.12) is given by

$$\begin{aligned}
\dot{\xi} &= -a(\mu)(2+3\xi+\xi^2+\mathcal{O}_4(\mu))\xi + \left(\frac{a(\mu)}{\alpha}\right)^2\left(\Delta\hat{f}_r - \phi'(\theta)\Delta\hat{f}_\theta\right) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w}\\
&\quad + \rho\sqrt{\frac{\alpha}{a(\mu)}}(\tilde{\Phi}_\xi(\xi,\theta,\mathbf{w}) + \mathcal{O}_4(\mu))p_{T,\iota}(t),\\
\dot{\theta} &= \hat{\omega} + \frac{\beta}{\alpha}a(\mu)(\xi+\phi(\theta)+2)(\xi+\phi(\theta)) + \left(\frac{a(\mu)}{\alpha}\right)^2 \hat{f}_\theta(\xi+\phi(\theta)+1,\theta) + \mathcal{O}_4(\sqrt{\mu})\mathbf{w}\\
&\quad + \rho\frac{1}{\xi+\phi(\theta)+1}\sqrt{\frac{\alpha}{a(\mu)}}\tilde{\Phi}_\theta(\xi,\theta,\mathbf{w})p_{T,\iota}(t),\\
\dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\tilde{\Phi}_{\mathbf{w}}(\xi,\theta,\mathbf{w})p_{T,\iota}(t),
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\Delta\hat{f}_r &= \hat{f}_r(\xi+\phi(\theta)+1,\theta)-\hat{f}_r(\phi(\theta)+1,\theta),\\ \Delta\hat{f}_\theta &= \hat{f}_\theta(\xi+\phi(\theta)+1,\theta)-\hat{f}_\theta(\phi(\theta)+1,\theta)\ ,\end{aligned}$$

and $\tilde{\Phi}_\xi$ in these coordinates is given by

$$\tilde{\Phi}_\xi(\xi,\theta,\mathbf{w}) \ = \ \hat{\Phi}_r\Big(\sqrt{\frac{a}{\alpha}}(\xi+1+\phi(\theta)),\ \theta,\ \sqrt{\frac{a}{\alpha}}\mathbf{w}\Big)\ , \tag{4.15}$$

with $\tilde{\Phi}_\theta$ and $\tilde{\Phi}_{\mathbf{w}}$ defined similarly. Using Lemma 4.2 and reasoning as in Lemma 4.1, we obtain $\Delta\hat{f}_r, \Delta\hat{f}_\theta = \mathcal{O}_4(\mu)\xi$. Hence, we may write equation (4.14) as

$$\begin{aligned}\dot{\xi} &= -a(\mu)(2+3\xi+\xi^2+\mathcal{O}_4(\mu))\xi+\mathcal{O}_4(\sqrt{\mu})\mathbf{w}+\rho\sqrt{\frac{\alpha}{a(\mu)}}(\tilde{\Phi}_\xi+\mathcal{O}_4(\mu))p_{T,\iota}(t),\\ \dot{\theta} &= \hat{\omega}+\mu^2 g(\theta)+\frac{\beta}{\alpha}a(\mu)(\xi+2+\mathcal{O}_4(\mu))\xi+\mathcal{O}_4(\sqrt{\mu})\mathbf{w}+\rho\left(\frac{1}{\xi+1}+\mathcal{O}_4(\mu)\right)\sqrt{\frac{\alpha}{a(\mu)}}\tilde{\Phi}_\theta p_{T,\iota}(t),\\ \dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w}+\mathcal{O}_4(\sqrt{\mu})\mathbf{w}+\rho\sqrt{\frac{\alpha}{a(\mu)}}\tilde{\Phi}_{\mathbf{w}}p_{T,\iota}(t)\end{aligned} \tag{4.16}$$

where

$$\mu^2 g(\theta) = \left(\frac{a(\mu)}{\alpha}\right)^2\hat{f}_\theta(\phi(\theta)+1,\theta)+\frac{\beta}{\alpha}a(\mu)(\phi(\theta)+2)\phi(\theta). \tag{4.17}$$

(iii) Our final coordinate change is an $\mathcal{O}_4(\mu)$-perturbation which makes the angular velocity on the limit cycle $\xi=0$ constant: Let

$$\Theta^* = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1+\hat{\omega}^{-1}\mu^2 g(\psi)}d\psi,$$

and let

$$\Theta = \frac{1}{\Theta^*}\int_0^{\theta}\frac{1}{1+\hat{\omega}^{-1}\mu^2 g(\psi)}d\psi. \tag{4.18}$$

In $(\xi,\Theta,\mathbf{w})$-coordinates, the second equation in (4.16) becomes

$$\dot{\Theta} = \frac{\hat{\omega}}{\Theta^*}+\frac{\beta}{\alpha}a(\mu)(\xi+2+\mathcal{O}_4(\mu))\xi+\mathcal{O}_4(\sqrt{\mu})\mathbf{w}+\Phi_\Theta(\xi,\Theta,\mathbf{w})p_{T,\iota}(t)\ . \tag{4.19}$$

Here we have rewritten $\tilde{\Phi}_\theta(\xi,\theta,\mathbf{w})$ as $\Phi_\Theta(\xi,\Theta,\mathbf{w})$. To obtain the corresponding equations for ξ and $\mathbf{w}$, we rewrite $\tilde{\Phi}_\xi$ as Φ_ξ, $\tilde{\Phi}_{\mathbf{w}}$ as $\Phi_{\mathbf{w}}$. The equations for ξ and $\mathbf{w}$ are otherwise unchanged. See (4.22).

We finish by computing a useful form of Φ_ξ, Φ_Θ and $\Phi_{\mathbf{w}}$. Let $\Phi=(\Phi_x,\Phi_y,\Phi_w)$ be as in Sect. 3.1, and let c_0 be such that $\tan c_0 = \frac{\Phi_y(0)}{\Phi_x(0)}$.

Lemma 4.3.

$$\begin{aligned}\Phi_\xi &= \sqrt{\Phi_x^2(0)+\Phi_y^2(0)}\ \cos(\Theta-c_0)+\mathcal{O}_4(\sqrt{\mu}),\\ \Phi_\Theta &= -\sqrt{\Phi_x^2(0)+\Phi_y^2(0)}\ \sin(\Theta-c_0)+\mathcal{O}_4(\sqrt{\mu}),\\ \Phi_{\mathbf{w}} &= \Phi_w(0)+\mathcal{O}_4(\sqrt{\mu}).\end{aligned} \tag{4.20}$$

Proof: By (4.9) and (4.15), $\tilde{\Phi}_\xi(\xi,\theta,\mathbf{w}) =$

$$\cos\theta\hat{\Phi}_x\Big(\sqrt{\frac{a(\mu)}{\alpha}}r\cos\theta, \sqrt{\frac{a(\mu)}{\alpha}}r\sin\theta, \sqrt{\frac{a(\mu)}{\alpha}}\bar{w}\Big) \;+\; \sin\theta\hat{\Phi}_y\Big(\sqrt{\frac{a(\mu)}{\alpha}}r\cos\theta, \sqrt{\frac{a(\mu)}{\alpha}}r\sin\theta, \sqrt{\frac{a(\mu)}{\alpha}}\bar{w}\Big)$$

where $r = \xi + 1 + \phi(\theta)$. Since $a(\mu) \sim \sqrt{\mu}$, it follows that the expression above is equal to

$$\cos\theta\hat{\Phi}_x(0) + \sin\theta\hat{\Phi}_y(0) + \mathcal{O}_4(\sqrt{\mu})\ .$$

Replacing $\tilde{\Phi}_\xi, \hat{\Phi}_x, \hat{\Phi}_y$ and θ by Φ_ξ, Φ_x, Φ_y and Θ respectively does not change this expression as the differences are absorbed into the $\mathcal{O}_4(\sqrt{\mu})$ term. The asserted formula for Φ_ξ follows from this in a straightforward way. The other two terms are treated similarly. □

4.4. **Summary of coordinate transformations and relevant domains.**

Let $\mathbb{D} = (-\frac{99}{100},\infty)\times\mathbb{S}^1\times E^s$ be the product of the cylinder $(-\frac{99}{100},\infty)\times\mathbb{S}^1$ and E^s with coordinates $(\xi,\Theta,\mathbf{w})$. We have described for each μ a μ-dependent diffeomorphism between $\mathbb{D}$ and an open subset of $\mathbb{E}$. All of the action of interest take place in $\mathbb{D}$; the more delicate parts of the dynamics revolve around the limit cycle $\{0\}\times\mathbb{S}^1\times\{0\}$. The $\mathcal{O}_4$-bounds in the formulas for our coordinate changes and transformed equations are valid on a bounded subset $\bar{V}\subset\mathbb{D}$. An explicit choice of $\bar{V}$ is given below, but first we summarize the results from previous subsections.

The coordinate changes in this section can be summarized as (4.18) together with

$$\begin{aligned} x + h_x(x,y) &= \sqrt{\frac{a}{\alpha}}\,(\xi - 1 - \phi(\Theta) + \mathcal{O}_4(\mu^2))\cos(\Theta + \mathcal{O}_4(\mu^2)) \\ y + h_y(x,y) &= \sqrt{\frac{a}{\alpha}}\,(\xi - 1 - \phi(\Theta) + \mathcal{O}_4(\mu^2))\sin(\Theta + \mathcal{O}_4(\mu^2)) \\ w - W(x,y) &= \sqrt{\frac{a}{\alpha}}\,\mathbf{w}, \end{aligned} \tag{4.21}$$

where $w = W(x,y)$ is the equation of the center manifold of the unforced equation (coordinate change #1), h_x and h_y are used in putting the central flow into normal form (change #2), and ϕ measures the deviation of the periodic solution from $\eta = 1$ (Lemma 4.1). The final form of the equation in $(\xi,\Theta,\mathbf{w})$ is

$$\begin{aligned} \dot{\xi} &= -a(\mu)(2 + 3\xi + \xi^2 + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}(\Phi_\xi + \mathcal{O}_4(\mu))p_{T,\iota}(t), \\ \dot{\Theta} &= \frac{\hat{\omega}}{\Theta^*} + \tau a(\mu)(\xi + 2 + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\left(\frac{1}{\xi+1} + \mathcal{O}_4(\mu)\right)\sqrt{\frac{\alpha}{a(\mu)}}\Phi_\Theta p_{T,\iota}(t), \\ \dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a(\mu)}}\Phi_{\mathbf{w}}p_{T,\iota}(t) \end{aligned} \tag{4.22}$$

where $(\Phi_\xi,\Phi_\Theta,\Phi_{\mathbf{w}})$ can be thought of as given by Lemma 4.3 and $\tau = \frac{\beta}{\alpha}$ is the twist number.

We finish by specifying a viable choice of $\bar{V}$ for purposes of establishing the estimates in the next two sections. Let $K_0 > 0$ and $C_0 > 1$ be two constants defined as follows:

(i) K_0: By (H3), $|P_0^c\Phi(0)| = 1$. Let

$$K_0 = \frac{1}{\sqrt{\Phi_x^2(0) + \Phi_y^2(0)}}|P^s\Phi(0)|$$

where Φ_x, Φ_y and P^s are defined using coordinates in $\mathbb{E}$. That is to say, K_0 is the ratio of the magnitudes of the forcing in the E^s- and E^c-directions.

(ii) C_0: We let $C_0 = (C\tilde{C})^2$ where $C \geq 1$ is the constant in (3.3) and $\tilde{C} \geq 1$ is the constant in the generalized Gronwall's inequality which says that if $u : [0, \infty) \to [0, \infty)$ satisfies

$$u(t) \leq M + K \int_0^t (t-s)^{-\sigma} u(s)\, ds,$$

then there is a constant $\tilde{C} > 1$ depending only on σ such that

$$u(t) \leq M\tilde{C}e^{qt}, \tag{4.23}$$

where $q = (K\Gamma(1-\sigma))^{1/(1-\sigma)}$ and Γ is the usual Γ-function. (See e.g. [H].)

Then define $\bar{V} \subset \mathbb{D}$ by

$$\bar{V} = \{(\xi, \Theta, \mathbf{w}) : \ -\frac{99}{100} < \xi < 100; \ \ |\mathbf{w}| < 100(K_0 + 1)C_0\}.$$

The constant K_1 at the end of Sect. 4.2 is chosen so that the set $\hat{V}$ there corresponds to a set in $\mathbb{D}$ containing $\bar{V}$.

5. Preliminary Estimates on Solutions of the Unforced Equation

The purpose of this section is to derive some relevant information on the time-t map of the *unforced* equation, which in the coordinates at the end of Section 4 may be written as

$$\begin{aligned} \dot{\xi} &= -a(\mu)(2 + 3\xi + \xi^2 + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w}, \\ \dot{\Theta} &= \frac{\hat{\omega}}{\Theta^*} + \tau a(\mu)(\xi + 2 + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w}, \\ \dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w}. \end{aligned} \tag{5.1}$$

We let $u_t = (\xi_t, \Theta_t, \mathbf{w}_t)$ denote the solution of (5.1) with initial condition $u_0 = (\xi_0, \Theta_0, \mathbf{w}_0)$. In order to apply the theory of rank one maps in [WY1], we need to estimate the derivatives of u_t with respect to u_0 and t up to order three for all large t.

5.1. C^0 estimates.

Let

$$V_{r_1,r_2} = \{(\xi, \Theta, \mathbf{w}) \in \mathbb{D} : |\xi| < r_1, |\mathbf{w}| < r_2\}.$$

Recall from Sect. 4.4 that the $\mathcal{O}_4(\sqrt{\mu})$ estimates in (5.1) are valid on a fixed region $\bar{V} \subset \mathbb{D}$, and let $C_0, C, \tilde{C}$ and K_0 be as defined there.

Proposition 5.1. *There exist $K_1 > 1$, $0 < c < \beta_0$,[2] and $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, the follow estimates hold for all $u_0 \in V_{\frac{3}{5}, C_0(1+K_0)}$ and all $t > 0$:*

$$|\mathbf{w}_t| < C_0 e^{-ct} |\mathbf{w}_0|, \qquad |\xi_t| < e^{-\frac{1}{3}a(\mu)t}(|\xi_0| + K_1\sqrt{\mu}). \tag{5.2}$$

In particular, (i) $u_t \in \bar{V}$ for all $t > 0$, and (ii) u_t converges to the limit cycle $\{\xi = 0, \mathbf{w} = 0\}$ as $t \to \infty$. It follows that if M_0 is large enough, then $u_t \in V_{\frac{1}{100}, \frac{1}{100}}$ for all $t > M_0\mu^{-1}$.

Proof: For $u_0 \in V_{\frac{3}{5}, C_0(1+K_0)}$, let

$$t_0 = \sup\{\tau \in [0, \infty) \ : \ u_t \in V_{\frac{31}{50}, C_0^2(1+K_0)} \text{ for } 0 \le t < \tau\}.$$

We first prove (5.2) for $t < t_0$, then notice at the end that $t_0 = \infty$.

From the third equation of (5.1), we have, for all $t < t_0$,

$$\mathbf{w}_t = e^{\mathcal{A}^s t}\mathbf{w}_0 + \int_0^t e^{\mathcal{A}^s(t-\tau)}\mathcal{O}_4(\sqrt{\mu})\mathbf{w}_\tau d\tau, \tag{5.3}$$

so that

$$|\mathbf{w}_t| \le Ce^{-\beta_0 t}|\mathbf{w}_0| + CK\sqrt{\mu}\int_0^t \frac{1}{(t-\tau)^\sigma} e^{-\beta_0(t-\tau)}|\mathbf{w}_\tau| d\tau.$$

Thus, by the generalized Gronwall's inequality, we have

$$|\mathbf{w}_t| \le C\tilde{C}|\mathbf{w}_0| e^{-\beta_0 t} e^{(CK\sqrt{\mu}\Gamma(1-\sigma))^{1/(1-\sigma)}\, t} \tag{5.4}$$

where $\tilde{C}$ is a constant depending only on σ. Now we choose μ_0 small enough so that for $\mu < \mu_0$,

$$-\beta_0 + (CK\sqrt{\mu}\Gamma(1-\sigma))^{1/(1-\sigma)} < -c < 0.$$

Since $C_0 = (C\tilde{C})^2$ by definition, we have

$$|\mathbf{w}_t| < C_0 e^{-ct}|\mathbf{w}_0| \le C_0^2(1 + K_0)\ . \tag{5.5}$$

As for ξ_t, from the first equation of (5.1) we have, for $t < t_0$,

$$\xi_t = e^{\int_0^t -a(\mu)(2+3\xi_s+\xi_s^2+\mathcal{O}(\mu))ds}\left(\xi_0 + \int_0^t \mathcal{O}(\sqrt{\mu})\mathbf{w}_s e^{\int_0^s a(\mu)(2+3\xi_\tau+\xi_\tau^2+\mathcal{O}(\mu))d\tau} ds\right).$$

Observe that for $|\xi_s| < \frac{31}{50}$,

$$2 + 3\xi_s + \xi_s^2 + \mathcal{O}(\mu) > \frac{1}{3}.$$

Using (5.5) for $|\mathbf{w}_t|$, we have, for μ_0 sufficiently small,

$$|\xi_t| \le e^{-\frac{1}{3}a(\mu)t}(|\xi_0| + K_1\sqrt{\mu}) < \frac{31}{50}\ . \tag{5.6}$$

Since the boundary of $V_{\frac{31}{50}, C_0^2(1+K_0)}$ is not reached in finite time, we conclude that $t_0 = \infty$. □

[2]See the exponential dichotomy in Sect. 3.2 for β_0.

5.2. C^3 **bounds.**

A. Derivatives with respect to ξ_0, Θ_0 and $\mathbf{w}_0$.

We first treat the C^3-norms of the mapping $u_0 \mapsto u_t$ for arbitrary but fixed $t > 0$. The notation ∂^k below represents a (specific) partial derivative of order k with respect to the components of u_0.

Proposition 5.2. *There exist $K_2 > 1$, and $\mu_0 > 0$ small enough depending on K_2 such that for each $\mu \in (0, \mu_0]$, the following estimates hold for $u_0 \in V_{\frac{3}{5}, C_0(1+K_0)}$ for all $t \geq 0$ and $1 \leq k \leq 3$:*

$$|\partial^k \xi_t| < K_2 e^{-\frac{a(\mu)t}{2}}, \quad |\partial^k \Theta_t| < K_2, \quad |\partial^k \mathbf{w}_t| < K_2 e^{-\frac{ct}{2}}. \tag{5.7}$$

Since the convergence to the limit cycle $\{\xi = 0, \mathbf{w} = 0\}$ is exponential, and bounds of the type in (5.7) are valid on the limit cycle, one may expect them to hold also for orbits in a bounded region of the basin. Proposition 5.2 asserts more: it asserts that the constant K_2 is independent of μ for μ small. This is somewhat more delicate, for $a(\mu)$, the rate of contraction in the ξ-direction, tends to zero as $\mu \to 0$.

Proof of Proposition 5.2: Our plan of proof is as follows. We will show that there is a constant $K^* > 1$ with the property that for any $R > 1$, there exists $\mu(R) > 0$ such that for every $\mu \in (0, \mu(R))$ and $t_0 > 0$, if

$$|\partial^k \xi_t| < R e^{-\frac{a(\mu)t}{2}}, \quad |\partial^k \Theta_t| < R, \quad |\partial^k \mathbf{w}_t| < R e^{-\frac{ct}{2}} \tag{5.8}$$

for all $t \in (0, t_0)$, then (5.8) holds with R replaced by K^* on the same t-interval. We stress that K^* is independent of R. Having proved this, we define

$$t^* = \sup\{\tau : (5.8) \text{ with } R = 2K^* \text{ holds for all } t \in (0, \tau) \text{ and } \mu \in (0, \mu(2K^*))\},$$

and claim that $t^* = \infty$: If not, then (5.8) with $R = 2K^*$ would hold for $t < t^*$ while one of the inequalities would fail at $t = t^*$. This is incompatible with the fact that (5.8) with $R = K^*$ is in fact valid for $t < t^*$. The Proposition is therefore proved with $K_2 = 2K^*$ and $\mu_0 = \mu(2K^*)$.

Notation: The generic constant K is assumed to be independent of R (as well as μ).

I. First derivatives. We pick an arbitrary R, assume (5.8) holds on the time interval $(0, t_0)$, and for $t \in (0, t_0)$, compute bounds on the partial derivatives of w_t, ξ_t and Θ_t with respect to ξ_0, Θ_0 and $\mathbf{w}_0$ in terms of R, μ and K. The values of K^* and $\mu(R)$ will become clear from these bounds.

For $\partial \mathbf{w}_t$, we have, from (5.3),

$$\begin{aligned} \partial \mathbf{w}_t &= e^{\mathcal{A}^s t} \partial \mathbf{w}_0 + \int_0^t e^{\mathcal{A}^s (t-\tau)} \left(\mathcal{O}_3(\sqrt{\mu}) \partial \xi_\tau + \mathcal{O}_3(\sqrt{\mu}) \partial \Theta_\tau + \mathcal{O}_3(\sqrt{\mu}) \partial \mathbf{w}_\tau \right) \mathbf{w}_\tau d\tau \\ &\quad + \int_0^t e^{\mathcal{A}^s (t-\tau)} \mathcal{O}_4(\sqrt{\mu}) \partial \mathbf{w}_\tau d\tau. \end{aligned}$$

We use (5.8) to bound $\partial\xi_t, \partial\Theta_t, \partial\mathbf{w}_t$ and Proposition 5.1 to bound $\mathbf{w}$ in the first integral, obtaining an upper bound of $K(1+R\sqrt{\mu})e^{-ct}$ for the sum of the first two terms. Applying the generalized Gronwall's inequality to

$$|\partial\mathbf{w}_t| \le K(1+R\sqrt{\mu})e^{-ct} + K\sqrt{\mu}\int_0^t \frac{1}{(t-\tau)^\sigma}e^{-c(t-\tau)}|\partial\mathbf{w}_\tau|d\tau,$$

we obtain

$$|\partial\mathbf{w}_t| \le K(1+R\sqrt{\mu})e^{-\frac{c}{2}t} \tag{5.9}$$

provided that $\mu << c$.

To estimate $\partial\xi_t$ we write the first equation of (5.1) as

$$\frac{d}{dt}\xi = -a(\mu)\Lambda\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} \qquad \text{and} \qquad \Lambda = 2+3\xi+\xi^2+\mathcal{O}_4(\mu).$$

Then

$$\xi_t = e^{-\int_0^t a(\mu)\Lambda ds}\left(\xi_0 + \int_0^t \mathcal{O}_4(\sqrt{\mu})\mathbf{w}_s e^{\int_0^s a(\mu)\Lambda d\hat{s}}ds\right), \tag{5.10}$$

and

$$\frac{d}{dt}\partial\xi = -a(\mu)\hat{\Lambda}\partial\xi + g_1 + g_2 \tag{5.11}$$

where

$$\begin{aligned}
\hat{\Lambda} &= 3\xi + 2\xi^2 + \Lambda \\
g_1 &= a(\mu)\xi\cdot(\mathcal{O}_3(\mu)\partial\xi + \mathcal{O}_3(\mu)\partial\Theta) \\
g_2 &= \mathcal{O}_4(\sqrt{\mu})\partial\mathbf{w} + (\mathcal{O}_3(\sqrt{\mu})\partial\xi + \mathcal{O}_3(\sqrt{\mu})\partial\Theta + \mathcal{O}_3(\sqrt{\mu})\partial\mathbf{w})\mathbf{w}.
\end{aligned}$$

From (5.11),

$$\partial\xi_t = e^{-a(\mu)\int_0^t \hat{\Lambda}ds}(\partial\xi_0 + G_1 + G_2) \tag{5.12}$$

where

$$G_i = \int_0^t g_i e^{a(\mu)\int_0^s \hat{\Lambda}d\hat{s}}ds, \qquad i = 1,2.$$

We now estimate G_1 and G_2. Substituting in (5.10) for ξ_s and writing $a(\mu) = \mathcal{O}(\mu)$, we have

$$G_1 = \int_0^t (\mathcal{O}_3(\mu^2)\partial\Theta_s + \mathcal{O}_3(\mu^2)\partial\xi_s)\cdot\left(\xi_0 + \int_0^s \mathcal{O}_4(\sqrt{\mu})\mathbf{w}_{\hat{s}}e^{\int_0^{\hat{s}} a(\mu)\Lambda d\hat{\hat{s}}}d\hat{s}\right)\cdot e^{a(\mu)\int_0^s(\hat{\Lambda}-\Lambda)d\hat{s}}\ ds.$$

Using our assumption on $|\partial\Theta_s|$ and $|\partial\xi_s|$, the first of the 3 factors in the integrand is $< KR\mu^2$. For $|\xi_t|$ and $|\mathbf{w}_t|$ in the other two factors, we use Proposition 5.1. Then the middle factor inside brackets is $\mathcal{O}(1)$, for the integral is $< K\sqrt{\mu}$. The third factor is also $\mathcal{O}(1)$, for $\hat{\Lambda}-\Lambda = 3\xi+2\xi^2$, so

$$a(\mu)\int_0^t |\hat{\Lambda}-\Lambda|ds < K\ .$$

Thus

$$|G_1| < KR\mu^2 t. \tag{5.13}$$

The estimates for G_2 are easier since the rates of decrease for $\mathbf{w}_t$ and $\partial \mathbf{w}_t$ in time are $\geq \frac{1}{2}c >> \mu$ and independent of μ. We have in fact

$$|G_2| < K\sqrt{\mu}R. \tag{5.14}$$

Putting (5.12), (5.13) and (5.14) together, we have

$$|\partial \xi_t| < e^{-a(\mu)\int_0^t \hat{\Lambda} ds}(1 + K\sqrt{\mu}R + K\mu^2 Rt) \tag{5.15}$$

for all $t \in (0, t_0)$.

We now estimate

$$\mathcal{E} := e^{-a(\mu)\int_0^t \hat{\Lambda} ds}$$

for $u_0 \in \mathcal{U}_{\frac{3}{5}, C_0(1+K_0)}$. First we note that $|\xi_t| < \frac{2}{3}$ for all $t > 0$ from Proposition 5.1. It follows that $\hat{\Lambda} > -2$. We consider separately the following two cases:

(i) For $t < 10a^{-1}(\mu)$, $\mathcal{E} < e^{20} < e^{30}e^{-a(\mu)t}$.

(ii) For $t > 10a^{-1}(\mu)$, we have $\mathcal{E} < e^{20}e^{\int_{10a^{-1}(\mu)}^t -a(\mu)\hat{\Lambda} dt}$. From Proposition 5.1, $|\xi_t| < \frac{1}{100}$ for $t > 10a^{-1}(\mu)$. This implies $\hat{\Lambda} > 1$ for $t > 10a^{-1}(\mu)$. Therefore we have

$$\mathcal{E} < Ke^{-a(\mu)(t-10a^{-1}(\mu))} < Ke^{-a(\mu)t}.$$

From these estimates on $\mathcal{E}$ and (5.15), we obtain

$$|\partial \xi_t| < e^{-a(\mu)\int_0^t \hat{\Lambda} ds}(1 + K\sqrt{\mu}R + K\mu^2 Rt) < KR\sqrt{\mu} \cdot e^{-\frac{1}{2}a(\mu)t}. \tag{5.16}$$

We point out that we have used here the fact that $\mu t e^{-a(\mu)t} < Ke^{-\frac{1}{2}a(\mu)t}$ for some K independent of μ. Since this inequality is not valid without the copy of μ on the left side, the assertion in (5.16) requires that the power of μ in our bound for G_1 be at least $\frac{3}{2}$.

We proceed to the estimate of $\partial\Theta_t$. From (4.19),

$$\frac{d}{dt}\partial\Theta = 2\tau a(\mu)(1 + \xi + \mathcal{O}_4(\mu))\partial\xi + g + g_2$$

where

$$g = \tau a(\mu)\xi \cdot \mathcal{O}_3(\mu)\partial\Theta$$

and g_2 is as in (5.11). It follows that

$$\partial\Theta_t = \int_0^t 2\tau a(\mu)(1 + \xi_s + \mathcal{O}_4(\mu))\partial\xi_s ds + \int_0^t g ds + \int_0^t g_2 ds.$$

The magnitude of the first term is $< 3\tau a(\mu)\int_0^t |\partial\xi_s| ds$. Using (5.16) for $\partial\xi_s$ and noticing that the factor of μ^{-1} from integrating $\partial\xi_s$ is cancelled by $a(\mu)$, we see that this term is $< K$. For the integral of g, we use R to bound $|\partial\Theta|$. Again using $a(\mu)\int_0^t \xi_s ds = \mathcal{O}(1)$, we obtain $|\int g| < KR\mu$. The third term is estimated as before. Altogether, we get

$$|\partial\Theta_t| < K(1 + R\sqrt{\mu}) \ . \tag{5.17}$$

From (5.9), (5.16) and (5.17), we see that as far as first derivatives go, it suffices to require $\mu(R) < \frac{1}{R}$, and to take $K^* = 2K$ where K is as in these estimates.

II. Higher derivatives. Let us start with $\partial^k \mathbf{w}_t$. From the third equation of (5.1),

$$\frac{d}{dt}\partial^k \mathbf{w} = \mathcal{A}^s \partial^k \mathbf{w} + \sum \mathcal{O}(\sqrt{\mu}) P_k$$

where the summation is over a finite collection of terms P_k each of which is a monomial in $\partial^{i_1}\xi, \partial^{i_2}\Theta$ and $\partial^{i_3}\mathbf{w}$ where i_1, i_2 and i_3 are all $\leq k$. Observe that for each term in the summation, P_k has either $\partial^{i_3}\mathbf{w}$, $1 \leq i_3 \leq k$, as a factor or it has a factor of $\mathbf{w}$. That is, we can rewrite the equation for $\partial^k \mathbf{w}$ as

$$\frac{d}{dt}\partial^k \mathbf{w} = \mathcal{A}^s \partial^k \mathbf{w} + \sum \mathcal{O}(\sqrt{\mu}) P_k \mathbf{w} + \sum \mathcal{O}(\sqrt{\mu}) P_{k-i_3} \partial^{i_3} \mathbf{w} \tag{5.18}$$

where $1 \leq i_3 \leq k$. From (5.18), for $k = 2, 3$,

$$\partial^k \mathbf{w}_t = \sum \int_0^t e^{\mathcal{A}^s(t-\tau)} \mathcal{O}(\sqrt{\mu}) P_k \mathbf{w}_\tau d\tau + \sum \int_0^t e^{\mathcal{A}^s(t-\tau)} \mathcal{O}(\sqrt{\mu}) P_{k-i_3} \partial^{i_3} \mathbf{w}_\tau d\tau. \tag{5.19}$$

Using (5.8) for $t \in (0, t_0)$, the terms $|\mathcal{O}(\sqrt{\mu}) P_k|$ and $|\mathcal{O}(\sqrt{\mu}) P_{k-i_3}|$ above are $< \mathcal{O}(\sqrt{\mu}) R^k$, and are easily made independent of R by shrinking μ_0. We therefore have

$$|\partial^k \mathbf{w}_t| < \sum \int_0^t e^{-c(t-\tau)} \mathcal{O}(\sqrt{\mu}) P_k |\mathbf{w}_\tau| d\tau + \sum \int_0^t e^{-c(t-\tau)} \mathcal{O}(\sqrt{\mu}) P_{k-i_3} |\partial^{i_3} \mathbf{w}_\tau| d\tau. \tag{5.20}$$

From the generalized Grownwall's inequality, we again have

$$|\partial^k \mathbf{w}_t| << K e^{-\frac{c}{2}t}$$

provided μ is small enough.

As before, estimates for $\partial^k \xi_t$ are trickier because the slow decay rate $a(\mu)$ leads to a factor of μ^{-1} when integrated. Let us start with $\partial^2 \xi_t$. From (5.11) we have

$$\frac{d}{dt}\partial^2 \xi = -a(\mu)\hat{\Lambda}\partial^2 \xi - a(\mu)\partial\hat{\Lambda}\partial\xi + \partial g_1 + \partial g_2, \tag{5.21}$$

from which we obtain

$$\partial^2 \xi_t = e^{-\int_0^t a(\mu)\hat{\Lambda} ds}((I) + (II) + (III)) \tag{5.22}$$

where

$$\begin{aligned} (I) &= -\int_0^t a(\mu)\partial\hat{\Lambda}\partial\xi_s e^{\int_0^s a(\mu)\hat{\Lambda} d\hat{s}} ds \\ (II) &= \int_0^t \partial g_1 e^{\int_0^s a(\mu)\hat{\Lambda} d\hat{s}} ds \\ (III) &= \int_0^t \partial g_2 e^{\int_0^s a(\mu)\hat{\Lambda} d\hat{s}} ds. \end{aligned} \tag{5.23}$$

We focus on the involvement of R in (I)-(III). (I) involves only first derivatives, so we may use previous estimates, i.e. there is no need to involve R. In (II) and (III), however, there are terms in $\partial^2\Theta$ and $\partial^2 \mathbf{w}_t$. For these terms we first use (5.8), then observe that there is always a factor of $\mu^{\frac{1}{2}}$

that cancels the effect of R from (5.8). (III) is easier because every term in ∂g_2 contains either a factor $\mathbf{w}$ or $\partial\mathbf{w}$, both of which decay with a rate bounded from below by a constant c independent of μ. This ensures that the factor $\sqrt{\mu}$, which appears in all terms of g_2, compensates for the effect R from (5.8). The estimate for (II) is trickier though the problem is by now familiar: the integration over ξ or $\partial\xi$ (with decay rate $\sim \mu$) contributes a factor μ^{-1}; this is taken care of by the μ^2 that appears in every term in g_1.

In general, we write the variational equations for $\partial^k\xi$ as

$$\frac{d}{dt}\partial^k\xi = -a(\mu)\hat{\Lambda}\partial^k\xi + \sum a(\mu)\hat{P}_{k-i_1}\partial^{i_1}\xi + \sum \mathcal{O}(\mu^2)P_{k-i_2}\partial^{i_2}\xi + \sum \mathcal{O}(\sqrt{\mu})P_{k-i_3}\partial^{i_3}\mathbf{w} \tag{5.24}$$

where $0 \le i_1, i_2, i_3 \le k-1$. Here $\hat{P}_{k-i_1}$ are monomials of $\partial^i\xi$, $1 \le i \le k-1$, for which the sums of the degrees of the derivatives are $k-i_1$; and P_{k-i_2}, P_{k-i_3} are monomials of $\partial^i\xi_t, \partial^i\Theta_t$ and $\partial^i\mathbf{w}_t$, for which the sums of the degrees of the derivatives are $k-i_2$ and $k-i_3$ respectively. Note that the terms in the second sum may involve $\partial^i\Theta$, for which the factor in front is, by direct computation, of order μ^2 (instead of μ). The involvement of $\partial^i\Theta$ in the terms in the third sum is of less concern because of the appearance of $\partial^{i_3}\mathbf{w}_t$. From (5.24), we have for $k = 2, 3$,

(5.25)

$$\begin{aligned}\partial^k\xi_t = e^{-\int_0^t a(\mu)\hat{\Lambda}ds}\Bigg(&\sum\int_0^t a(\mu)\hat{P}_{k-i_1}\partial^{i_1}\xi_s e^{\int_0^s a(\mu)\hat{\Lambda}d\hat{s}}ds + \sum\int_0^t \mathcal{O}(\mu^2)P_{k-i_2}\partial^{i_2}\xi_s e^{\int_0^s a(\mu)\hat{\Lambda}d\hat{s}}ds \\ &+\sum\int_0^t \mathcal{O}(\sqrt{\mu})P_{k-i_3}\partial^{i_3}\mathbf{w}_s e^{\int_0^s a(\mu)\hat{\Lambda}d\hat{s}}ds\Bigg).\end{aligned}$$

We will prove inductively that for $1 \le i \le 3$,

$$\partial^i\xi_t = e^{-\int_0^t a(\mu)\hat{\Lambda}ds}(K + \sum_{j=1}^{i}\mathcal{O}(\mu^{j+\frac{1}{4}})t^j). \tag{5.26}$$

First, observe that under the assumption that (5.26) holds for $1 \le i \le k-1$ and our previous computation for $\mathcal{E} := e^{-\int_0^t a(\mu)\hat{\Lambda}ds}$, we have

$$|\partial^i\xi_t| < Ke^{-\frac{a(\mu)}{2}t} \tag{5.27}$$

for $1 \le i \le k-1$. Second, observe that the following hold for $i = k$: The first sum in (5.25) is independent of R and it is bounded by a constant K (using (5.26) for $\partial^{i_1}\xi$ and (5.27) for $\hat{P}_{k-i_1}$). The third sum, thanks to the occurrence of $\partial^{i_3}\mathbf{w}$, is easily bounded by $\mathcal{O}(\sqrt{\mu})R^3 < K\mu^{\frac{1}{4}} < 1$. These two sums contribute to the constant in (5.26). Using (5.26) for $\partial^{i_2}\xi_t$, the second sum is clearly in form of (5.26): $\partial^i\Theta_t$ contributes a factor $< R^3$ and this is absorbed by one copy of $\mu^{\frac{3}{4}}$. (Here we need μ^2 instead of μ in front). Note that we also need to distinguish the case of $i_1, i_2 = 0$ from $i_1, i_2 \neq 0$, treating the differences between $\hat{\Lambda}$ and Λ for the former. This has been considered in our earlier estimates of the first derivatives.

To compute $\partial^k \Theta_t$ we differentiate (4.19), obtaining

$$\partial^k \Theta_t = \partial^k \Theta_0 + \int_0^t P_k ds$$

where P_k is a sum of finitely many terms from differentiating (i) $\mathcal{O}(\mu^2)\xi$, (ii) $a(\mu)(\xi+2)\xi$, and (iii) $\mathcal{O}(\sqrt{\mu})\mathbf{w}$. Terms from (i) give quantities of $\mathcal{O}(\mu)R^3$, and those from (iii) are bounded by $\mathcal{O}(\sqrt{\mu})R^3$. These bounds are easily made independent of R by shrinking μ_0. For contributions from (ii), there is no dependency on R at all because we can now use (5.27) for $\partial^i \xi$. Altogether this gives

$$|\partial^k \Theta_t| < K \tag{5.28}$$

where K is independent of R. □

B. Derivatives with respect to $\xi_0, \Theta_0, \mathbf{w}_0$ and t. Derivatives with respect to t may not be defined at $t = 0$.

Proposition 5.3. *There exist K_3, μ_0, and $n_0 > 0$ such that for each $\mu \in (0, \mu_0]$ and $t \geq n_0$, the following estimates hold for all $u_0 \in V_{\frac{11}{20}, C_0(1+K_0)}$:*

$$|\partial^k \xi_t| < K_3 e^{-\frac{a(\mu)t}{2}}, \quad |\partial^k \Theta_t| < K_3, \quad |\partial^k \mathbf{w}_t| < K_3 e^{-\frac{ct}{2}}. \tag{5.29}$$

Here $\partial^k, k = 1, 2, 3,$ is any kth partial derivative in $\xi_0, \Theta_0, \mathbf{w}_0$ or t.

Proof: Instead of reproving Proposition 5.2 to include derivatives in t, we combine the results of Proposition 5.2 with bounds on derivatives in t for a *bounded* range of t. As before, let G_t be the time-t map of the unforced equation (5.1). Note that G_t is 2π-periodic in Θ_0. By Corollary 3.4.6 of [H] (pp 66), the mapping $(\xi_0, \Theta_0, \mathbf{w_0}, t) \mapsto G_t(\xi_0, \Theta_0, \mathbf{w_0})$ is C^3 for all $t > 0$. The proof of Theorem 3.4.4 in [H] implies that all partial derivatives of order up to 3 are uniformly bounded for $|\xi_0| \leq r_1$, $\Theta_0 \in \mathbb{R}$, $|\mathbf{w}_0| \leq r_2$, $0 < t_1 \leq t \leq t_2$, and $\mu \in (0, \mu_0)$, where r_1, r_2, t_1, and t_2 are positive constants.

Let μ_0 be as in Proposition 5.2 and with $\frac{11}{20} + K_1\sqrt{\mu_0} < 3/5$ where K_1 is as in Proposition 5.1. By Proposition 5.1, there exists n_0 large enough so that for each fixed $\mu \in (0, \mu_0]$, G_{n_0} maps $V_{\frac{11}{20}, C_0(1+K_0)}$ into $V_{\frac{3}{5}, C_0(1+K_0)}$. Let arbitrary $t_0 \geq n_0$ be fixed. Then writing $s = t - (t_0 - n_0)$, we have

$$\begin{aligned} \partial_t \xi_t \mid_{t=t_0} &= \partial_s (\xi_{t_0-n_0} \circ G_s) \mid_{s=n_0} \\ &= \{(\partial_\xi \xi_{t_0-n_0} \circ G_s)\partial_s \xi_s \ + \ (\partial_\Theta \xi_{t_0-n_0} \circ G_s)\partial_s \Theta_s \ + \ (\partial_{\mathbf{w}} \xi_{t_0-n_0} \circ G_s)\partial_s \mathbf{w}_s\} \mid_{s=n_0} \ . \end{aligned}$$

Thus, using Proposition 5.1 and Proposition 5.2, we obtain

$$|\partial_t \xi_t \mid_{t=t_0} \mid \leq K_2 e^{-\frac{a(\mu)}{2}t_0} e^{\frac{a(\mu)}{2}n_0} \max\{|\partial_s \xi_s| \mid_{s=n_0}, |\partial_s \Theta_s| \mid_{s=n_0}, |\partial_s \mathbf{w}_s| \mid_{s=n_0}\}.$$

Note that $\max\{|\partial_s \xi_s| \mid_{s=n_0}, |\partial_s \Theta_s| \mid_{s=n_0}, |\partial_s \mathbf{w}_s| \mid_{s=n_0}\}$ is uniformly bounded for $u_0 \in V_{\frac{11}{20}, C_0(1+K_0)}$. Other derivatives can be treated the same way. Moreover, in $\partial^k \xi_t \mid_{t=t_0}$, since each term involves a factor of the form $\partial^j \xi_{t_0-n_0}$, we also have the asserted exponential decay with respect to t_0. Derivatives of Θ and $\mathbf{w}$ are treated similarly. □

5.3. **Approximate form of Θ_t for large t.**

For $t >> 1$, we estimate Θ_t as follows: From equation (5.1),

$$\Theta_t = \Theta_0 + \frac{\hat{\omega}}{\Theta^*}t + \tau a(\mu)\int_0^t (\xi_s + 2 + \mathcal{O}_4(\mu))\xi_s ds + \int_0^t \mathcal{O}_4(\sqrt{\mu})\mathbf{w}ds. \tag{5.30}$$

We introduce a new function

$$\Theta_{t,\infty} = \Theta_0 + \frac{\hat{\omega}}{\Theta^*}t + \tau a(\mu)\int_0^\infty (\xi_s + 2 + \mathcal{O}_4(\mu))\xi_s ds + \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\mathbf{w}ds. \tag{5.31}$$

In view of Proposition 5.1, these improper integrals make sense for $(\xi_0, \Theta_0, \mathbf{w}_0) \in V_{\frac{11}{20}, C_0(K+1)}$.

Proposition 5.4. *There exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ and $t > 0$,*

$$\Theta_{t,\infty} = \Theta_0 + \frac{\hat{\omega}}{\Theta^*}t + \tau \ln(\xi_0 + 1) + \mathcal{O}_3(\sqrt{\mu}),$$

where $\mathcal{O}_3(\sqrt{\mu})$ is a function of $(\xi_0, \Theta_0, \mathbf{w}_0)$.

Proof: The first equation in (5.1) implies

$$\frac{d\xi}{\xi + 1} = -a(\mu)(\xi + 2)\xi ds + \mathcal{O}_4(\mu^2)\xi ds + \mathcal{O}_4(\sqrt{\mu})\frac{\mathbf{w}}{\xi + 1}ds,$$

from which we deduce

$$\int_0^\infty a(\mu)(\xi + 2)\xi ds = \ln(\xi_0 + 1) + \int_0^\infty \mathcal{O}_4(\mu^2)\xi ds + \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\frac{\mathbf{w}}{\xi + 1}ds\ .$$

We have, therefore,

$$\begin{aligned}\Theta_{t,\infty} = \Theta_0 + \frac{\hat{\omega}}{\Theta^*}t + \tau\left(\ln(\xi_0 + 1) + \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\frac{\mathbf{w}}{\xi + 1}ds\right)\\ + \int_0^\infty \mathcal{O}_4(\mu^2)\xi ds + \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\mathbf{w}ds.\end{aligned} \tag{5.32}$$

The assertion in the proposition follows from (5.32) together with

$$\int_0^\infty \mathcal{O}_4(\mu^2)\xi ds = \mathcal{O}_3(\mu), \quad \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\mathbf{w}ds = \mathcal{O}_3(\sqrt{\mu}), \quad \int_0^\infty \mathcal{O}_4(\sqrt{\mu})\frac{\mathbf{w}}{1 + \xi}ds = \mathcal{O}_3(\sqrt{\mu}),$$

which follow from Propositions 5.1-5.3. □

We also have the following estimate:

Proposition 5.5. *Viewed as a function of $t, (\xi_0, \Theta_0, \mathbf{w}_0)$ on $[M_0\mu^{-1}, \infty) \times V_{\frac{11}{20}, C_0(K_0+1)}$,*

$$\|\Theta_{t,\infty} - \Theta_t\|_{C^3} < K\mu e^{-\frac{a(\mu)}{4}t}.$$

Proof: This result essentially follows from Propositions 5.1–5.4: derivative estimates of $\Theta_{t,\infty} - \Theta_t$ are relatively straightforward exercises and are left to the reader. □

Remark. Propositions 5.4 and 5.5 have the following geometric interpretation: Since all eigenvalues of the equation linearized along the limit cycle $\Omega = \{\xi = 0, \mathbf{w} = 0\}$ have strictly negative real parts,

and $V_{\frac{11}{20},C_0(1+K_0)}$ is contained in its basin of attraction, by standard theory there is on $V_{\frac{11}{20},C_0(1+K_0)}$ a codimension 1 stable foliation $\mathcal{W}^s$ whose leaves are defined by

$$\mathcal{W}^s_{\bar{u}_0} = \{u_0 : |u_t - \bar{u}_t| \to 0 \text{ as } t \to \infty\}, \quad \bar{u}_0 \in \Omega \ .$$

We claim that the approximate form of $\mathcal{W}^s_{\bar{u}_0}$ can be deduced from Propositions 5.4 and 5.5. Let $\bar{u}_0 = (0, \bar{\Theta}_0, 0) \in \Omega$. By the quasi-invariance of $\mathcal{W}^s$, a point $u_0 = (\xi_0, \Theta_0, \mathbf{w}_0)$ is in $\mathcal{W}^s_{\bar{u}_0}$ if and only if $\Theta_{n(2\pi\Theta^*\hat{\omega}^{-1})} \to \bar{\Theta}$ as $n \to \infty$, $2\pi\Theta^*\hat{\omega}^{-1}$ being the period of the limit cycle. By Proposition 5.5, $\Theta_{n(2\pi\Theta^*\hat{\omega}^{-1})} \approx \Theta_{n(2\pi\Theta^*\hat{\omega}^{-1}),\infty}$ for large n, and by Proposition 5.4, $\Theta_{n(2\pi\Theta^*\hat{\omega}^{-1}),\infty} \approx \Theta_0 + \tau \ln(\xi_0 + 1)$ for small μ. Thus as $\mu \to 0$, $\mathcal{W}^s_{\bar{u}_0}$ tends to the hypersurface $\{(\xi, \Theta, \mathbf{w}) : \Theta + \tau \ln(\xi + 1) = \bar{\Theta}\}$.

6. Time-T Map of Forced Equation and Derived 2-D System

In Sects. 6.1 and 6.2, we derive an approximate form of the time-T map F_T of the forced system. By restricting F_T to a suitable region of a center manifold, we obtain a 2-dimensional self-map of an annulus. As explained in Sect. 3.3, these 2-D systems hold the key to all of our results. Proofs of Theorems 1 and 3 are given in Sect. 6.3. The last subsection contains some technical preparation needed for the proof Theorem 2.

6.1. **Approximate form of kick map.**

First, we complete the definition of the *kick ratio* $\gamma := \rho(d\sqrt{\mu})^{-1}$ introduced in Sect. 2.1 by specifying the number d. Returning to $\mathbb{H}^\sigma$, let P^c_μ and P^s_μ denote the projections of $\mathbb{H}^\sigma$ onto E^c_μ and E^s_μ respectively, and let E^Φ denote the codimension 1 subspace spanned by $(E^c_0)^\perp$ and $P^c_0(\Phi(0))$. Let $p_\pm(\mu)$ be the points of intersection of E^Φ with the Hopf limit cycle. Then

$$d := \lim_{\mu \to 0} \frac{|p_\pm(\mu)|}{\sqrt{\mu}}.$$

It is easy to verify that d is well defined and $\neq 0$ under Conditions (H1) and (H2). The next lemma gives an approximation of γ in terms of familiar quantities.

Lemma 6.1. *Let* $\tilde{\gamma} := \rho\sqrt{\frac{\alpha}{a}}\sqrt{\Phi_x^2(0) + \Phi_y^2(0)}$. *Then* $\tilde{\gamma} = \gamma \cdot (1 + \mathcal{O}(\sqrt{\mu}))$.

Proof: Let v_1 and v_2 be as in Sect. 3.1, and $v_1^\perp$ and $v_2^\perp$ be unit vectors in E^c orthogonal to v_1 and v_2. Let $x_\pm, y_\pm$ be the (x, y)-coordinates of $p_\pm$ in $\mathbb{E}$. By definition we have

$$x_\pm = \frac{p^\pm \cdot v_1^\perp}{v_2 \cdot v_1^\perp}|p_\pm| \left(\frac{P^c_0(\Phi(0)) \cdot v_1^\perp}{v_2 \cdot v_1^\perp} + \mathcal{O}(\sqrt{\mu}) \right) = |p_\pm|(\Phi_x(0) + \mathcal{O}(\sqrt{\mu})).$$

Similarly, $y_\pm = |p_\pm|(\Phi_y(0) + \mathcal{O}(\sqrt{\mu}))$. Thus

$$|(x_\pm, y_\pm)| = |p_\pm|(\sqrt{\Phi_x^2(0) + \Phi_y^2(0)} + \mathcal{O}(\sqrt{\mu})). \tag{6.1}$$

On the other hand, from (4.21) and equation (5.1) we have,

$$|(x_\pm, y_\pm)| = \sqrt{\frac{a}{\alpha}}(1 + \mathcal{O}(\sqrt{\mu})). \tag{6.2}$$

The desired result follows from the last two lines combined with the definition of d. □

Provisional assumption (2) in Sect. 3.3 follows from Lemma 6.1 and equation (3.1).

We return now to the coordinates at the end of Section 4 to study the effect of the forcing. For $0 < t \le \iota$, equation (4.22) can be written as

$$\begin{aligned}
\dot{\xi} &= -a(\mu)(2+3\xi+\xi^2+\mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \frac{1}{\iota}\rho\sqrt{\frac{\alpha}{a(\mu)}}(\Phi_\xi + \mathcal{O}_4(\mu)),\\
\dot{\Theta} &= \frac{\hat{\omega}}{\Theta^*} + \tau a(\mu)(\xi+2+\mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \frac{1}{\iota}\rho\left(\frac{1}{\xi+1}+\mathcal{O}_4(\mu)\right)\sqrt{\frac{\alpha}{a(\mu)}}\Phi_\Theta,\\
\dot{\mathbf{w}} &= \mathcal{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \frac{1}{\iota}\rho\sqrt{\frac{\alpha}{a(\mu)}}\Phi_{\mathbf{w}}.
\end{aligned} \tag{6.3}$$

We let $(\hat{\xi}_t, \hat{\Theta}_t, \hat{\mathbf{w}}_t)$ denote the solution of this equation with initial condition $(\xi_0, \Theta_0, \mathbf{w}_0)$. Recall that our $\mathcal{O}_4$-bounds are valid on a domain $\bar{V}$ (see Sect. 4.4). For $u_0 = (\xi_0, \Theta_0, \mathbf{w}_0) \in \bar{V}$, we let $\kappa(u_0) = (\hat{\xi}_\iota, \hat{\Theta}_\iota, \hat{\mathbf{w}}_\iota)$ provided the solution starting from u_0 remains in $\bar{V}$ up to time ι; κ is called the "kick map". Let $V_{r_1,r_2} = \{|\xi| < r_1, |\mathbf{w}| < r_2\}$ as in Sect. 5.1.

Proposition 6.1. *Assume that μ and ι are sufficiently small. Then*

$$\kappa(V_{\frac{1}{100},\frac{1}{100}}) \subset V_{\frac{11}{20},C_0(1+K_0)},$$

and for $u_0 \in V_{\frac{1}{100},\frac{1}{100}}$, we have

$$\begin{aligned}
\hat{\xi}_\iota &= \sqrt{(1+\xi_0)^2 + \tilde{\gamma}(\tilde{\gamma} + 2(1+\xi_0)\cos(\Theta_0 - c_0))} - 1 + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}) \ ,\\
\tan\hat{\Theta}_\iota &= \frac{(1+\xi_0)\sin\Theta_0 + \tilde{\gamma}\sin c_0}{(1+\xi_0)\cos\Theta_0 + \tilde{\gamma}\cos c_0} + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}) \ ,
\end{aligned} \tag{6.4}$$

where $\tan c_0 = \frac{\Phi_y(0)}{\Phi_x(0)}$, *and*

$$|\hat{\mathbf{w}}_\iota| \le C_0(|\mathbf{w_0}| + \rho\sqrt{\frac{\alpha}{a}}(|P_0^s\Phi(0)| + \mathcal{O}_4(\sqrt{\mu}))) \ .$$

Proof: For $\hat{\xi}_\iota$ and $\hat{\Theta}_\iota$, by Lemmas 4.3 and 6.1, we may write equation (6.3) as

$$\begin{aligned}
\dot{\xi} &= \mathcal{O}_4(1) + \tilde{\gamma}\cdot\frac{1}{\iota}(\cos(\Theta - c_0) + \mathcal{O}_4(\sqrt{\mu}))\\
\dot{\Theta} &= \mathcal{O}_4(1) - \tilde{\gamma}\cdot\frac{1}{\iota}(\sin(\Theta - c_0) + \mathcal{O}_4(\sqrt{\mu}))
\end{aligned}$$

where $\tilde{\gamma}$ is as in Lemma 6.1. Observe that because we are integrating over a time interval of length ι, the contribution from the $\mathcal{O}_4(1)$ terms are $\mathcal{O}_4(\iota)$. In addition, the contributions from the $\mathcal{O}_4(\sqrt{\mu})$ terms in the forcing part of the equation are of the form $\mathcal{O}_4(\sqrt{\mu})$. Hence for the desired estimates, it suffices for us to consider the equations without these perturbation terms and to add an error term of the form $\mathcal{O}_4(\iota + \sqrt{\mu})$ to the result afterwards. To integrate these two equations, it is easier to go back to (x, y)-coordinates, in which the equations are

$$\dot{x} = \frac{1}{\iota}\rho\Phi_x(0), \quad \dot{y} = \frac{1}{\iota}\rho\Phi_y(0) \tag{6.5}$$

and their solutions are $x(\iota) = x_0 + \rho\Phi_x(0)$ and $y(\iota) = y_0 + \rho\Phi_y(0)$. It remains to go back to (ξ, Θ)-coordinates.

Applying the variation of constants formula to the equation for $\mathbf{w}$, we obtain

$$\hat{\mathbf{w}}_\iota = e^{\mathcal{A}^s\iota}\mathbf{w_0} + \int_0^\iota e^{\mathcal{A}^s(\iota-\tau)}\Big(\mathcal{O}_4(\sqrt{\mu})\mathbf{w} + \rho\sqrt{\frac{\alpha}{a}}\cdot\frac{1}{\iota}(P_0^s\Phi(0) + \mathcal{O}_4(\sqrt{\mu}))\Big)d\tau, \tag{6.6}$$

from which the bound in the proposition follows via Gronwall's inequality.

To prove $\kappa(V_{\frac{1}{100},\frac{1}{100}}) \subset V_{\frac{11}{20},C_0(1+K_0)}$, we substitute $|\mathbf{w}_0| < \frac{1}{100}$ and the formula for $\hat{\gamma}$ into this bound, getting

$$\begin{aligned} |\hat{\mathbf{w}}_\iota| &< C_0\big(\frac{1}{100} + \hat{\gamma}\frac{1}{\sqrt{\Phi_x^2(0)+\Phi_y^2(0)}}|P_0^s(\Phi(0))| + \mathcal{O}_4(\sqrt{\mu})\big) \\ &< C_0\big(\frac{2}{100} + \frac{1}{2}K_0\big) \ < \ C_0(1+K_0). \end{aligned}$$

The Assumption (H4) is used in the second inequality. As for the ξ-direction, one verifies easily from (6.4) that for $\iota, \sqrt{\mu} << 1$ and $|\xi_0| < \frac{1}{100}$, we have $|\hat{\xi}_\iota| < \frac{11}{20}$. □

6.2. **The map F_T and a derived 2-D system.**

Let us continue to use F_T to denote the time-T map of the forced equation (in spite of the many coordinate changes in between). Then $F_T = G \circ \kappa$ where κ is the kick map in Sect. 6.1, and $G = G_{T-\iota}$ where G_t is the time-t map of the unforced system. Let $V = V_{\frac{1}{100},\frac{1}{100}}$. We have shown that $F_T(V) \subset V$ for $T > M_0\mu^{-1}$: by Proposition 6.1, $\kappa(V) \subset V_{\frac{11}{20},C_0(1+K_0)}$, and by Proposition 5.1, $G(V_{\frac{11}{20},C_0(1+K_0)}) \subset V$.

We now combine the results in Section 5 and Sect. 6.1 to derive a more explicit expression for the map F_T. Writing its component functions as $F_T = ((F_T)_\xi, (F_T)_\Theta, (F_T)_{\mathbf{w}})$, we have, first, that F_T contracts in the ξ and $\mathbf{w}$ directions, namely

$$\begin{aligned} (F_T)_\xi(\xi,\Theta,\mathbf{w}) &= e^{-\frac{a(\mu)}{4}T}\cdot H_\xi(\xi,\Theta,\mathbf{w};T)\ , \\ (F_T)_{\mathbf{w}}(\xi,\Theta,\mathbf{w}) &= e^{-\frac{c}{4}T}\cdot H_{\mathbf{w}}(\xi,\Theta,\mathbf{w};T)\ . \end{aligned}$$

As functions of $((\xi,\Theta,\mathbf{w}),T) \in V \times [M_0\mu^{-1},\infty)$, H_ξ and $H_{\mathbf{w}}$ are (uniformly) bounded in C^3; the precise form of these two mappings is unimportant.

The component $(F_T)_\Theta$ is more interesting: Let $(\hat{\xi}_\iota, \hat{\Theta}_\iota, \hat{\mathbf{w}}_\iota) = \kappa(\xi,\Theta,\mathbf{w})$. Then by Propositions 5.4 and 5.5, we may write

$$(F_T)_\Theta(\xi,\Theta,\mathbf{w}) = \Theta_{T-\iota,\infty}(\hat{\xi}_\iota,\hat{\Theta}_\iota,\hat{\mathbf{w}}_\iota) + e^{-\frac{a(\mu)}{4}T}\cdot\hat{H}_\Theta(\hat{\xi}_\iota,\hat{\Theta}_\iota,\hat{\mathbf{w}}_\iota;T)$$

where

$$\Theta_{t,\infty}(\hat{\xi}_\iota,\hat{\Theta}_\iota,\hat{\mathbf{w}}_\iota) = \hat{\Theta}_\iota + \frac{\hat{\omega}}{\Theta^*}t + \tau(\ln(1+\hat{\xi}_\iota) + \mathcal{O}_3(\sqrt{\mu}))\ , \tag{6.7}$$

the $\mathcal{O}_3(\sqrt{\mu})$ term here being a function of $(\hat{\xi}, \hat{\Theta}, \hat{\mathbf{w}})$, and $\hat{H}_\Theta$ is uniformly bounded in C^3. Substituting in the formulas for $(\hat{\xi}_\iota, \hat{\Theta}_\iota, \hat{\mathbf{w}}_\iota)$ from Proposition 6.1, we obtain

$$(F_T)_\Theta(\xi, \Theta, \mathbf{w}) = \phi(\xi, \Theta, \mathbf{w}) + \frac{\hat{\omega}}{\Theta^*}(T - \iota) + \tau \ln \psi(\xi, \Theta, \mathbf{w}) + e^{-\frac{a(\mu)}{4}T} H_\Theta(\Theta, \xi, \mathbf{w}; T) \tag{6.8}$$

where H_Θ has bounded C^3 norm with respect to the four variables $\xi, \Theta, \mathbf{w}$ and T, and the functions ϕ and ψ are defined by

$$\tan \phi = \frac{(1+\xi)\sin\Theta + \tilde{\gamma}\sin c_0}{(1+\xi)\cos\Theta + \tilde{\gamma}\cos c_0} + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}) \tag{6.9}$$

and

$$\psi = \sqrt{(1+\xi)^2 + \tilde{\gamma}(\tilde{\gamma} + 2(1+\xi)\cos(\Theta - c_0))} + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}). \tag{6.10}$$

Here $\tilde{\gamma} = \gamma(1 + \mathcal{O}(\sqrt{\mu}))$ (Lemma 6.1). We remark that both ϕ and ψ depend on the variable $\mathbf{w}$ through the $\mathcal{O}_4(\iota + \sqrt{\mu})$ terms on the right-hand side of (6.9) and (6.10).

We now introduce the 2-dimensional systems $\mathcal{F}_T : \mathbb{A} \to \mathbb{A}$ equivalent to the restriction of F_T to a region of a center manifold. Let $\mathbb{A} = \{(\Theta, \xi) : |\xi| < \frac{1}{100}\}$, and let $\mathcal{W}^c_T$ be a family of center manifolds for F_T (such as those given by Proposition 3.1)[3] defined on $\mathbb{A}$, i.e. $\mathcal{W}^c_T = \text{graph}(\mathbf{W}_T)$ for some $\mathbf{W}_T : \mathbb{A} \to E^s$. We let $\ell^c_T : \mathbb{A} \to \mathbb{A} \times E^s$ be the lift from $\mathbb{A}$ to $\mathcal{W}^c_T$, i.e. $\ell^c_T(\Theta, \xi) = (\xi, \Theta, \mathbf{W}_T(\Theta, \xi))$, and let $\pi^c : \mathbb{A} \times E^s \to \mathbb{A}$ be the projection map. Then the family of interest $\mathcal{F}_T$ is given by

$$\mathcal{F}_T = \pi^c \circ F_T \circ \ell^c_T \ .$$

Notice that we have written Θ as the leading coordinate in $\mathbb{A}$; this is to reflect the fact that from this point on, Θ will play a more important role than ξ.

To get a sense of the geometric properties of $\mathcal{F}_T$, let $\mathcal{W}^c$ be given by Proposition 3.1 for the moment. Since $\|\mathbf{W}_T\|_{C^1} << 1$, it follows that $\mathcal{F}_T \approx ((F_T)_\Theta, (F_T)_\xi) \mid_{V \cap \{\mathbf{w}=0\}}$ in the C^1 metric. In particular, when T is sufficiently large relative to $a(\mu)^{-1}$, $\mathcal{F}_T$ is strongly contractive in the ξ-direction. When this contraction is sufficiently strong, and ι and μ are sufficiently small, $\mathcal{F}_T$ is approximated by a 1-dimensional map of the form

$$\Theta \mapsto \tan^{-1}\left(\frac{\sin\Theta + \tilde{\gamma}\sin c_0}{\cos\Theta + \tilde{\gamma}\cos c_0}\right) + \Omega_0 + \tau \ln \sqrt{1 + \tilde{\gamma}(\tilde{\gamma} + 2\cos(\Theta - c_0))} \tag{6.11}$$

where $\Omega_0, c_0, \tilde{\gamma}$ and τ are constants. Notice that when $\tau = 0$, the mapping in (6.11) is a circle diffeomorphism. In particular, it remains injective when $|\tau|\tilde{\gamma}$ is small, and loses its injectivity (developing two critical points) as $|\tau|\tilde{\gamma}$ is increased. Larger values of $|\tau|\tilde{\gamma}$ lead to more expansion in the circle map. We point out that this is consistent with the assumptions on τ in the hypotheses of Theorems 1, 2 and 3; see Sect. 2.1.

[3]Center manifolds are, in general, not unique.

6.3. **Proofs of Theorems 1 and 3.**

The C^1 information on $\mathcal{F}_T$ from the last subsection is sufficient for proving Theorems 1 and 3. First, we have two estimates that are results of straightforward computations:

Lemma 6.2. *For $(\Theta, \xi) \in \mathbb{A}$, under the assumption that $\gamma < \frac{1}{2}$, we have*
(i) $\frac{1}{2} < \left|\frac{\partial}{\partial\Theta}\phi(\xi, \Theta, \mathbf{W}_T(\Theta, \xi))\right| < 2$.
(ii) $\frac{\partial}{\partial\Theta} \ln \psi(\xi, \Theta, \mathbf{W}_T(\Theta, \xi)) = -\tilde{\gamma}(1 + \xi) \sin(\Theta - c_0)\psi^{-2} + \mathcal{O}(\iota + \sqrt{\mu})$.

Proof of Theorem 1: Let $A_1 = \{(\Theta, \xi) \in \mathbb{A} : |\Theta - (\frac{1}{2}\pi - c_0)| < \frac{\pi}{4}\}$ and $A_2 = \{(\Theta, \xi) \in \mathbb{A} : |\Theta - (\frac{3}{2}\pi - c_0)| < \frac{\pi}{4}\}$. Assuming $\gamma < \frac{1}{2}$ and $|\tau| > 20\gamma^{-1}$, Lemma 6.2 says that for $(\Theta, \xi) \in (A_1 \cup A_2)$,

$$\left|\frac{\partial}{\partial\Theta}(\phi + \tau \ln \psi)(\Theta, \xi, \mathbf{W}_T(\Theta, \xi))\right| > 5.$$

Consequently, $\mathcal{F}_T$ wraps each of the rectangles A_1 and A_2 all the way around $\mathbb{A}$ in the Θ-direction. Assuming $\frac{a(\mu)}{4}T$ is large enough so that the contraction in the ξ-direction is sufficiently strong, a standard construction gives an invariant set conjugate to the full 2-shift, i.e. a horseshoe. This in turn implies F_T has a topological horseshoe in the Hopf attractor Λ. □

Proof of Theorem 3: With $|\tau| < (100\gamma)^{-1}$, the term $\tau \ln \psi$ is dominated by ϕ. By Lemma 6.2(i), $|\frac{\partial}{\partial\Theta}\phi| > \frac{1}{2}$. If contraction in the ξ-direction is sufficiently strong, then by standard textbook arguments [HPS], $\mathcal{F}_T$ has a center manifold which is an invariant circle C^1-near $\{\xi = 0\}$ and a transversal stable foliation with 1-dimensional leaves defined on a neighborhood of the invariant circle. By iterating forward, it follows that every point in $\mathbb{A}$ lies on one such stable leaf. We denote the ℓ_T^c-image of this invariant circle by $\tilde{W}^c$, and the ℓ_T^c-image of the 1-dimensional stable foliation by $\tilde{W}^s$. Then $\Lambda = \tilde{W}^c$ is the Hopf attractor for F_T. To complete the proof of Theorem 3, it remains to produce for every $u_0 \in \mathcal{U}$, a point $v_0 \in \Lambda$ such that $|F_T^n(u_0) - F_T^n(v_0)| \to 0$ as $n \to \infty$. By Proposition 3.4, there exist $n \in \mathbb{Z}^+$ and $\hat{u}_0 \in V \cap \mathcal{W}^c$ such that $F_T^n(u_0) \in \mathcal{W}^{ss}(\hat{u}_0)$. Then $v_0 \in \Lambda \cap \tilde{W}^s(\hat{u}_0)$ has the desired properties. □

6.4. **Further analytic preparation for Theorem 2.**

There are two remaining estimates needed for the proof of Theorem 2 that we wish to dispose of before moving to the next set of ideas:

Proposition 6.2. *There exists $C_1 > 1$ independent of T (but possibly depending on μ) such that for all $z, z' \in \mathbb{A}$,*

$$\frac{|\det(D\mathcal{F}_T(z))|}{|\det(D\mathcal{F}_T(z'))|} < C_1 \ .$$

Proof: We will work with $F_T|_{\mathcal{W}^c\cap V}$ instead $\mathcal{F}_T$, which clearly suffices. The crux of the idea lies in the fact that all orbits of the unforced flow G_t are asymptotic to the limit cycle $\Omega = \{\xi = 0, \mathbf{w} = 0\}$. We divide the proof into two parts:

Part 1. Let $G_t^c = G_t|_{W^c}$ be the 2-dimensional flow on $\{\mathbf{w} = 0\}$. We claim that for all $u_0, u_0' \in W^c \cap V$ and $t > 0$,

$$\frac{\det(DG_t^c(u_0))}{\det(DG_t^c(u_0'))} < C' \tag{6.12}$$

for some C' depending possibly on μ. To see this, observe first that (6.12) holds for $u_0, u_0' \in \Omega$. This is because $u_0' = G_{t_0}(u_0)$ for some $t_0 < p$, the period of Ω, so

$$DG_t^c(u_0) = (DG_{t_0}^c)^{-1}(G_t^c(u_0')) \cdot DG_t^c(u_0') \cdot DG_{t_0}^c(u_0),$$

and $\det(DG_s^c), |s| < p$, are bounded above and below. Next we know from the last paragraph of Sect. 5.3 that (i) every point $v_0 \in W^c \cap V$ lies in a 1-dimensional stable curve $W_{u_0}^s$ for some $u_0 \in \Omega$, (ii) $|G_t(u_0) - G_t(v_0)| \to 0$ exponentially fast, and (iii) this convergence is uniform in v_0. We therefore have (6.12) for the pair u_0 and v_0, with the constant depending possibly on μ, the convergence rate. Finally, to prove (6.12) for any $v_0, v_0' \in W^c \cap V$, we first compare v_0 to $u_0 \in \Omega$ with $v_0 \in W_{u_0}^s$, then v_0' to $u_0' \in \Omega$ with $v_0' \in W_{u_0'}^s$, and finally u_0 and u_0'.

Part 2. We view F_T via the following sequence of transformations: Starting from u_0, we first apply the kick map. Then slide along a $\mathcal{W}^{ss}$-leaf to obtain $v_0 \in W^c \cap \mathcal{W}^{ss}(\kappa(u_0))$. Next we apply $G_{T-\iota}$ to v_0, and finally slide along $\mathcal{W}^{ss}$ at $G_{T-\iota}(v_0)$ back to $\mathcal{W}^c$ to obtain $F_T(u_0)$. The $G_{T-\iota}$ part is treated in Part 1. It remains to show that each of the other transformations changes infinitesimal area (on the relevant surfaces) by factors that are bounded above and below. From Sect. 6.1, we see that κ restricted to $\mathcal{W}^c \cap V$ has the desired property, and it maps $\mathcal{W}^c \cap V$ to a surface that is a graph of a function from $\mathbb{A}$ to E^s with bounded slope. Next, we apply Proposition 3.3(b) to conclude that since the holonomy map between $\kappa(\mathcal{W}^c \cap V)$ and W^c obtained by sliding along $\mathcal{W}^{ss}$-leaves is bi-Lipschitz, it transforms area in a bounded way. The same reasoning applies to the last sliding map. □

Unlike Theorems 1 and 3, the proof of Theorem 2 requires C^3 estimates for the maps $\mathcal{F}_T$. We have obtained the needed estimates for F_T but not yet for ℓ_T^c, or equivalently, $\mathbf{W}_T$.

Proposition 6.3. *There is a family of center manifolds $\mathcal{W}_T^c$ for F_T with the properties that*

(a) $\mathcal{W}_T^c = graph(\mathbf{W}_T(\Theta, \xi))$ for some $\mathbf{W}_T : \mathbb{A} \to E^s$;

(b) as a function on $\mathbb{A} \times (M_0\mu^{-1}, \infty)$,

$$\|\mathbf{W}_T\|_{C^3} < Ke^{-\frac{c}{3}T}$$

where c is as in Proposition 5.1.

Proposition 6.3 is proved in Appendix B. Notice that the center manifold of Proposition 6.3 is not necessarily part of the center manifold $\mathcal{W}^c$ in Proposition 3.1, as center manifolds are not necessarily unique. The family $\mathcal{F}_T$ depends on the choice of $\mathcal{W}^c$, and we will assume from here on that they have the smoothness in Proposition 6.3. The results in Proposition 3.4 carry over easily to these new center manifolds since two different center manifolds are connected by $\mathcal{W}^{ss}$-leaves and the Lebesgue measure classes on them are preserved by the holonomy map defined by sliding along $\mathcal{W}^{ss}$-leaves (Proposition 3.3(b)).

7. Strange Attractors with SRB Measures

This section is devoted to the proof of Theorem 2. Via the use of invariant manifolds, the problem has been reduced to considering the 2-dimensional family $\mathcal{F}_T : \mathbb{A} \to \mathbb{A}$ introduced in Sect. 6.2. Results for this family are proved, in turn, by appealing to the general theory of rank one maps developed in [WY1] and [WY4]. In these two papers, conditions guaranteeing the existence of strange attractors with SRB measures and other dynamical properties are identified. These conditions are reviewed in Sect. 7.1 and verified for the family $\mathcal{F}_T$ in Sect. 7.2. The logic of the entire proof is recapitulated at the end of Sect. 7.2.

7.1. **Review of results from** [WY1] **and** [WY4]**.**

The results below are valid in n dimensions for any finite $n \geq 2$. We review only the 2-dimensional version since it is all that is needed for present purposes. Notation in this subsection is independent of the rest of this paper, although (Θ, ξ) will correspond exactly to the variables in $\mathbb{A}$.

The setting is as follows. Let $M = \mathbb{S}^1 \times [-1, 1]$, and let $F_{a,b} : M \to M$ be a family of maps with parameters a, b. Here $a \in [a_0, a_1] \subset \mathbb{R}$, and $b \in B_0$ where $B_0 \subset \mathbb{R} \setminus \{0\}$ is an arbitrary subset with 0 as an accumulation point. Points in M are denoted by (Θ, ξ) with $\Theta \in \mathbb{S}^1$ and $\xi \in [-1, 1]$. A number of conditions are imposed on this family.

(C0) Regularity conditions

(i) *For each $b \in B_0$, the function $(\Theta, \xi, a) \mapsto F_{a,b}(\Theta, \xi)$ is C^3;*

(ii) *for each $a \in [a_0, a_1]$ and $b \in B_0$, $F_{a,b}$ is an embedding of M into itself;*

(iii) *there exists $C > 0$ independent of b such that for all (a, b), $b \neq 0$,*

$$\frac{|\det DF_{a,b}(\Theta, \xi)|}{|\det DF_{a,b}(\Theta', \xi')|} \leq C \qquad \forall (\Theta, \xi), (\Theta', \xi') \in M.$$

(C1) Existence of singular limit *There exist $F_{a,0} : M \to \mathbb{S}^1 \times \{0\}$, $a \in [a_0, a_1]$, such that the maps $(\Theta, \xi, a) \mapsto F_{a,b}(\Theta, \xi)$ converge in the C^3 norm to the map $(\Theta, \xi, a) \mapsto F_{a,0}(\Theta, \xi)$ as $b \to 0$.*

Identifying $\mathbb{S}^1 \times \{0\}$ with $\mathbb{S}^1$, we refer to $F_{a,0}$ as well as its restriction to $\mathbb{S}^1 \times \{0\}$, i.e. the family of 1D maps $f_a : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f_a(\Theta) = F_{a,0}(\Theta, 0)$, as the *singular limit* of $F_{a,b}$. The rest of our conditions are imposed on the singular limit alone.

(C2) Existence of a sufficiently expanding map from which to perturb *There exists $a^* \in (a_0, a_1)$ such that $f = f_{a^*}$ has the following properties: f has nondegenerate critical points, so that its critical set $\mathcal{C} = \{f' = 0\}$ is finite. We assume there are numbers $c_1 > 0$, $N_1 \in \mathbb{Z}^+$, and a neighborhood I of $\mathcal{C}$ such that*

(i) *f is expanding on $\mathbb{S}^1 \setminus I$ in the following sense:*
  (a) *if $\Theta, f\Theta, \cdots, f^{n-1}\Theta \notin I$, $n \geq N_1$, then $|(f^n)'\Theta| \geq e^{c_1 n}$;*
  (b) *if $\Theta, f\Theta, \cdots, f^{n-1}\Theta \notin I$ and $f^n\Theta \in I$, any n, then $|(f^n)'\Theta| \geq e^{c_1 n}$;*

(ii) *$f^n\Theta \notin I$ $\forall \Theta \in C$ and $n > 0$;*

(iii) *derivatives on I are controlled as follows:*
  (a) *$|f''|$ is bounded away from 0;*
  (b) *by following a critical orbit, every $\Theta \in I \setminus \mathcal{C}$ has a recovery time $n(\Theta) \geq 1$ with the property that $f^j\Theta \notin I$ for $0 < j < n(\Theta)$ and $|(f^{n(\Theta)})'\Theta| \geq e^{c_1 n(\Theta)}$.*

We remark that **(C2)** is not assumed in [WY1]. Instead, the authors of [WY1] had originally used the assumption that f_{a^*} is a Misiurewicz map. This assumption involves a number of properties some of which are hard to check or not needed. In [WY2], this condition was replaced by **(C2)**, which is more directly checkable though cumbersome to state. That the results in [WY1] remain valid when the Misiurewicz condition is replaced by **(C2)** is proved in Lemma A.1 in the Appendix of [WY2]. **(C2)** is also used in [WY4], which contains generalizations of some of the results in [WY1] to higher dimensions.

To state the next condition, we need a notion of *smooth continuation* for certain orbits. Let $\mathcal{C}_a$ denote the critical set of f_a. For $\hat{\Theta} = \hat{\Theta}(a^*) \in \mathcal{C}_{a^*}$, the continuation $\hat{\Theta}(a)$ of $\hat{\Theta}$ to a near a^* is defined to be the unique critical point of f_a near $\hat{\Theta}$. If p is a hyperbolic periodic point of f_{a^*}, then $p(a)$ is the unique periodic point of f_a near p having the same period. It is a fact that in general, if p is a point whose f_{a^*}-orbit is bounded away from $\mathcal{C}_{a^*}$, then for a sufficiently near a^*, there is a unique point $p(a)$ with the same symbolic itinerary under f_a (see Sect. 4.2 of [WY5]).

(C3) Parameter transversality *For each $\hat{\Theta} \in \mathcal{C}_{a^*}$, let $p = f_{a^*}(\hat{\Theta})$, and let $\hat{\Theta}(a)$ and $p(a)$ denote the continuations of $\hat{\Theta}$ and p respectively. Then*

$$\frac{d}{da} f_a(\hat{\Theta}(a)) \neq \frac{d}{da} p(a) \qquad \text{at } a = a^*.$$

(C4) Nondegeneracy at "turns"

$$\frac{\partial}{\partial \xi} F_{a^*,0}(\hat{\Theta}, 0) \neq 0 \qquad \forall \hat{\Theta} \in \mathcal{C}_{a^*}.$$

We summarize below those parts of [WY1] that are relevant for Theorem 2.

Theorem A (Existence of SRB measures and implications [WY1]**)** *Assume the family* $\{F_{a,b}\}$ *satisfies conditions* **(C0)-(C4)**. *Then for each* $b \neq 0$ *for which* $|b|$ *is small enough, there is a subset* $\Delta_b \subset [a_0, a_1]$ *of positive Lebesgue measure such that for all* $a \in \Delta_b$, *the following hold for* $F_{a,b}$*:*

(i) $F_{a,b}$ *admits at least one and at most finitely many ergodic SRB measures* $\nu_1, \cdots, \nu_r$*;*

(ii) *with respect to Lebesgue measure, the orbit starting from almost every* $(\Theta, \xi) \in M$ *has a positive Lyapunov exponent and is generic with respect to* ν_i *for some* i.

The parameter selection part of Theorem A borrows ideas from [BC]. It is in fact proved in [WY1] that with respect to Lebesgue measure, a.e. $(\Theta, \xi) \in M$ lies in a stable manifold of a point that is typical with respect to some ν_i. Genericity with respect to ν_i follows. The positivity of Lyapunov exponents follows also as SRB measures have positive Lyapunov exponents by definition.

In the rest of the discussion, let us refer to the set of parameters (a, b) for which the conclusions of Theorem A hold as the set of "good parameters".

Conditions **(C1)**–**(C4)** alone do not imply the uniqueness of SRB measures; additional assumptions are need. One approach is to mimick conditions for Markov chains. Let $J_1, \cdots, J_\ell$ be the intervals of monotonicity of f_{a^*}, and let $P = (p_{i,j})$ be the matrix defined by

$$p_{i,j} = \begin{cases} 1 & \text{if } \ f(J_i) \supset J_j, \\ 0 & \text{otherwise}\ , \end{cases}$$

and let $P^n = (p^n_{i,j})$ denote the nth power of the matrix P.

Theorem B ([WY1], [WY2]) *The setting and hypotheses are as in Theorem A. Additionally, we assume* $e^{c_1} > 2$ *where* c_1 *is as in* **(C2)**. *Then the following hold for all "good parameters":*

(i) If for each $i, j \in \{1, \cdots, \ell\}$, *there exists* $n = n(i, j)$ *such that* $p^n_{i,j} > 0$, *then* $F_{a,b}$ *has a unique SRB measure* ν, *which is therefore ergodic.*

(ii) If there exists n *such that* $p^n_{i,j} > 0$ *for all* $i, j \in \{1, \cdots, \ell\}$, *then* $(F_{a,b}, \nu)$ *is mixing.*

This result can be interpreted to say that when $e^{c_1} > 2$, the maps $F_{a,b}$ with "good" (a, b) have no local complexity, so that their ergodic and mixing properties can be expressed in terms of sets corresponding to intervals of monotonicity for f_{a^*}. In the cited papers, assertion (i) is in fact not explicitly stated, but it follows from the proof of assertion (ii).

Finally, we remark that [WY1] contains not only the results cited in Theorems A and B but a comprehensive dynamical profile for the maps $F_{a,b}$ corresponding to "good parameters", including geometric structures of the attractors (such as approximations by horseshoes) and statistical properties (such as correlation decay and central limit theorems). The corresponding results in n-dimensions, $n \geq 2$, are proved in [WY4] and in a forthcoming preprint by the same authors. We have opted to limit the statement of results in the present paper to SRB measures and Lyapunov

exponents, but all aspects of this larger dynamical picture in fact apply once the requisite conditions are checked.

7.2. **Proof of Theorem 2.**

We now fix ρ, μ and ι, and let $\mathcal{F}_T : \mathbb{A} \to \mathbb{A}$, $T > M_0\mu^{-1}$, be the family introduced in Sect. 6.2. To apply the results in the last subsection, first we put this family into the form $\{F_{a,b}\}$. Given T, there exist unique $a \in [0, 2\pi)$ and $b \in B_0 = \{\frac{1}{n}\}_{n\in\mathbb{Z}^+}$ such that

$$T - \iota = \Theta^*\hat{\omega}^{-1}(2\pi b^{-1} + a) \ .$$

We define $\mathcal{F}_T = F_{a,b}$, and verify **(C0)-(C4)** for the family $\{F_{a,b}\}$ so defined:

Verification of (C0): **(C0)**(i) is equivalent to $\mathcal{F}_T$ being C^3 as a function of Θ, ξ and T. This is true because $\mathcal{F}_T = \pi^c \circ F_T \circ \ell^c_T$ (Sect. 6.2), ℓ^c_T is C^3 (Proposition 6.3), F_T is C^3 by smooth dependence of solution on initial condition and on T (see [H]), and π^c is clearly C^3. **(C0)**(ii) is true because F_T restricted to $\mathcal{W}^c$ is an embedding, and **(C0)**(iii) is proved in Proposition 6.2.

Verification of (C1): We claim that the required convergence holds with

$$F_{a,0}(\Theta, \xi) = \phi(\Theta, \xi, \mathbf{0}) + a + \tau \ln \psi(\Theta, \xi, \mathbf{0})$$

where ϕ, ψ are as in (6.9) and (6.10). This is because $F_{a,b} = \pi^c \circ G_{\Theta^*\hat{\omega}^{-1}(2\pi b^{-1}+a)} \circ \kappa \circ \ell^c_T$, and we have proved the following convergence as functions of Θ, ξ and T: (i) ℓ^c_T converges in C^3 to ℓ^c_∞ where $\ell^c_\infty(\Theta, \xi) = (\Theta, \xi, 0)$ (Proposition 6.3) and (ii) $G_{\Theta^*\hat{\omega}^{-1}(2\pi b^{-1}+a)}$ converges in C^3 as $b \to 0$ (Proposition 5.5).

Verification of (C2): Let $f_a : \mathbb{S}^1 \to \mathbb{S}^1$ be given by $f_a(\Theta) = F_{a,0}(\Theta, 0)$. Then

$$f_a(\Theta) = \phi(\Theta) + a + \tau \ln \psi(\Theta) \tag{7.1}$$

where

$$\phi(\Theta) = \tan^{-1} \frac{\sin\Theta + \tilde{\gamma}\sin c_0}{\cos\Theta + \tilde{\gamma}\cos c_0} + \mathcal{O}_4(\iota + \sqrt{\mu})$$
$$\psi(\Theta) = \sqrt{1 + 2\tilde{\gamma}\cos(\Theta - c_0) + \tilde{\gamma}^2} + \mathcal{O}_4(\iota + \sqrt{\mu}).$$

When $L := |\tau|\gamma$ is sufficiently large, $\tau \ln \psi$ is the dominating term in f_a. As noted in Lemma 6.2, f_a has two critical points, c_1 and c_2, near c_0 and $c_0 + \pi$.

The following notation is used below: We let $C = \{c_1, c_2\}$ be the critical set of f_a and let C_δ denote the δ-neighborhood of C in $\mathbb{S}^1$. The following hold for all $L > L_0$ for some large L_0 to be determined.

Proposition 7.1. *Let $\sigma = L^{-\frac{1}{4}}$. Then for any given interval $\Delta_0 \subset [0, 2\pi)$ of length $= 5\sigma$, there exists $a^* \in \Delta_0$ so that $\{f^n_{a^*}(C)\}_{n\geq 1} \cap C_\sigma = \emptyset$.*

Lemma 7.1. *Let $I = C_{\delta_0}$ where $\delta_0 = L^{-\frac{3}{4}}$.*
(a) For $\Theta \in \mathbb{S}^1 \setminus I$, $|f_a'(\Theta)| > \frac{2}{5}L^{\frac{1}{4}}$.
(b) Let $f = f_{a*}$ be as in Proposition 7.1, and let Θ and $c \in C$ be such that $|\Theta - c| < \delta_0$. Let $n(\Theta)$ be the smallest n such that $|f^n(\Theta) - f^n(c)| > \frac{1}{4}\sigma$. Then $(f^n)'(\Theta) > L^{\frac{1}{6}}$.

Proofs of Proposition 7.1 and Lemma 7.1 are similar to proofs of analogous results in [WY2]. They are included in Appendix C for completeness. **(C2)** follows with $e^{c_1} = L^{\frac{1}{6}}$.

Verification of (C3): Let a^* be as in Proposition 7.1, and let $p(a)$ be as in **(C3)**. We use the following fact proved in Sect. 4.2 of [WY5]: For all $c \in C$, the critical set of f_{a^*},

$$\frac{d}{da}p(a)\bigg|_{a=a^*} = -\sum_{k=1}^{\infty} \frac{[(\partial_a f_a)(f^k c)]_{a=a^*}}{(f_{a^*}^k)'(f_{a^*}c)}.$$

Because the functions $\Theta \mapsto \phi(\Theta)$ and $\psi(\Theta)$ do not depend on a, we have $\partial_a f_a(\Theta) = 1$ for all $\Theta \in \mathbb{S}^1$. The result above implies therefore that for $c \in C$,

$$\left|\frac{d}{da}f_a(c) - \frac{d}{da}p(a)\right|_{a=a*} > 1 - \sum_{k=1}^{\infty} \frac{1}{L^{\frac{k}{6}}} \neq 0$$

provided L is large enough.

Verification of (C4): Since $|\partial_\xi \phi(\Theta, \xi, 0)| < 2$, $|\partial_\xi \ln \psi(\Theta, \xi, 0)| > \frac{1}{10}\gamma$ and $|\tau| = L\gamma^{-1}$, we have $|\partial_\xi F_{a,0}(\Theta, \xi)| > \frac{1}{10}L$.

This completes the verification of the conditions in Theorem A. As for the hypotheses of Theorem B, clearly we may assume $e^{c_1} = L^{\frac{1}{6}} > 2$. Increasing L further, one guarantees easily that $p_{i,j} = 1$ for all i, j.

Proof of Theorem 2: The main steps of the proof are as follows.

(1) First we standardize coordinates by, among other things, blowing up the phase space by a factor $\sim \mu^{-\frac{1}{2}}$ so that the limit cycle is a circle of radius 1; see Section 4.

(2) For each admissible μ, ρ, ι and T, we show that the time-T map F_T of the forced equation sends an annular region of the center manifold, namely $V \cap \mathcal{W}_T^c$, into itself, and derive a fairly explicit form of F_T for large T; see Sects. 5.1–6.1.

(3) We then project F_T to the annulus $\mathbb{A}$, obtaining a family $\mathcal{F}_T$ smoothly conjugate to F_T but defined on a single domain; see Sect. 6.2.

(4) In this section, previous results on rank one maps are applied to the family $\mathcal{F}_T$, giving, for a positive measure set of T, the existence of SRB measures and certain properties for Lebesgue-*a.e.* initial condition. These properties include positive Lyapunov exponents and genericity with respect to SRB measures. Moreover, by the correspondence between T and a, the set of "good parameters" have the properties asserted in the last line of Theorem 2. These results are easily passed back to $F_T|_{\mathcal{W}_T^c \cap V}$.

(5) Finally, via the use of invariant manifolds, we pass the results for $F_T|_{\mathcal{W}^c_T \cap V}$ to solutions starting from "almost every" initial condition in an open set $\mathcal{U}$ in the phase space, "a.e." here referring to "a.e. transversal to $\mathcal{W}^{ss}$"; see Section 3, Proposition 3.4 in particular. □

8. Application: The Brusselator

In this section we apply the general results proved in Sections 3–7 to a concrete model, namely the Brusselator in one physical dimension. The Brusselator is well known to undergo a Hopf bifurcation when certain parameters are varied; see e.g. [HKW]. To apply Theorems 1 and 2, we need an additional piece of information not known previously, namely that the twist factor τ is sufficiently large. We will carry out the normal form computation in the Dirichlet case, leaving the Neumann case (which is straightforward) to the reader.

8.1. **The Dirichlet case.**

The model is as defined in Sect. 2.4, with $d_1 = \pi^{-2}$. We let $U = u - a$, $V = v - ba^{-1}$, and study the system

$$\begin{aligned} U_t &= \pi^{-2} U_{xx} + (b-1)U + a^2 V + h(U,V) + \rho \sin \pi x \; p_{T,\iota}(t), \\ V_t &= \theta \pi^{-2} V_{xx} - bU - a^2 V - h(U,V), \end{aligned} \tag{8.1}$$

where

$$h(U,V) = ba^{-1}U^2 + 2aUV + U^2 V$$

with boundary conditions $U(0,t) = U(1,t) = 0$ and $V(0,t) = V(1,t) = 0$.

A. Equations for Fourier coefficients and Hopf bifurcation

Using $\{\sin\ n\pi x\}_{n\geq 1}$ as a basis of $L^2(0,1)$, we expand $U(x,t)$ and $V(x,t)$ as

$$U(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x), \qquad V(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin(n\pi x). \tag{8.2}$$

Then (8.1) is equivalent to (u_n, v_n) satisfying

$$\begin{pmatrix} \dot{u}_1 \\ \dot{v}_1 \end{pmatrix} = \mathcal{L}_1 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \int_0^1 h(U,V) \sin(\pi x) dx + \begin{pmatrix} \rho \\ 0 \end{pmatrix} \; p_{T,\iota}(t), \tag{8.3}$$

$$\begin{pmatrix} \dot{u}_n \\ \dot{v}_n \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \int_0^1 h(U,V) \sin(n\pi x) dx, \qquad n \geq 2\ , \tag{8.4}$$

where

$$\mathcal{L}_n = \begin{pmatrix} b - 1 - n^2 & a^2 \\ -b & -a^2 - \theta n^2 \end{pmatrix}\ , \qquad n \geq 1. \tag{8.5}$$

To put the equations above in the form of (2.3), i.e.

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u) + \rho \Phi(u)\; p_{T,\iota}(t),$$

we fix a and θ, and let $\mu = \frac{1}{2}(b-2-a^2-\theta)$ be our bifurcation parameter. Then $u = (u_1, v_1, u_2, v_2, \cdots, u_n, v_n, \cdots)$, $\mathcal{A}_\mu$ is represented by an infinite matrix whose diagonal terms are $\mathcal{L}_n$, $\Phi(u) = (1, 0, 0, 0, \cdots)$, and so on.

Next we compute the eigenvalues of $\mathcal{A}_\mu$. Clearly, λ is an eigenvalue of $\mathcal{A}_\mu$ if and only if λ is an eigenvalue of $\mathcal{L}_n$ for some $n \geq 1$. By a direct computation, we find that the eigenvalues of $\mathcal{L}_n$ are

$$\lambda_n^{\pm} = \frac{1}{2}\left(\xi_n \pm \sqrt{\xi_n^2 - 4\eta_n}\right)$$

where

$$\xi_n = 2\mu + (1+\theta)(1-n^2), \qquad \eta_n = a^2(n^2+1) - \theta n^2(2\mu + 1 + \theta + a^2 - n^2).$$

Lemma 8.1. *Assume $a \geq 1$ and $0 < \theta << 1$. When $\mu = 0$, we have that*

(i) $\lambda_1^{\pm} = \pm i\sqrt{(2-\theta)a^2 - \theta^2}$ *and*

(ii) *there exists a constant $c > 0$ independent of a such that $Re(\lambda_n^{\pm}) < -c$ for all $n \geq 2$.*

Moreover, $\frac{d}{d\mu}Re(\lambda_1) = 1$ at $\mu = 0$.

Proof: (i) is straightforward, as is $\frac{d}{d\mu}Re(\lambda_1) = 1$. For (ii) we let $n \geq 2$. At $\mu = 0$ we have (i) $\xi_n < -3$ and (ii) $\eta_n > 0$ because $0 < \theta << 1$. If $\xi_n^2 - 4\eta_n$, i.e., the term inside of the square root sign for $\lambda_n^{\pm}$, is negative, then the real part of $\lambda_n^{\pm}$ is less than $-\frac{3}{2}$. Otherwise $\lambda_n^{\pm}$ are real and we have

$$\lambda_n^- < \lambda_n^+ = \frac{2\eta_n}{\xi_n - \sqrt{\xi_n^2 - 4\eta_n}} < \frac{\eta_n}{\xi_n}.$$

Using $a \geq 1$ and $0 < \theta << 1$ we have at $\mu = 0$,

$$\frac{\eta_n}{\xi_n} = \frac{a^2(n^2+1) + \theta n^4 - \theta n^2(1+\theta+a^2)}{(1+\theta)(1-n^2)} \leq \frac{\frac{1}{2}a^2n^2}{2(1-n^2)} < -\frac{1}{4}.$$

This proves (ii). □

It follows that E^c is the subspace spanned by (u_1, v_1) while the stable subspace E^s is its orthogonal complement. It remains to compute the normal form of the unforced flow restricted to its center manifold.

B. Canonical form of linear and high order terms

We first transform equation (8.3) to an equation with a standard linear part. We introduce new coordinates $\begin{pmatrix} u \\ v \end{pmatrix}$ by $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}$ where

$$P = \begin{pmatrix} 1 & 0 \\ -a^{-2}(\mu + a^2 + \theta) & -a^{-2}\omega \end{pmatrix}, \quad P^{-1} = -\frac{a^2}{\omega}\begin{pmatrix} -a^{-2}\omega & 0 \\ a^{-2}(\mu + a^2 + \theta) & 1 \end{pmatrix},$$

and

$$\omega = \sqrt{(2-\theta)a^2 - \theta^2 - \mu^2 - 2\mu\theta}.$$

This change of coordinate can be written explicitly as

$$u_1 = u, \quad v_1 = -a^{-2}(\mu + a^2 + \theta)u - a^{-2}\omega v\ .$$

Then with $w = (u_2, v_2, \cdots)$, we have

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -\frac{\mu+\theta}{\omega} \end{pmatrix} \mathcal{F}(u,v,w) + \rho \begin{pmatrix} 1 \\ -\frac{\mu+a^2+\theta}{\omega} \end{pmatrix} p_{T,\iota}(t) \tag{8.6}$$

$$\begin{pmatrix} \dot{u}_n \\ \dot{v}_n \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathcal{G}_n(u,v,w), \tag{8.7}$$

$$\mathcal{F}(u,v,w) = \int_0^1 h(U,V)\sin(\pi x)dx, \qquad \text{and} \qquad \mathcal{G}_n(u,v,w) = \int_0^1 h(U,V)\sin(n\pi x)dx.$$

Note that both $\mathcal{F}(u,v,w)$ and $\mathcal{G}_n(u,v,w)$ are terms of order ≥ 2 in u, v and w.

Next we compute $\mathcal{F}$ and $\mathcal{G}_n$. We are going to retain only the terms needed for the coefficient k_1 in (1.4). Let $z = u + iv$. Note that equations (8.3) and (8.4) without the forcing has a center manifold given by the graph of a function $w = w(z, \bar{z})$ with $w(z, \bar{z}) = \mathcal{O}(|z|^2)$. In what follows, we regard a term in z, w as $\mathcal{O}(|z|^m)$ if, by letting $w = \mathcal{O}(|z|^2)$, it is $\mathcal{O}(|z|^m)$. We also set $\mu = 0$. It follows from a straightforward computation that

$$\begin{aligned} \mathcal{F}(u,v,w) &= \frac{4}{3\pi}(2a^{-1} - \theta a^{-1} - a)u^2 - \frac{8}{3\pi}a^{-1}\omega uv - \frac{3}{8}u^2\left((1+\theta a^{-2})u + a^{-2}\omega v\right) \\ &\quad + \sum_{n=2}^{\infty} I_n\left(4a^{-1}uu_n + 2auv_n - 2a^{-1}\omega vu_n\right) + \mathcal{O}(|z|^4), \\ \mathcal{G}_n(u,v,w) &= I_n\left(\left(2a^{-1} - \theta a^{-1} - a\right)u^2 - 2a^{-1}\omega uv\right) + \mathcal{O}(|z|^3) \end{aligned}$$

where

$$I_n = \int_0^1 \sin^2(\pi x)\sin(\pi n x)dx = \begin{cases} 0, & n = 2k, \\ \frac{4}{\pi n(4-n^2)}, & n = 2k+1. \end{cases}$$

C. Computation of normal forms

To compute the normal form up to the order three we need to retain only the second order terms in $w(z, \bar{z})$. The equations for $(u_n, v_n), n \geq 2$, are as follows

$$\begin{aligned} \begin{pmatrix} \dot{u}_n \\ \dot{v}_n \end{pmatrix} &= \mathcal{L}_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} + I_n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Big(\frac{1}{2}\left(2a^{-1} - \theta a^{-1} - a + 2a^{-1}\omega i\right) z^2 \\ &\quad + (2a^{-1} - \theta a^{-1} - a)z\bar{z} + \frac{1}{2}\left(2a^{-1} - \theta a^{-1} - a - 2a^{-1}\omega i\right)\bar{z}^2 \Big) + \cdots \end{aligned}$$

Let $w = w(z, \bar{z}) = (w_2(z, \bar{z}), w_3(z, \bar{z}), \cdots, w_n(z, \bar{z}), \cdots)$ be the function whose graph is the center manifold where

$$w_n(z, \bar{z}) = \frac{1}{2}w_{20,n}z^2 + w_{11,n}z\bar{z} + \frac{1}{2}w_{02,n}\bar{z}^2 + \cdots.$$

We obtain the following equations that determine w_n:

$$\begin{aligned}(2\omega i I-\mathcal{L}_n)w_{20,n} &= I_n(2a^{-1}-\theta a^{-1}-a+2a^{-1}\omega i)\begin{pmatrix}1\\-1\end{pmatrix},\\ -\mathcal{L}_n w_{11,n} &= I_n(2a^{-1}-\theta a^{-1}-a)\begin{pmatrix}1\\-1\end{pmatrix},\\ (-2\omega i I-\mathcal{L}_n)w_{02,n} &= I_n(2a^{-1}-\theta a^{-1}-a-2a^{-1}\omega i)\begin{pmatrix}1\\-1\end{pmatrix},\end{aligned}$$

from which we solve for $w_{20,n}, w_{11,n}$ and $w_{02,n}$. Note that $w_{02,n}=\bar{w}_{20,n}$. For $w_{20,n}$ and $w_{11,n}$ let us denote

$$w_{20,n}=\begin{pmatrix}w^u_{20,n}\\ w^v_{20,n}\end{pmatrix},\qquad w_{11,n}=\begin{pmatrix}w^u_{11,n}\\ w^v_{11,n}\end{pmatrix}.$$

Then, we have

$$\begin{aligned}
Re(w^u_{20,n}) &= \frac{I_n}{D_{20,n}}[16a(4a^2-1-(a^2-1)n^2)+\mathcal{O}(\theta n^6)],\\
Im(w^u_{20,n}) &= \frac{I_n}{D_{20,n}}[2\sqrt{2}a((10a-a^3)n^2-22a+7a^3)+\mathcal{O}(\sqrt{\theta}n^6)],\\
Re(w^v_{20,n}) &= \frac{I_n}{D_{20,n}}[(a^3-10a)n^4+(10a^3-4a)n^2-(71a^3-38a)+\mathcal{O}(\theta n^6)],\\
Im(w^v_{20,n}) &= \frac{I_n}{D_{20,n}}[-2\sqrt{2}(2(a^2-1)n^4+(-a^4+4a^2)n^2+(7a^4-30a^2+2))+\mathcal{O}(\sqrt{\theta}n^6)],\\
Re(w^u_{11,n}) &= \frac{I_n}{D_{11,n}}[(2a^{-1}-\theta a^{-1}-a)\theta n^2],\\
Im(w^u_{11,n}) &= 0,\\
Re(w^v_{11,n}) &= \frac{I_n}{D_{11,n}}[(2a^{-1}-\theta a^{-1}-a)(-1-n^2)],\\
Im(w^v_{11,n}) &= 0,
\end{aligned}\tag{8.8}$$

where

$$\begin{aligned}D_{20,n} &= \left(a^2(n^2-7+4\theta)+\theta n^4-\theta(1+\theta+a^2)n^2+4\theta^2\right)^2\\ &\quad +4\omega^2(n^2-1)^2(1+\theta)^2,\\ D_{11,n} &= a^2(n^2+1)+\theta n^4-\theta n^2(1+\theta+a^2),\end{aligned}$$

and $\mathcal{O}(\theta n^6)$ ($\mathcal{O}(\sqrt{\theta})n^6$) represents a term of magnitude $< K(A_1)\theta n^6$ ($K(A_1)\sqrt{\theta}n^6$) for all $a\in[1,A_1]$ and $n>1$. We note that for the purpose of proving Proposition 2.4 and Theorem 5, there is no need for further explicit details of the $\mathcal{O}(\theta n^6)$ terms. Let us also observe that the effects of θn^4 in both $D_{20,n}$ and $D_{11,n}$ are to increase positively their magnitude.

We are now ready to write the equation of the central flow as

$$\dot{z}=i\omega z+\frac{1}{2}c_{20}z^2+c_{11}z\bar{z}+\frac{1}{2}c_{02}\bar{z}^2+\frac{1}{2}c_{21}z^2\bar{z}+\cdots\tag{8.9}$$

where the functions on the left are obtained by letting $u=\frac{1}{2}(z+\bar{z})$, $v=-\frac{i}{2}(z-\bar{z})$ in $\mathcal{F}$. Note that we need to replace u_n and v_n by $w_n(z,\bar{z})$ obtained in the last subsection in computing c_{21}. For the

terms of second order, we easily obtain

$$\begin{aligned} c_{20} &= \frac{4}{3\pi}(1-\frac{\theta}{\omega}i)\left(2a^{-1}-\theta a^{-1}-a+2a^{-1}\omega i\right), \\ c_{11} &= \frac{4}{3\pi}(1-\frac{\theta}{\omega}i)\left(2a^{-1}-\theta a^{-1}-a\right), \\ c_{02} &= \frac{4}{3\pi}(1-\frac{\theta}{\omega}i)\left(2a^{-1}-\theta a^{-1}-a-2a^{-1}\omega i\right). \end{aligned}$$

c_{21} is less straightforward to compute because of the relevancy of the terms in u_n and v_n. Let us write

$$c_{21} = \hat{c}_{21} + \hat{\hat{c}}_{21}$$

where

$$\hat{c}_{21} = (1-\frac{\theta}{\omega}i)\frac{3}{16}\left(-3(1+\theta a^{-2})+a^{-2}\omega i\right)$$

is from the third order terms in $\mathcal{F}$. $\hat{\hat{c}}_{21}$ is from writing

$$\begin{aligned} \hat{h}_3(z,\bar{z},w) &= 2(1-\frac{\theta}{\omega}i)\sum_{n=2}^{\infty} I_n\left((2a^{-1}u_n+av_n)(z+\bar{z})+a^{-1}\omega u_n(z-\bar{z})i\right) \\ &:= \frac{1}{2}\hat{\hat{c}}_{21}z^2\bar{z}+\cdots. \end{aligned}$$

We now put the equation of the center manifold into (8.9) to compute $\hat{\hat{c}}_{21}$. It follows that

$$\hat{\hat{c}}_{21} = 4(1-\frac{\theta}{\omega}i)\sum_{n=2}^{\infty} I_n\left(a^{-1}w^u_{20,n}+\frac{1}{2}aw^v_{20,n}+2a^{-1}w^u_{11,n}+aw^v_{11,n}+a^{-1}\omega(w^u_{11,n}-\frac{1}{2}w^u_{20,n})i\right). \tag{8.10}$$

Lemma 8.2. *For any $A_1 > 1$, there exists $\theta_0(A_1) > 0$ such that the sum in (8.10) converges uniformly on $(a,\theta) \in [1,A_1]\times[0,\theta_0(A_1)]$. Consequently, $\hat{\hat{c}}_{21}$ as a function of parameters a and θ is continuous on $[1,A_1]\times[0,\theta_0(A_1)]$.*

Proof: First let us show that

$$\begin{aligned} \mathcal{I} &:= \sum_{n=2}^{\infty} I_n Re(w^u_{20,n}) = \sum_{n=2}^{\infty}\frac{I_n^2}{D_{20,n}}[16a(4a^2-1-(a^2-1)n^2)+\mathcal{O}(\theta n^6)] \\ &= \sum_{k=1}^{\infty}\frac{16[16a(4a^2-1-(a^2-1)(2k+1)^2)+\mathcal{O}(\theta(2k+1)^6)]}{\pi^2(2k+1)^2(1-(2k+1)^2)^2D_{20,2k+1}} \end{aligned}$$

converges uniformly as claimed. This is because we have

$$D_{20,n} = \left(a^2(n^2-7+4\theta)+\theta n^4-\theta(1+\theta+a^2)n^2+4\theta^2\right)^2+4\omega^2(n^2-1)^2(1+\theta)^2 > n^4$$

for large n uniformly with respect to $0 \le \theta << 1$ and $a \ge 1$. So the denominator for the k-th term in $\mathcal{I}$ is $> k^{10}$. On the other hand, the $\mathcal{O}(\theta n^6)$ terms in numerators are $< K(A_1)\theta k^6$, which can be made $< k^6$ by our letting $\theta_0 < K(A_1)^{-1}$. Therefore on $(a,\theta) \in [1,A_1]\times[0,\theta_0(A_1)]$,

$$\mathcal{I} < K(A_1)\sum_{k=1}^{\infty}\frac{1}{k^4}.$$

Estimates are similar for

$$\sum_{n>1} I_n Im(w^u_{20,n}), \quad \sum_{n>1} I_n Re(w^v_{20,n}), \quad \sum_{n>1} I_n Im(w^v_{20,n}).$$

For terms related to $w_{11,n}$, we first estimate

$$\begin{aligned} \mathcal{I}I &:= \sum_{n>1} \frac{I_n}{D_{11,n}} Re(w^v_{11,n}) = -\sum_{n>1} \frac{I_n}{D_{11,n}} [((2-\theta)a^{-1} - a)(1+n^2)] \\ &= -\sum_{k\geq 1} \frac{16[((2-\theta)a^{-1} - a)(1+(2k+1)^2)]}{\pi^2(2k+1)^2(4-(2k+1)^2)^2 D_{11,2k+1}}. \end{aligned}$$

Observe that for $a \geq 1$ and $0 \leq \theta << 1$,

$$D_{11,2k+1} = a^2((2k+1)^2+1) + \theta(2k+1)^4 - \theta(2k+1)^2(1+\theta+a^2) > k^2.$$

$\mathcal{I}I$ is obviously bounded by $K(A_1)\sum_{k\geq 1} \frac{1}{k^6}$. Estimates for other sums related to $w_{11,n}$ are similar. □

Finally, using the normal form theory, one has that k_1 in (1.4) is given by

$$k_1(a,\theta) = \frac{i}{2\omega}(c_{20}c_{11} - 2|c_{11}|^2 - \frac{1}{3}|c_{02}|^2) + \frac{1}{2}c_{21}.$$

Note that all c_{20}, c_{11}, c_{02}, and c_{21} are continuous in $\theta \in [0, \theta_0(A_1)]$. For $\theta = 0$, we have

$$Re(k_1(a,0)) = -\left(\frac{4}{3\pi}\right)^2 (2a^{-2} - 1) - \frac{9}{32} + \frac{1}{2} Re(\hat{\hat{c}}_{21}(a,0)), \tag{8.11}$$

$$Im(k_1(a,0)) = \frac{-\sqrt{2}}{3a}\left(\frac{4}{3\pi}\right)^2 ((2a^{-1} - a)^2 + 2) + \frac{3\sqrt{2}}{32}a^{-1} + \frac{1}{2} Im(\hat{\hat{c}}_{21}(a,0)). \tag{8.12}$$

D. Proofs of Proposition 2.4 and Theorem 5

Proof of Proposition 2.4: Let θ be fixed and satisfy $0 < \theta << 1$. For fixed $a \in [1, \infty)$, Condition (H1) follows from Lemma 8.1. We will show

(i) there exists $A_0 > 1$ such that condition (H2) holds for all fixed $a \in [1, A_0)$ at $\mu = 0$;
(ii) $Re(k_1(0)) = 0$ and $Im(k_1(0)) \neq 0$ at $a = A_0$.

Consider first the case $\theta = 0$. Using (8.8) and (8.10), we obtain

$$Re(\hat{\hat{c}}_{21}(a,0)) = \sum_{n=2}^{\infty} \frac{4I_n^2\left((\frac{3}{2}a^4 + a^2 - 16)n^4 + (-11a^4 + 14a^2 + 48)n^2 + \frac{55}{2}a^4 - 51a^2 - 32\right)}{a^4(n^2-7)^2 + 8a^2(n^2-1)^2}, \tag{8.13}$$

$$Im(\hat{\hat{c}}_{21}(a,0)) = \sum_{n=2}^{\infty} \frac{4\sqrt{2}I_n^2(-2(a^3-a)n^4 + (a^5 + 2a^3 + 12a)n^2 - 38a + 12a^3 - 7a^5)}{a^4(n^2-7)^2 + 8a^2(n^2-1)^2}. \tag{8.14}$$

From (8.11) and (8.13), by an elementary computation, we have

$$Re(k_1(1,0)) < -\left(\frac{4}{3\pi}\right)^2 - \frac{9}{32} < 0$$

and $Re(k_1(a,0)) > 0$ for a sufficiently large. This implies that there is a A_0 such that $Re(k_1(a,0)) < 0$ for $a \in [1, A_0)$ and $Re(k_1(A_0,0)) = 0$. Hence (i) and the first part of (ii) hold for $\theta = 0$. As for the second part of (ii), we claim that $Im(k_1(a,0)) < 0$ for all $a \geq 1$. For simplicity, we denote

$$B_n(a) = \frac{-2(a^3-a)n^4 + (a^5+2a^3+12a)n^2 - 38a + 12a^3 - 7a^5}{n^2(4-n^2)^2\big(a^4(n^2-7)^2 + 8a^2(n^2-1)^2\big)},$$

and write

$$\begin{aligned} Im(k_1(a,0)) = {} & \frac{-\sqrt{2}}{3a}\left(\frac{4}{3\pi}\right)^2((2a^{-1}-a)^2+2) + \frac{3\sqrt{2}}{32}a^{-1} \\ & + 2\sqrt{2}\left(\frac{4}{\pi}\right)^2\Big(B_3(a) + B_5(a) + B_7(a) + \sum_{n=9,\text{ odd}}^{\infty} B_n(a)\Big). \end{aligned} \tag{8.15}$$

Estimating $B_n(a)$, we obtain

$$\begin{aligned} B_3(a) &\leq \frac{1}{2\times 3^2\times 5^2}a + \frac{100}{3^2\times 5^2(2^2+8^3)}a^{-1}, \\ B_5(a) &\leq \frac{18}{5^2\times 21^2\times 18^2}a + \frac{324}{5^2\times 21^2(18^2+8\times 24^2)}a^{-1}, \\ B_7(a) &\leq \frac{42}{7^2\times 45^2\times 42^2}a + \frac{660}{7^2\times 45^2(42^2+8\times 48^2)}a^{-1}, \\ B_n(a) &\leq \frac{27}{(n-\sqrt{7})^8}a, \quad \text{for } n \geq 9. \end{aligned}$$

Using an integral estimate, we have

$$\sum_{n=9,\text{ odd}}^{\infty} B_n(a) \leq \frac{1}{2}\int_7^{\infty} \frac{27a}{(x-\sqrt{7})^8}\,dx = \frac{27}{2\times 7\times(7-\sqrt{7})^7}\,a.$$

Let

$$\begin{aligned} G(a) = {} & \frac{-\sqrt{2}}{3a}\left(\frac{4}{3\pi}\right)^2((2a^{-1}-a)^2+2) + \frac{3\sqrt{2}}{32}a^{-1} \\ & + 2\sqrt{2}\left(\frac{4}{\pi}\right)^2\Bigg[\left(\frac{1}{2\times 3^2\times 5^2} + \frac{18}{5^2\times 21^2\times 18^2} + \frac{42}{7^2\times 45^2\times 42^2} + \frac{27}{2\times 7\times(7-\sqrt{7})^7}\right)a \\ & + \left(\frac{100}{3^2\times 5^2(2^2+8^3)} + \frac{324}{5^2\times 21^2(18^2+8\times 24^2)} + \frac{660}{7^2\times 45^2(42^2+8\times 48^2)}\right)a^{-1}\Bigg]. \end{aligned}$$

Minimizing $G(a)$ over $a \in [1,\infty)$, we have for all $a \geq 1$

$$G(a) \leq -\frac{75}{10000}.$$

Hence, $Im(k_1(a,0)) \leq -75/10000$. This completes the proof of (i) and (ii) for $\theta = 0$.

To prove (i) and (ii) for positive but small θ, we choose $1 < A_1 < \infty$ such that $Re(k_1(A_1,0)) > 0$. By Lemma 8.2, we have for (a,θ) on $[1,A_1]\times[0,\theta_0(A_1)]$, both $Re(k_1)$ and $Im(k_1)$ are continuous in terms of a and θ. Thus (i) and (ii) hold for $\theta > 0$ sufficiently small. □

Proof of Theorem 5: With Proposition 2.4 and the fact that $\Phi(u) = (1, 0, \cdots) \in E^c$ and has norm 1, this theorem is a straightforward application of Theorems 1 and 2. □

8.2. **The Neumann case.**

As mentioned earlier, this case is considerably simpler (see also [HKW]). We give only results of the main computations: Here we let $\mu = \frac{1}{2}(b - 1 - a^2)$, and obtain

$$Re(k_1) = -\frac{1}{2} - a^{-2}, \qquad Im(k_1) = -\frac{1}{3a}(2a^2 + 2a^{-2} + \frac{5}{2}),$$

from which it follows that $|\tau| = |\frac{Im(k_1)}{Re(k_1)}| \to \infty$ as $a \to \infty$.

Appendix A. Proofs of Propositions 3.1-3.3

As usual, we deduce the desired local results from global versions of these propositions using cut-off functions for the nonlinear term. Let $\chi(s)$ be a C^∞ function from $(-\infty, \infty)$ to $[0, 1]$ with

$$\chi(s) = 1 \text{ for } |s| \le 1, \quad \chi(s) = 0 \text{ for } |s| \ge 2, \quad \sup_{s\in\mathbb{R}} |\chi'(s)| \le 2.$$

For $\varepsilon > 0$, we define

$$f_\varepsilon(u) = \chi\left(\frac{|u|}{\varepsilon}\right) f(u).$$

Clearly, $f_\varepsilon : \mathbb{E} \to \check{\mathbb{E}}$ is as smooth as f, and $\max_{u\in\mathbb{E}} \|f_\varepsilon(u)\|$, $\max_{u\in\mathbb{E}} \|\partial f_\varepsilon(u)\| \to 0$ as $\varepsilon \to 0$. Writing $u \in \mathbb{E}$ as $u = (v, w), v \in E^c, w \in E^s$, we consider the following modified version of (3.1) with an addition parameter $q \in \mathbb{R}$:

$$\begin{aligned} \dot{v} &= \mathcal{A}^c v + f_{\varepsilon,v}(v, w) + \rho\Phi_v(v, w)\ p_{T,\iota}(t + q) \\ \dot{w} &= \mathcal{A}^s w + f_{\varepsilon,\check{w}}(v, w) + \rho\Phi_w(v, w) p_{T,\iota}(t + q). \end{aligned} \tag{A.1}$$

Let $u(t, u_0, q)$ denote the mild solution of (A.1) with initial condition u_0, i.e. $u(0, u_0, q) = u_0$. A family of sets $\mathcal{M}(q), q \in \mathbb{R}$, of $\mathbb{E}$ is said to be *equivariant* if

$$u(t, \mathcal{M}(q), q) \subseteq \mathcal{M}(q + t), \quad t \ge 0.$$

To prove a global version of Proposition 3.1, one shows the existence of an equivariant family of two dimensional manifolds $\mathcal{W}^c(q)$ given by

$$\mathcal{W}^c(q) = \{v + h^c(v, q) \mid v \in E^c\}$$

where $h^c(\cdot, q) : E^c \to E^s$ is a C^1 mapping satisfying $h^c(v, q + T) = h^c(v, q)$. Specifically, let $a_0 > 0$ be such that $a(\mu) < a_0$ for all $\mu < \mu_0$, and fix $\gamma > 0$ so that $a_0 < \gamma < 2\gamma < \beta_0$. Following the method of Lyapunov and Perron, we consider the Banach Space

$$\mathcal{C}^-_\gamma = \left\{\varphi \mid \varphi : \mathbb{R}^- \to \mathbb{E} \text{ is continuous and } \sup_{t\le 0} |e^{\gamma t}\varphi(t)| < \infty\right\}$$

with norm $|\varphi|^-_\gamma = \sup_{t\le 0} |e^{\gamma t}\varphi(t)|$, and let

$$\mathcal{W}^c(q) = \left\{u^0 \mid u(t, u^0, q) \text{ is defined for all } t \le 0 \text{ and } u^-(\cdot, u^0, q) \in \mathcal{C}^-_\gamma\right\}.$$

Here $u^-(\cdot, u^0, q)$ denotes the restriction of $u(\cdot, u^0, q)$ to $\mathbb{R}^-$. One then shows that given any $q \in \mathbb{R}$ and $\eta \in E^c$, there is a unique $u^0 = (v^0, w^0) \in \mathcal{W}^c(q)$ with $v^0 = \eta$, and verifies that

$$\begin{aligned} u(t, u^0, q) = e^{\mathcal{A}^c t}\eta &+ \int_0^t e^{\mathcal{A}^c(t-\tau)}\Big(f_{\varepsilon,v}(u(\tau)) + \rho\Phi_v(u(\tau))\ p_{T,\iota}(\tau+q)\Big)d\tau \\ &+ \int_{-\infty}^t e^{\mathcal{A}^s(t-\tau)}\Big(f_{\varepsilon,\check{w}}(u(\tau)) + \rho\Phi_w(u(\tau))\ p_{T,\iota}(\tau+q)\Big)d\tau \end{aligned} \tag{A.2}$$

provided $\varepsilon \in (0, \varepsilon_0)$ and $\rho \in (0, \rho_0)$ for sufficiently small ε_0 and ρ_0.

A further argument shows that $\eta \mapsto u(\cdot; u^0, q)$ is differentiable, and $\eta \mapsto \partial_\eta u(\cdot; u^0, q)$ as a function from E^c to $L(E^c, \mathcal{C}^-_{2\gamma})$ is continuous. The existence of a region of $\mathcal{W}^c(0)$ invariant under F_T is assured by taking μ sufficiently small and T sufficiently large. We omit details as the proof outlined above is quite standard.

Proof of Proposition 3.2: To estimate $h^c_\rho(v) - h^c_0(v)$, it is convenient to work in coordinates in which the center manifold of the unforced system is $w = 0$. Thus we let $\hat{w} = w - h^c_0(v)$, and rename $\hat{w}$ as w. In these new coordinates $f_{\varepsilon,\check{w}}$ has a special form, namely

$$f_{\varepsilon,\check{w}}(u, w) = B_{\varepsilon,\check{w}}(u, w)w \qquad \text{where } B_{\varepsilon,\check{w}} \in L(E^s, \check{E}^s) \text{ with } \mathrm{B}_{\varepsilon,\check{\mathrm{w}}}(0,0) = 0\ .$$

For more detail, see Sect. 4.1. This in turn implies the following estimates: Let $\hat{\beta}_0 < \beta_0$. Then there exists C^*_1 such that for all continuous functions $(\bar{v}(\cdot), \bar{w}(\cdot))$, if $\Psi(t,s)w_0$, $s \le t$, denotes the solution of the linear equation

$$\dot{w} = \mathcal{A}^s w + B_{\varepsilon,\check{w}}(\bar{v}(t), \bar{w}(t))w \tag{A.3}$$

with initial condition $\Psi(s,s)w_0 = w_0$, then

$$|\Psi(t,s)w| < C^*_1 e^{-\hat{\beta}_0(t-s)}|w|, \qquad |\Psi(t,s)w| < C^*_1 \frac{1}{(t-s)^\sigma} e^{-\hat{\beta}_0(t-s)}\|w\|, \quad t \ge s\ . \tag{A.4}$$

Now fix $\eta \in E^c$, and consider the solution $u(t) = (v(t), w(t))$ with $u(0) \in \mathcal{W}^c(0)$ and $v(0) = \eta$. As before,

$$w(t) = \int_{-\infty}^t \Psi(t,s)\rho\Phi_w(v(s), w(s))p_{T,\iota}(s)ds. \tag{A.5}$$

Since $p_{T,\iota}(s) = 0$ for $s \in (-T+\iota, 0)$, we have, for $-T+\iota \le t \le 0$,

$$w(t) = \int_{-\infty}^{-T+\iota} \Psi(t,s)\rho\Phi_w(v(s), w(s))p_{T,\iota}(s)ds.$$

By (A.4), we have $|w(t)| < K\rho$ for all $t \le 0$, and $|w(t)| < K\rho e^{-(t+T-\iota)\hat{\beta}_0}$ for $t \in (-T+\iota, 0]$. In particular, $|h^c(\eta, 0)| = |w(0)| < K\rho e^{-(T-\iota)\hat{\beta}_0}$.

For the first derivative, we have

$$\partial_\eta w(t) = \int_{-\infty}^t \Psi(t,s)\rho\partial_\eta\Phi_w(v(s), w(s))p_{T,\iota}(s)ds + \int_{-\infty}^t \Psi(t,s)\partial_\eta B_{\varepsilon,\check{w}}(v(s), w(s))w(s)ds\ .$$

The first integral is treated as above. The norm of the second term is

$$< \int_{\infty}^t C^*_1 \frac{1}{(t-s)^\sigma} e^{-\hat{\beta}_0(t-s)}\|\partial_\eta B_{\varepsilon,w}\||w(s)|ds\ .$$

Substituting in the bound of $|w(s)|$ from above and evaluating at $t = 0$, this integral is

$$< \text{ const} \cdot \left\{ \int_0^{T-\iota} \frac{1}{s^\sigma} e^{-\hat{\beta}_0 s} K\rho e^{-(T-\iota-s)\hat{\beta}_0} ds + \int_{T-\iota}^\infty \frac{1}{s^\sigma} e^{-\hat{\beta}_0 s} K\rho ds \right\} \ < \ K'\rho T e^{-(T-\iota)\hat{\beta}_0}\ .$$

□

Proof of Proposition 3.3: Define

$$\mathcal{C}^+_\gamma = \left\{ \varphi | \varphi : \mathbb{R}^+ \to \mathbb{E} \text{ is continuous and } \sup_{t\ge 0} |e^{\gamma t}\varphi(t)| < \infty \right\}$$

with the norm $|\varphi|_\gamma^+ = \sup_{t\geq 0}|e^{\gamma t}\varphi(t)|$. For a given solution $u(t,u^0,q)(t\geq 0)$ of equation (A.1), let

$$\mathcal{W}^{ss}(u^0,q) = \{\bar{u}^0 \ : \ u(\cdot,\bar{u}^0,q) - u(\cdot,u^0,q) \in \mathcal{C}_\gamma^+\}$$

We will show that $\mathcal{W}^{ss}(u^0,q)$ is the graph of a function over E^s. We now find all solutions $u(t)$ of (A.1) such that

$$z(t) = u(t) - u(t,u^0,q) \in \mathcal{C}_\gamma^+,$$

which is equivalent to finding all $z(\cdot)\in\mathcal{C}_\gamma^+$ satisfying

$$\begin{aligned} z(t) = e^{\mathcal{A}^s t}\zeta &+ \int_0^t e^{\mathcal{A}^s(t-\tau)}\Big(f_{\varepsilon,\tilde{w}}(z(\tau)+u(\tau,u^0,q)) - f_{\varepsilon,\tilde{w}}(u(\tau,u^0,q)) \\ &\quad + \rho\big(\Phi_w(z(\tau)+u(\tau,u^0,q)) - \Phi_w(u(\tau,u^0,q))\big)\ p_{T,\iota}(\tau+q)\Big)d\tau \\ &+ \int_{+\infty}^t e^{\mathcal{A}^c(t-\tau)}\Big(f_{\varepsilon,v}(z(\tau)+u(\tau,u^0,q)) - f_{\varepsilon,v}(u(\tau,u^0,q)) \\ &\quad + \rho\big(\Phi_v(z(\tau)+u(\tau,u^0,q)) - \Phi_v(u(\tau,u^0,q))\big)\ p_{T,\iota}(\tau+q)\Big)d\tau, \end{aligned} \tag{A.6}$$

where $\zeta = z_w(0)$, the w component of z. We will show that for each $(\zeta,u^0)\in E^s\times\mathbb{E}$, equation (A.6) has a unique solution in $\mathcal{C}_\gamma^+$. To see this, let $\mathcal{J}^s(z,\zeta,u^0,q)$ be the right hand side of (A.6). A simple calculation gives that $\mathcal{J}^s:\varphi\mapsto\mathcal{J}^s(\varphi,\zeta,u^0,q)$ is well-defined from $\mathcal{C}_\gamma^+\times E^s\times\mathbb{E}\times\mathbb{R}$ to $\mathcal{C}_\gamma^+$. For any $\varphi,\bar{\varphi}\in\mathcal{C}_\gamma^+$ we have

$$|\mathcal{J}^s(\varphi,\zeta,u^0,q) - \mathcal{J}^s(\bar{\varphi},\zeta,u^0,q)|_\gamma^+ \leq L(\varepsilon,\rho,\sigma,a_0,\beta_0,\gamma)|\varphi-\bar{\varphi}|_\gamma^+ \leq \frac{1}{2}|\varphi-\bar{\varphi}|_\gamma^+ \tag{A.7}$$

provided that ε_0 and ρ_0 are small enough. It is clear that $\mathcal{J}^s$ is Lipschitz continuous in ζ. However it is not known that $\mathcal{J}^s$ is continuous in u^0 because of the lack of compactness. Using the uniform contraction principle, we have that for each $(\zeta,u^0,q)\in E^s\times\mathbb{E}\times\mathbb{R}$, equation (A.6) has a unique solution $z(\cdot;\zeta,u^0,q)\in\mathcal{C}_\gamma^+$ which is Lipschitz continuous in ζ and satisfies

$$|z(\cdot;\zeta,u^0,q) - z(\cdot;\bar{\zeta},u^0,q)|_\gamma^+ \leq 2C|\zeta-\bar{\zeta}|. \tag{A.8}$$

To see that z is C^1 in ζ, we first have that $\mathcal{J}^s$ is a uniform contraction from $\mathcal{C}_{\gamma+\nu}^+$, $0\leq\nu\leq\nu_0$, to itself with respect to the parameters ζ, u^0, and q. Thus, $\mathcal{J}^s$ has a unique fixed point $z_\nu(\cdot;\zeta,u^0,q)\in\mathcal{C}_{\gamma+\nu}^+$ which is a solution of (A.6). Since $\mathcal{C}_{\gamma+\nu}^+\subset\mathcal{C}_\gamma^+$, by the uniqueness of solutions of equation (A.6), we have that $z=z_\nu$. Hence $z(\cdot;\zeta,u^0,q)\in\mathcal{C}_{\gamma+\nu}^+$. In other words, $z(t;\zeta,u^0,q)$ decays much faster than $e^{-\gamma t}$ as $t\to\infty$. Using the same argument as in Proposition 3.1, we obtain that $\zeta\mapsto z(\cdot;\zeta,u^0,q)$ is C^1 from E^s to $\mathcal{C}_\gamma^+$.

To show $z(\cdot,\zeta,u^0,q)$ is Lipschitz continuous in u^0, we first notice that from the variations of constants formula, using (3.2) and (3.3) and choosing ε_0 and ρ_0 to be small enough, we have

$$\|\partial_{u^0}u(t,u^0,q)\|_{L(\mathbb{E},\mathbb{E})} \leq 2\max\{2,C\}e^{\frac{\nu_0}{2}t}, t\geq 0. \tag{A.9}$$

For each $u^0,\bar{u}^0\in\mathbb{E}$, using (A.6), we obtain

$$\begin{aligned} &|z(\cdot,\zeta,\bar{u}^0,q) - z(\cdot,\zeta,u^0,q)|_{\mathcal{C}_{\gamma-\nu_0/2}^+} \\ &\leq L(\varepsilon,\rho,\sigma,a_0,\beta_0,\gamma)|z(\cdot,\zeta,\bar{u}^0,q) - z(\cdot,\zeta,u^0,q)|_{\mathcal{C}_{\gamma-\nu_0/2}^+} \\ &+ \Big(\frac{2\|\partial^2 f_\varepsilon\|}{\gamma-\nu_0/2-a_0} + \frac{4\rho\|\partial^2\Phi\|}{1-e^{-(\gamma-\nu_0/2-a_0)}} \\ &+ \frac{(2-\sigma)C\|\partial^2 f_\varepsilon\|}{(1-\sigma)(\beta_0-\gamma+\nu_0/2)^{1-\sigma}} + \frac{2C\rho\|\partial^2\Phi\|}{1-e^{-(\beta_0-\gamma+\nu_0/2)}}\Big)|z(\cdot,\zeta,u^0,q)|_{\mathcal{C}_\gamma^+}|\bar{u}^0-u^0|, \end{aligned} \tag{A.10}$$

which yields

$$
\begin{aligned}
&|z(\cdot,\zeta,\bar{u}^0,q)-z(\cdot,\zeta,u^0,q)|_{\mathcal{C}^+_{\gamma-\nu_0/2}} \le 2\Big(\frac{2\|\partial^2 f_\varepsilon\|}{\gamma-\nu_0/2-a_0}+\frac{4\rho\|\partial^2\Phi\|}{1-e^{-(\gamma-\nu_0/2-a_0)}}\\
&+\frac{(2-\sigma)C\|\partial^2 f_\varepsilon\|}{(1-\sigma)(\beta_0-\gamma+\nu_0/2)^{1-\sigma}}+\frac{2C\rho\|\partial^2\Phi\|}{1-e^{-(\beta_0-\gamma+\nu_0/2)}}\Big)|z(\cdot,\zeta,u^0,q)|_{\mathcal{C}^+_\gamma}|\bar{u}^0-u^0|.
\end{aligned}
\tag{A.11}
$$

Hence, $u^0 \to z(\cdot;\zeta,u^0,q)$ is Lipschitz continuous.

Derivative with respect to ζ. Let

$$l^s(u^0,\zeta,q) := v_0 + z_v(0;\zeta-w_0,u^0,q),$$

where $u^0=(v_0,w_0)$ and z_v is the v-component of z. Then, $l^s(u^0,\zeta,q)$ is C^1 in ζ, Lipschitz continuous in u^0, and

$$l^s(u^0,\zeta,q+T)=l^s(u^0,\zeta,q).$$

By (A.6), we have

$$
\begin{aligned}
z_v(0;\zeta,u^0,q) = \int_{+\infty}^{0} e^{-\mathcal{A}^c\tau}\Big(&f_{\varepsilon,v}\big(z(\tau;\zeta,u^0,q)+u(\tau,u^0,q)\big)-f_{\varepsilon,v}(u(\tau,u^0,q))\\
&+\rho\big(\Phi_v\big(z(\tau;\zeta,u^0,q)+u(\tau,u^0,q)\big)-\Phi_v(u(\tau,u^0,q))\big)\ p_{T,\iota}(\tau+q)\Big)d\tau.
\end{aligned}
$$

Using (A.8), we have

$$\|\partial_\zeta l^s(u^0,\zeta,q)\|_{L(E^s,E^c)} \le 4C\left(\frac{\mathrm{Lip} f_\varepsilon}{\gamma-a_0}+\frac{2\rho\mathrm{Lip}\Phi}{1-e^{\gamma-a_0}}\right) < K(\rho+\varepsilon) < 1$$

provided that ε_0 and ρ_0 are small enough.

Lipschitz continuity with respect to u^0. Let $\Lambda_{u^0}(q)$ be the stable leave defined by the graph of $l^s(u^0,\zeta,q)$. For $\bar{u}^0\in\mathbb{E}$, let the w component of $\bar{u}^0-u^0$ be ζ. By the fact that $\bar{u}^0\in\Lambda_{u^0}(q)$ if and only if $z(\cdot,\zeta,u^0,q)\in\mathcal{C}^+_\gamma$ and satisfies (A.6), we have that $\bar{u}^0\in\mathcal{W}^{ss}(u^0,q)$ if and only if $\bar{u}^0=l^s(\zeta,u^0,q)+\zeta$. Hence,

$$\Lambda_{u^0}(q)=\mathcal{W}^{ss}(u^0,q)=\{l^s(u^0,\zeta,q)+\zeta|\zeta\in E^s\}.$$

From (A.11), we have that $l^s(\zeta,u^0,q)$ is Lipschitz continuous in u^0. It follows that, for each $u^0(v_0,w_0)\in E$ and $\zeta\in E^s$,

$$\phi(u^0,\zeta):=\Lambda_{u^0}(q)\cap\{\zeta+E^c\}l^s(u^0,\zeta,q)+\zeta.$$

That $\eta(u^0,\zeta)$ is Lipschitz continuous follows from (A.11).

Finally, we prove that this foliation is equivariant for the dynamical system generated by (A.1). To see this, taking a leaf of the stable foliation $\Lambda_{u^0}(q)$, we will show that $u(\tau,\cdot,q)$ maps it into the leave $\Lambda_{u(\tau,u^0,q)}(q+\tau)$. Let $\bar{u}^0\in\lambda_{u^0}(q)$. Then $\big(u(\cdot,\bar{u}^0,q)-u(\cdot,u^0,q)\big)\in\mathcal{C}^+_\gamma$, which implies$u(\cdot+\tau,\bar{u}^0,q)-u(\cdot+\tau,u^0,q)\in\mathcal{C}^+_\gamma$. Thus, by using the cocycle property, $u(t+\tau,\bar{u}^0,q)=u(t,u(\tau,\bar{u}^0,q),q+\tau)$ and $u(t+\tau,u^0,q)=u(t,u(\tau,u^0,q),q+\tau)$, we have that $u(\tau,\bar{u}^0,q)\in\Lambda_{u(\tau,u^0,q)}(q+\tau)$. Let $G(u,\mathbf{w})=l^s(u,\mathbf{w},0)$ and $\mathcal{W}^{ss}_u=\Lambda_u(0)$. Proposition 3.3 then follows from the above discussion. This completes the proof. □

Appendix B. Proof of Proposition 6.3

Existing results may be cited to give the existence of center manifolds but not the estimate

$$\|\mathbf{W}(\xi,\Theta;T)\|_{C^3(\mathbb{A}\times(M_0\mu_0^{-1},\infty))} < Ke^{-\frac{\varepsilon}{3}T}\ , \tag{B.1}$$

which contains uniform C^3 bounds for all large t. We will follow the approach in [BLZ]. However, to get the estimate (B.1), one needs to go through the entire process of proving the existence of invariant manifolds. We first modify the system using bump functions and show that the manifold $\tilde{\mathcal{W}}$ is approximately normally hyperbolic. Then, we prove that F_T has a true invariant manifold nearby it that satisfies the estimate

(B.1) by using the Lyapunov and Perron approach. The existence of such manifold can also be obtained by Hadamard's graph transform, but the proof of smoothness is much more complicated.

B.1. **Extending the domains of κ and G.**

We use a cut-off function to extend the domains of κ and G to $\mathbb{E}$. Let $\sigma : \mathbb{R} \to [0,1]$ be a C^∞ function satisfying $\sigma(s) = 1$ for $|s| < 1$; $\sigma(s) = 0$ for $|s| > 2$. We further assume that $\sigma(s)$ is such that $|\sigma'(s)| \le 2$.

B.1.1. *Modifications for forcing period: κ.*

Recall that $\kappa : V_{\frac{1}{100},\frac{1}{100}} \to V_{\frac{11}{20},C_0(1+K_0)}$ is such that

$$\begin{aligned}
\hat{\xi}_\iota &= \sqrt{(1+\xi_0)^2 + \tilde{\gamma}(\tilde{\gamma} + 2(1+\xi_0)\cos(\Theta_0 - c_0))} - 1 + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}),\\
\tan\hat{\Theta}_\iota &= \frac{(1+\xi_0)\sin\Theta_0 + \tilde{\gamma}\sin c_0}{(1+\xi_0)\cos\Theta_0 + \tilde{\gamma}\cos c_0} + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}),\\
\hat{\mathbf{w}}_\iota &= e^{\mathcal{A}^s\iota}\mathbf{w_0} + \rho\sqrt{\frac{\alpha}{a}}\cdot\frac{1}{\iota}\int_0^\iota e^{\mathcal{A}^s(\iota-\tau)}P_0^s\Phi(0)d\tau + \mathcal{O}_4(\iota + \mu^{\frac{1}{2}}),
\end{aligned} \tag{B.2}$$

where $\tan c_0 = \frac{\Phi_y(0)}{\Phi_x(0)}$ and C_0 is given by (4.23). We modify κ to obtain a function $\tilde{\kappa}$ defined on $\mathbb{E}$ by replacing $\mathcal{O}_4(\iota + \mu^{\frac{1}{2}})$ in (B.2) by $\mathcal{O}_4(\iota + \mu^{\frac{1}{2}})\sigma(100|\xi|)\sigma(100|\mathbf{w}|)$.

B.1.2. *Modified equation for relaxing period: $\tilde{G}$.*

From Proposition 5.1, the solutions of equation (5.1) initiated from $V_{\frac{3}{5},C_0(K_0+1)}$ will stay inside of $V_{\frac{31}{50},C_0^2(K_0+1)}$ for all $t > 0$. To extend the domain of equation (5.1) from $V_{\frac{31}{50},C_0^2(K_0+1)}$ to $\mathbb{E}$, we re-write (5.1) as

$$\begin{aligned}
\dot{\xi} &= -a(\mu)(2 + c_1(\xi) + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w},\\
\dot{\Theta} &= \frac{\hat{\omega}}{\Theta^*} + \tau a(\mu)(2 + c_2(\xi) + \mathcal{O}_4(\mu))\xi + \mathcal{O}_4(\sqrt{\mu})\mathbf{w},\\
\dot{\mathbf{w}} &= \tilde{A}^s\mathbf{w} + \mathcal{O}_4(\sqrt{\mu})\mathbf{w},
\end{aligned} \tag{B.3}$$

where (i) $c_1(\xi) = (3\xi + \xi^2)\sigma(\frac{50}{31}\xi)$, $c_2(\xi) = \xi\sigma(\frac{50}{31}\xi)$; (ii) $\mathcal{O}_4(\mu), \mathcal{O}_4(\sqrt{\mu})$ in the above are obtained from their corresponding terms in equation (5.1) multiplied by $\sigma(\frac{50}{31}|\xi|)\sigma((C_0^2(K_0+1))^{-1}|\mathbf{w}|)$.

Let $\tilde{G}$ be the time-$T-\iota$ map of equation (B.3). Then,

(a) $\tilde{G} = G$ on $V_{\frac{3}{5},C_0(K_0+1)}$.

(b) Propositions 5.1, 5.2, which we proved in Sect. 5.1 originally for equation (5.1), remain valid for the modified equation (B.3).

To prove (b) we need to go through the entire proof for Propositions 5.1 and 5.2 for equation (B.3). We caution that, in deriving the equations of variations up to order four used in the proof of Proposition 5.2, there are additional terms from derivatives of $\sigma(\frac{50}{31}\xi)$ and $\sigma((C_0^2(K_0+1))^{-1}|\mathbf{w}|)$. If one such term is from a term of $\mathcal{O}_4(\mu^k)$ in (B.3), we would have extra terms in the same form. We, however, can not absorb terms originated from differentiating $c_1(\xi)$ and $c_2(\xi)$ in the same way. These terms do not alter the conclusions of Propositions 5.2 but they will make the constant K_2 in this proposition larger.

B.1.3. *Estimates on contraction for unforced system.*

Let $\hat{u}_\iota = (\hat{\xi}_\iota, \hat{\Theta}_\iota, \hat{\mathbf{w}}_\iota)$, $\tilde{G}_{T-\iota}(\hat{u}_\iota) = (\xi_{T-\iota}, \Theta_{T-\iota}, \mathbf{w}_{T-\iota})$. Denote the first two of equation (B.3) as

$$\frac{d}{dt}\xi = \zeta(\xi,\Theta,\mathbf{w}), \qquad \frac{d}{dt}\Theta = \chi(\xi,\Theta,\mathbf{w}), \tag{B.4}$$

and let

$$\mathcal{M}^{(1)} = \begin{pmatrix} \frac{\partial\xi_{T-\iota}}{\partial\hat{\xi}_\iota} & \frac{\partial\xi_{T-\iota}}{\partial\hat{\Theta}_\iota} \\ \frac{\partial\Theta_{T-\iota}}{\partial\hat{\xi}_\iota} & \frac{\partial\Theta_{T-\iota}}{\partial\hat{\Theta}_\iota} \end{pmatrix}, \quad \mathcal{M}^{(2)} = \begin{pmatrix} \frac{\partial\xi_{T-\iota}}{\partial\hat{\xi}_\iota} & \frac{\partial\xi_{T-\iota}}{\partial\hat{\mathbf{w}}_\iota} \\ \frac{\partial\Theta_{T-\iota}}{\partial\hat{\xi}_\iota} & \frac{\partial\Theta_{T-\iota}}{\partial\hat{\mathbf{w}}_\iota} \end{pmatrix}, \quad \mathcal{M}^{(3)} = \begin{pmatrix} \frac{\partial\xi_{T-\iota}}{\partial\hat{\Theta}_\iota} & \frac{\partial\xi_{T-\iota}}{\partial\hat{\mathbf{w}}_\iota} \\ \frac{\partial\Theta_{T-\iota}}{\partial\hat{\Theta}_\iota} & \frac{\partial\Theta_{T-\iota}}{\partial\hat{\mathbf{w}}_\iota} \end{pmatrix}.$$

By a direct computation, we obtain

$$\frac{d}{dt}\det \mathcal{M}_t^{(i)} = \left(\frac{\partial \zeta}{\partial \xi} + \frac{\partial \chi}{\partial \Theta}\right)\det \mathcal{M}_t^{(i)} + \mathcal{W}^{(i)} \tag{B.5}$$

where

$$\mathcal{W}^{(1)} = \frac{\partial \zeta}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\xi}_\iota}\frac{\partial \Theta_t}{\partial \hat{\Theta}_\iota} - \frac{\partial \mathbf{w}}{\partial \hat{\Theta}_\iota}\frac{\partial \Theta_t}{\partial \hat{\xi}_\iota}\right) + \frac{\partial \chi}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\Theta}_\iota}\frac{\partial \xi_t}{\partial \hat{\xi}_\iota} - \frac{\partial \mathbf{w}_t}{\partial \hat{\xi}_\iota}\frac{\partial \xi_t}{\partial \hat{\Theta}_\iota}\right), \tag{B.6}$$

$$\mathcal{W}^{(2)} = \frac{\partial \zeta}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\xi}_\iota}\frac{\partial \Theta_t}{\partial \hat{\mathbf{w}}_\iota} - \frac{\partial \mathbf{w}}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \Theta_t}{\partial \hat{\xi}_\iota}\right) + \frac{\partial \chi}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \xi_t}{\partial \hat{\xi}_\iota} - \frac{\partial \mathbf{w}_t}{\partial \hat{\xi}_\iota}\frac{\partial \xi_t}{\partial \hat{\mathbf{w}}_\iota}\right), \tag{B.7}$$

$$\mathcal{W}^{(3)} = \frac{\partial \zeta}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \Theta_t}{\partial \hat{\Theta}_\iota} - \frac{\partial \mathbf{w}_t}{\partial \hat{\Theta}_\iota}\frac{\partial \Theta_t}{\partial \hat{\mathbf{w}}_\iota}\right) + \frac{\partial \chi}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}_t}{\partial \hat{\Theta}_\iota}\frac{\partial \xi_t}{\partial \hat{\mathbf{w}}_\iota} - \frac{\partial \mathbf{w}_t}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \xi_t}{\partial \hat{\Theta}_\iota}\right). \tag{B.8}$$

From (B.5) we obtain

$$\det \mathcal{M}^{(i)} = e^{\int_0^{T-\iota}(\frac{\partial \zeta}{\partial \xi}+\frac{\partial \chi}{\partial \Theta})dt}\left(\delta_{1,i} + \int_0^{T-\iota}\mathcal{W}^{(i)}e^{-\int_0^t(\frac{\partial \zeta}{\partial \xi}+\frac{\partial \chi}{\partial \Theta})ds}dt\right), \tag{B.9}$$

where $\delta_{1,i} = 1$ if $i = 1$ and $\delta_{1,i} = 0$ otherwise. We also have from (B.3) and (B.4) that

$$\frac{\partial \zeta}{\partial \xi} + \frac{\partial \chi}{\partial \Theta} = -a(\mu)(2 + c_1'(\xi)\xi + c_1(\xi)) + \mathcal{O}(\mu^2) + \mathcal{O}(\sqrt{\mu})\mathbf{w}. \tag{B.10}$$

We have

Lemma B.1. *(i)* $-10 < 2 + c_1'(\xi)\xi + c_1(\xi) < 10$, *(ii)* $|\mathcal{W}^{(i)}| < \mathcal{O}_4(\sqrt{\mu})e^{-\frac{c}{2}t}$.

Proof: (i) follows from a direct computation. (ii) follows from Proposition 5.2 for equation (B.3). □

B.1.4. *A lower bound estimate on the contraction of $\tilde{G}\circ\tilde{\kappa}$ in center directions.*

For $u_0 = (\xi, \Theta, \mathbf{w}) \in \mathbb{R}\times\mathbb{R}\times E_0^s$, let $\hat{u}_\iota = (\hat{\xi}_\iota, \hat{\Theta}_\iota, \hat{\mathbf{w}}_\iota) := \tilde{\kappa}(u_0)$ and $(\xi_{T-\iota}, \Theta_{T-\iota}, \mathbf{w}_{T-\iota}) := \tilde{G}\circ\tilde{\kappa}(u_0)$. We further denote

$$\mathcal{M} = \begin{pmatrix} \frac{\partial \xi_{T-\iota}}{\partial \xi} & \frac{\partial \xi_{T-\iota}}{\partial \Theta} \\ \frac{\partial \Theta_{T-\iota}}{\partial \xi} & \frac{\partial \Theta_{T-\iota}}{\partial \Theta} \end{pmatrix}.$$

Proposition B.1. $\det \mathcal{M} > \frac{1}{100}e^{-10a(\mu)T}$.

Proof: From

$$\frac{\partial \xi_{T-\iota}}{\partial \xi} = \frac{\partial \xi_{T-\iota}}{\partial \hat{\xi}_\iota}\frac{\partial \hat{\xi}_\iota}{\partial \xi} + \frac{\partial \xi_{T-\iota}}{\partial \hat{\Theta}_\iota}\frac{\partial \hat{\Theta}_\iota}{\partial \xi} + \frac{\partial \xi_{T-\iota}}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \hat{\mathbf{w}}_\iota}{\partial \xi},$$
$$\frac{\partial \xi_{T-\iota}}{\partial \Theta} = \frac{\partial \xi_{T-\iota}}{\partial \hat{\xi}_\iota}\frac{\partial \hat{\xi}_\iota}{\partial \Theta} + \frac{\partial \xi_{T-\iota}}{\partial \hat{\Theta}_\iota}\frac{\partial \hat{\Theta}_\iota}{\partial \Theta} + \frac{\partial \xi_{T-\iota}}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \hat{\mathbf{w}}_\iota}{\partial \Theta},$$
$$\frac{\partial \Theta_{T-\iota}}{\partial \xi} = \frac{\partial \Theta_{T-\iota}}{\partial \hat{\xi}_\iota}\frac{\partial \hat{\xi}_\iota}{\partial \xi} + \frac{\partial \Theta_{T-\iota}}{\partial \hat{\Theta}_\iota}\frac{\partial \hat{\Theta}_\iota}{\partial \xi} + \frac{\partial \Theta_{T-\iota}}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \hat{\mathbf{w}}_\iota}{\partial \xi},$$
$$\frac{\partial \Theta_{T-\iota}}{\partial \Theta} = \frac{\partial \Theta_{T-\iota}}{\partial \hat{\xi}_\iota}\frac{\partial \hat{\xi}_\iota}{\partial \Theta} + \frac{\partial \Theta_{T-\iota}}{\partial \hat{\Theta}_\iota}\frac{\partial \hat{\Theta}_\iota}{\partial \Theta} + \frac{\partial \Theta_{T-\iota}}{\partial \hat{\mathbf{w}}_\iota}\frac{\partial \hat{\mathbf{w}}_\iota}{\partial \Theta},$$

we have

$$\det \mathcal{M} = \det \mathcal{M}^{(1)}A_1 + \det \mathcal{M}^{(2)}A_2 + \det \mathcal{M}^{(3)}A_3 \tag{B.11}$$

where

$$A_1 = \frac{\partial \hat{\xi}_\iota}{\partial \xi}\frac{\partial \hat{\Theta}_\iota}{\partial \Theta} - \frac{\partial \hat{\xi}_\iota}{\partial \Theta}\frac{\partial \hat{\Theta}_\iota}{\partial \xi}, \quad A_2 = \frac{\partial \hat{\mathbf{w}}_\iota}{\partial \Theta}\frac{\partial \hat{\xi}_\iota}{\partial \xi} - \frac{\partial \hat{\mathbf{w}}_\iota}{\partial \xi}\frac{\partial \hat{\xi}_\iota}{\partial \Theta}, \quad A_3 = \frac{\partial \hat{\mathbf{w}}_\iota}{\partial \Theta}\frac{\partial \hat{\Theta}_\iota}{\partial \xi} - \frac{\partial \hat{\mathbf{w}}_\iota}{\partial \xi}\frac{\partial \hat{\Theta}_\iota}{\partial \Theta}.$$

A_1, A_2 and A_3 are contributions from $\tilde{\kappa}$. From (B.2),

$$A_1 > \frac{1}{10}, \quad |A_2|, \quad |A_3| < 1. \tag{B.12}$$

From (B.9) and (B.11) we have

$$\det \mathcal{M} = e^{\int_0^{T-\iota}(\frac{\partial \zeta}{\partial \xi}+\frac{\partial \chi}{\partial \Theta})dt}\left(A_1 + \sum_{i=1}^{3}\int_0^{T-\iota} A_i \mathcal{W}^{(i)} e^{-\int_0^t(\frac{\partial \zeta}{\partial \xi}+\frac{\partial \chi}{\partial \Theta})ds}dt\right). \tag{B.13}$$

The assertion then follows from (B.10), (B.12) and Lemma B.1. □

B.2. **2-D Invariant manifold.**

In this subsection, we show that the modified time-T map $\tilde{F}_T$ has a two-dimensional invariant manifold and the manifold satisfies the estimate (B.1).

B.2.1. *A canonical form for the modified time-T map.*

For the rest of this section we denote

$$x = (\xi, \Theta), \quad y = \mathbf{w}.$$

For $u_0 = (\xi, \Theta, \mathbf{w}) = (x, y)$, we write the modified time-T map $\tilde{F}_T := \tilde{G} \circ \tilde{\kappa}$ as

$$\tilde{F}_T(x, y) = (f_T(x, y), e^{\tilde{A}^s T} y + g_T(x, y))$$

where

$$\begin{aligned} f_T(x, y) &= (\xi_{T-\iota}(x, y), \Theta_{T-\iota}(x, y)), \\ g_T(x, y) &= \mathbf{w}_{T-\iota}(x, y) - e^{\tilde{A}^s T}\hat{\mathbf{w}}_\iota(x, y) + e^{\tilde{A}^s T}(\hat{\mathbf{w}}_\iota(x, y) - y). \end{aligned}$$

Lemma B.2. *For each $y \in E_0^s$, $f_T(\cdot, y)$ is a C^3 diffeomorphism from $\mathbb{R}^2$ onto itself. Furthermore, there is a positive constant K_4 independent μ and T such that*

$$|\,(D_x f_T(x, y))^{-1}\,| \le K_4 e^{10a(\mu)T}. \tag{B.14}$$

Proof: That $f_T(\cdot, y)$ is a C^3-diffeomorphism follows from (B.14), which is in a simple rewriting of Proposition B.1. □

Lemma B.3. *There is a positive constant K_5 independent of μ and T such that*

(i) $|D_u^k f_T(x, y)| \le K_5$ *for* $1 \le k \le 3$, *where* $u = (x, y)$;
(ii) $|D_y^k f_T(x, y)| \le \mathcal{O}(\sqrt{\mu})$, *for* $1 \le k \le 3$;
(iii) $|D_u^k g_T(x, y)| \le K_5 e^{-\frac{c}{2}T}$ *for* $0 \le k \le 3$;
(iv) $|D_{T,u}^k f_T(x, y)| \le K_5(1 + |\xi| + |y|)$ *for* $1 \le k \le 3$;
(v) $|D_{T,u}^k g_T(x, y)| \le K_5(1 + |\xi| + |y|)e^{-\frac{c}{2}T}$ *for* $1 \le k \le 3$.

Proof: These estimates follows from the chain rule in taking derivatives, the explicit formula for $\tilde{\kappa}$, and Propositions 5.1 and 5.2 for $\tilde{G}$. □

B.2.2. *Existence of invariant manifold.*

Let

$$\tilde{F}_T = \tilde{G}_T \circ \tilde{\kappa}(x, y) = (f_T(x, y), e^{\tilde{A}^s T} y + g_T(x, y))$$

be the map in the above. For simplicity, we denote $X = \mathbb{R}^2$ and $Y = E_0^s$. We note that X is a normally hyperbolic invariant manifold for $\tilde{G}_T$ due to Lemma B.2, which is an approximately normally hyperbolic invariant manifold for $\tilde{F}_T$ in the sense of [BLZ]. We want to show that for μ sufficiently small, $\tilde{F}_T$ has a two dimensional invariant manifold.

Proposition B.2. *Assume that f and g are C^3. Then, there exists $\mu_0 > 0$ such that for $0 < \mu < \mu_0$, there exists $T_0(\mu)$ sufficiently large such that for $T > T_0(\mu)$, $\tilde{F}_T$ has a C^3 invariant manifold*

$$\mathcal{W}(T) = \cup_{x\in X}\{x + \mathbf{W}(T,x)\}$$

where $\mathbf{W}(T,x) = \mathbf{W}(T,\xi,\Theta)$ is periodic in Θ and $\mathbf{W}_T : (T_0,\infty)\times X \to Y$ is a C^3 map and satisfies for $1 \le k \le 3$

$$||D_x^k \mathbf{W}(T,x)|| \le K_6 e^{-\frac{c}{2}T} \tag{B.15}$$

and

$$||D_{T,x}^k \mathbf{W}(T,x)|| \le K_6(1+|x|)e^{-\frac{c}{3}T} \tag{B.16}$$

where K_6 is a constant independent of T and μ.

B.3. **Proof of Proposition B.2.**

Let $\gamma > 0$. We define the following Banach spaces

$$C_\gamma(X) = \big\{\mathbf{x} = \{x_n\}_{n\le 0} \mid x_n \in X,\ \sup_{n\le 0}|x_n|e^{\gamma n} < \infty\big\}$$

with the norm

$$|\mathbf{x}|_{C_\gamma(X)} = \sup_{n\le 0}|x_n|e^{\gamma n}$$

and

$$C_\gamma(Y) = \big\{\mathbf{y} = \{y_n\}_{n\le 0} \mid y_n \in Y,\ \sup_{n\le 0}|y_n|e^{\gamma n} < \infty\big\}$$

with the norm

$$|\mathbf{y}|_{C_\gamma(Y)} = \sup_{n\le 0}|y_n|e^{\gamma n}.$$

Let $\mu \in (0,\mu_0)$ be fixed. In what follows we let α_1, γ, σ, and β_1 be such that

$$\alpha_1 = 4a(\mu)T, \quad \sigma_0 = \frac{c}{100}T, \quad \gamma = \frac{c}{32}T, \quad \beta_1 = \frac{c}{4}T, \tag{B.17}$$

where $c \in (0,\beta_0)$ is a constant given in Proposition 5.1. We note that $\gamma - \sigma_0 < \gamma + \sigma_0 < \frac{1}{4}\beta_1$

Since $a(\mu) \to 0$ as $\mu \to 0$, we may choose that $\mu_0 > 0$ sufficiently small such that for each $\mu \in (0,\mu_0)$, $\alpha_1 < \gamma - \sigma_0$. We also choose $T_0(\mu)$ large enough so that

$$K_4 e^{3a(\mu)T} < e^{\alpha_1} \ \text{ and } \ 2Ce^{-\beta_0 T} < e^{-\beta_1},$$

where C is from (3.3).

Lemma B.4. *Let $\eta \in X$ and $\tilde{\mathbf{y}} = \{\tilde{y}_n\}_{n\le 0}$ be a sequence in Y. Let $\mathbf{x} = \{x_n\}_{n\le 0} = \mathbf{x}(T,\eta,\tilde{\mathbf{y}})$ be a sequence determined by*

$$x_{n+1} = f_T(x_n,\tilde{y}_n), \ \ x_0 = \eta.$$

Then, there exists T_0 independent of η and $\tilde{\mathbf{y}}$ such that if $T > T_0$, then

$$y_{n+1} = e^{\tilde{A}^s T}y_n + g_T(x_n(T,\eta,\tilde{\mathbf{y}}),y_n), \quad n \le 0, \tag{B.18}$$

has a unique bounded solution $\mathbf{y}(T,\eta,\tilde{\mathbf{y}})$ given by

$$y_n = \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s} g_T(x_i(T,\eta,\tilde{\mathbf{y}}),y_i). \tag{B.19}$$

Proof: Let $\eta \in X$ and $\tilde{\mathbf{y}} \in C_0(Y)$. We first show that equation (B.18) has a solution $\mathbf{y} \in C_0(Y)$ if and only if $\mathbf{y} \in C_0(Y)$ and satisfies equation (B.19). Suppose $\mathbf{y} \in C_0(Y)$ satisfies equation (B.18). Then by the discrete variation of constants formula, we have

$$y_n = e^{(n-k)T\tilde{A}^s} y_k + \sum_{i=k}^{n-1} e^{(n-1-i)T\tilde{A}^s} g_T(x_i(T,\eta,\tilde{\mathbf{y}}), y_i) \tag{B.20}$$

Since $||e^{nT\tilde{A}^s}|| \le e^{-\beta_1 n}$ for $n \ge 0$, for $k \le n$

$$|e^{(n-k)T\tilde{A}^s} y_k| \le e^{-\beta_1(n-k)} |\mathbf{y}|_{C_0(Y)} \to 0, \quad \text{as } k \to -\infty,$$

which yields (B.19). Clearly, if $\mathbf{y}$ satisfies equation (B.19), then $\mathbf{y}$ satisfies equation (B.18). Next, we show that for each $\eta \in X$, equation (B.18) has a unique solution $\mathbf{y} \in C_0(Y)$. Define a map $\mathcal{J} = \{\mathcal{J}_n\}$ from $C_0(Y)$ to itself by

$$J_n(\mathbf{y}) = \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s} g_T(x_i(T,\eta,\tilde{\mathbf{y}}), y_i).$$

It is easy to see that this is well-defined. Next, we show that $\mathcal{J}$ is a contraction. For $\mathbf{y}, \mathbf{z} \in C_0(Y)$, we have that

$$|\mathcal{J}_n(\mathbf{y}) - \mathcal{J}_n(\mathbf{z})| \le \sum_{i=-\infty}^{n-1} e^{-\beta_1(n-1-i)} \text{Lip } g_T |\mathbf{y} - \mathbf{z}|_{C_0} \le \frac{K_5 e^{-\frac{c}{2}T}}{1 - e^{-\beta_1}} |\mathbf{y} - \mathbf{z}|_{C_0}$$

which is contraction provided that T_0 is large enough. Here, we used Lemma B.3. Hence, equation (B.19) has a unique bounded solution. The proof is complete. □

Lemma B.5. *Assume that f and g are C^3. Then, there exists $\mu_0 > 0$ such that for each $0 < \mu < \mu_0$ there is $T_0(\mu)$ sufficiently large such that for $T > T_0$ and for each $\eta \in X$ there exists a unique solution $(\mathbf{x}(\mathbf{T},\eta), \mathbf{y}(\mathbf{T},\eta))$ of*

$$\begin{cases} x_{n+1} = f_T(x_n, y_n), \quad x_0 = \eta, \\ y_n = \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s} g_T(x_i, y_i) \end{cases} \tag{B.21}$$

with $\mathbf{y}(T,\cdot) \in C_0(Y)$. Furthermore, $\mathbf{y} : (T_0, \infty) \times X \to C_{3\gamma}(Y)$ is C^3 and

$$||D^k_{T,\eta}\mathbf{y}|| \le K_6 e^{-\frac{c}{3}T}, \quad 1 \le k \le 3, \tag{B.22}$$

where K_6 is a constant independent of T and μ.

Proof: We prove this lemma in three steps.

Step 1. *Existence of a Lipschitz Continuous Solution.*

For $\xi, \eta \in X$, and $\tilde{\mathbf{y}}, \tilde{\mathbf{z}} \in C_\gamma(Y)$, let $\mathbf{x} = \mathbf{x}(T,\xi,\tilde{\mathbf{y}}) = \{x_n\}_{n\le 0}$ and $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(T,\eta,\tilde{\mathbf{z}}) = \{\tilde{x}_n\}_{n\le 0}$ be given by

$$\begin{aligned} x_{n+1} &= f_T(x_n, \tilde{y}_n), \quad x_0 = \xi; \\ \tilde{x}_{n+1} &= f_T(\tilde{x}_n, \tilde{z}_n), \quad \tilde{x}_0 = \eta. \end{aligned}$$

Let $w_n = x_n - \tilde{x}_n$. Then

$$w_{n+1} = D_x f_T(\tau x_n + (1-\tau)\tilde{x}_n, \tilde{y}_n) w_n + f_T(\tilde{x}_n, \tilde{y}_n) - f_T(\tilde{x}_n, \tilde{z}_n).$$

Using Lemma B.2, we have that

$$|w_n| \le e^{\alpha_1} |w_{n+1}| + e^{\alpha_1} \text{Lip }_y f_T |\tilde{y}_n - \tilde{z}_n|.$$

Then, we have for $n \le 0$

$$|w_n| \le e^{-\alpha_1 n} |\xi - \eta| + \text{Lip }_y f_T \sum_{i=n}^{-1} e^{\alpha_1(-n+i)} |\tilde{y}_i - \tilde{z}_i| \tag{B.23}$$

Hence,

$$|w_n| \le e^{-\alpha_1 n}|\xi-\eta| + \frac{\text{Lip }_y f_T}{1-e^{\alpha_1-\gamma}} e^{-\gamma n}|\tilde{\mathbf{y}}-\tilde{\mathbf{z}}|_{C_\gamma}. \tag{B.24}$$

We now are ready to show that equation (B.21) has a solution. For $\eta \in X$, $\tilde{\mathbf{y}}, \tilde{\mathbf{z}} \in C_\gamma(Y)$, using Lemma B.4, equation (B.19) has solutions $\mathbf{y}(T,\eta,\tilde{\mathbf{y}})$ and $\mathbf{y}(T,\eta,\tilde{\mathbf{z}})$, respectively. Thus, we have

$$y_n(T,\eta,\tilde{\mathbf{y}}) - y_n(T,\eta,\tilde{\mathbf{z}}) = \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s}\Big(g_T(x_i(\eta,\tilde{\mathbf{y}}), y_i(\eta,\tilde{\mathbf{y}})) - g_T(x_i(\eta,\tilde{\mathbf{z}}), y_i(\eta,\tilde{\mathbf{z}}))\Big). \tag{B.25}$$

Then, using (B.24) and Lemma B.3, we have

$$\begin{aligned} |\mathbf{y}(T,\eta,\tilde{\mathbf{y}}) - \mathbf{y}(T,\eta,\tilde{\mathbf{z}})|_{C_\gamma(Y)} &\le \Big(1 - \frac{\text{Lip}(g_T)}{1-e^{\gamma-\beta_1}}\Big)^{-1} \frac{\text{Lip}(g_T)\text{Lip}_y(f_T)}{(1-e^{\alpha_1-\gamma})(1-e^{\gamma-\beta_1})}|\tilde{\mathbf{y}}-\tilde{\mathbf{z}}|_{C_\gamma(Y)} \\ &\le \frac{1}{2}|\tilde{\mathbf{y}}-\tilde{\mathbf{z}}|_{C_\gamma(Y)} \end{aligned} \tag{B.26}$$

provided T_0 is sufficiently large. Thus, $\mathbf{y}(T,\eta,\cdot)$ is a contraction in $C_\gamma(Y)$. By the contraction mapping principle, $\mathbf{y}(T,\eta,\cdot)$ has a unique fixed point $\mathbf{y}(T,\eta) = \{y_n(T,\eta)\}_{n\le 0}$ in $C_\gamma(Y)$. Lemma B.4 implies that $\{y_n\}$ is bounded. Therefore, equation (B.21) has a unique solution $(\mathbf{x}(T,\eta), \mathbf{y}(T,\eta))$ with $\mathbf{y}(T,\eta)) \in C_0(Y)$, where $\mathbf{x}(T,\eta)$ is given by $x_{n+1} = f_T(x_n, y_n(T,\eta))$. Furthermore, we have

$$\begin{aligned} |x_n| &\le e^{-\alpha_1 n}|\eta| + \frac{\text{Lip}_y(f_T)e^{-\gamma n}}{(1-e^{\alpha_1-\gamma})}\left(\frac{K_5 e^{-\frac{c}{2}T}}{1-e^{\beta_1}} + |f_T(0,0)|\right), \\ |y_n| &\le \frac{K_5 e^{-\frac{c}{2}T}}{1-e^{\beta_1}}. \end{aligned} \tag{B.27}$$

which together with Lemma B.3 gives that

$$\begin{aligned} |\mathbf{x}|_{C_\gamma(X)} &\le |\eta| + K_5 + |f_T(0,0)| \\ |\mathbf{y}|_{C_0(Y)} &\le K_5 \end{aligned} \tag{B.28}$$

provided μ_0 is small enough and T_0 is sufficiently large.

Next, we show that $\mathbf{y}(T,\cdot)$ is a Lipschitz continuous from X to $C_\gamma(Y)$. Let $\xi,\eta \in X$. Using (B.24), we obtain

$$\begin{aligned} &|y_n(T,\xi) - y_n(T,\eta)| \\ &\le \sum_{i=-\infty}^{n-1} e^{-(n-1-i)\beta_1}\text{Lip}(g_T)\Big(|x_i(T,\xi,\mathbf{y}(T,\xi)) - x_i(T,\eta,\mathbf{y}(T,\eta))| + |y_i(T,\xi)) - y_i(T,\eta)|\Big) \\ &\le \sum_{i=-\infty}^{n-1} e^{-(n-1-i)\beta_1 - i\gamma}\text{Lip}(g_T)\Big(|\xi-\eta| + \big(1 + \frac{\text{Lip }_y f_T}{1-e^{\alpha_1-\gamma}}\big)|\mathbf{y}(T,\xi)) - \mathbf{y}(T,\eta)|_{C_\gamma}(Y)\Big) \end{aligned}$$

which implies that

$$|\mathbf{y}(T,\xi)) - \mathbf{y}(T,\eta)|_{C_\gamma(Y)} \le 4K_5 e^{-\frac{c}{2}T}|\xi-\eta|,$$

provided that T_0 is large enough.

Step 2. $\mathbf{y}(T,\eta)$ *is* C^1.

We first show that for each fixed $T > T_0$, $\mathbf{y}(T,\eta)$ is C^1 in η. Since $C_{\gamma_1}(Y) \subset C_{\gamma_2}(Y)$ for $0 < \gamma_1 < \gamma_2$, the uniqueness of the fixed point of a contraction mapping implies that $\mathbf{y}(T,\eta) \in C_{\gamma-\sigma}(Y)$ for all $\sigma \in [-\sigma_0, \sigma_0]$. We first claim that $\mathbf{y}(T,\cdot) : X \to C_{\gamma-\sigma_0/2}(Y)$ is differentiable. We consider the following equations

$$\begin{cases} U_{n+1} &= D_x f_T\big(x_n(T,\eta), y_n(T,\eta))\big)U_n + D_y f_T(x_n(T,\eta), y_n(T,\eta))V_n \quad U_0 = I; \\ V_n &= \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s}\Big(D_x g_T(x_i(T,\eta), y_i(T,\eta))U_i + D_y g_T(x_i(T,\eta), y_i(T,\eta))V_i\Big). \end{cases}$$

Using the same argument as we used in Step 1, we have that the above equation has a unique solution $(\mathbf{U}(T,\eta),\mathbf{V}(T,\eta)) \in L(X, C_{\gamma-\sigma}(X)) \times L(X, C_{\gamma-\sigma}(Y))$ for all $\sigma \in [-\sigma_0, \sigma_0]$. Next, we show that $\mathbf{V}(T,\eta)$ is the derivative of $\mathbf{y}(T,\eta)$ in η. We want to show that

$$\mathbf{y}(T,\xi) - \mathbf{y}(T,\eta) - \mathbf{V}(T,\eta)(\xi-\eta)o(|\xi-\eta|), \quad \text{as } \xi \to \eta.$$

As we estimated (B.24), we have

$$\begin{aligned} &|x_n(T,\xi) - x_n(T,\eta) - U_n(T,\eta)(\xi-\eta)| \\ &\le e^{-n(\gamma-\sigma_0/2)}\Big(o(|\xi-\eta|)\big(|\mathbf{U}|_{C_{\gamma-\sigma_0}(X)} + |\mathbf{V}|_{C_{\gamma-\sigma_0}(Y)}\big) \\ &\quad + \frac{\text{Lip }_y(f_L)}{1-e^{\alpha_1-\gamma+\sigma_0/2}}\big|\mathbf{y}(T,\xi) - \mathbf{y}(T,\eta) - \mathbf{V}(T,\eta)(\xi-\eta)\big|_{C_{\gamma-\sigma_0/2}(Y)}\Big) \end{aligned} \tag{B.29}$$

Now, we estimate

$$\begin{aligned} &\big|y_n(T,\xi) - y_n(T,\eta) - V_n(T,\eta)(\xi-\eta)\big| \\ &\le \Big|\sum_{i=-\infty}^{n-1} e^{\beta_1(n-1-i)}\Big(\|D_x g_T\|\,|x_i(T,\xi) - x_i(T,\eta) - U_i(T,\eta)(\xi-\eta)| \\ &\quad + \|D_y g_T\|\,|y_i(T,\xi) - y_i(T,\eta) - V_i(T,\eta)(\xi-\eta)| \\ &\quad + o(|\xi-\eta|)e^{-(\gamma-\sigma_0/2)i}\big(|\mathbf{U}|_{C_{\gamma-\sigma_0}} + |\mathbf{V}|_{C_{\gamma-\sigma_0}}\big)\Big) \end{aligned} \tag{B.30}$$

which together with (B.29) implies that

$$|\mathbf{y}(T,\xi) - \mathbf{y}(T,\eta) - \mathbf{V}(T,\eta)(\xi-\eta)\big|_{C_{\gamma-\sigma_0/2}} = o(|\xi-\eta|)\big(|\mathbf{U}|_{C_{\gamma-\sigma_0}} + |\mathbf{V}|_{C_{\gamma-\sigma_0}}\big) = o(|\xi-\eta|).$$

Hence, $\mathbf{y}(T,\cdot)$ is differentiable from X to $L(X, C_{\gamma-\sigma_0}(Y))$ and $D\mathbf{y}(\eta) = \mathbf{V}(\eta)$. Furthermore,

$$|D_\eta \mathbf{y}(T,\eta)|_{C_\gamma(Y)} \le 4K_5 e^{-\frac{c}{2}T}.$$

Similarly, we may show that $D_\eta \mathbf{y}(T,\eta)$ is continuous from X to $L(X, C_\gamma(Y))$.

We now fix η and show that for each $T > T_0$, $\mathbf{y}(\cdot,\eta)$ is differentiable at T in $C_{\gamma-\sigma_0/2}(Y)$ and $D_T\mathbf{y}(\cdot,\eta)$ is continuous at T in $C_\gamma(Y)$). In the following, we fix $\alpha_1, \delta_0, \gamma$, and β_1 given by (B.17) at T. Since $C_{\gamma_1}(Y) \subset C_{\gamma_2}(Y)$ for $0 < \gamma_1 < \gamma_2$, the uniqueness of the fixed point of a contraction mapping gives that $\mathbf{y}(\tilde{T},\cdot) \in C_\gamma(Y))$ for $\tilde{T}$ sufficiently close to T.

We first show that $\mathbf{y}(T,\eta)$ is Lipschitz continuous in T. Let $(\mathbf{x},\mathbf{y})$ and $(\tilde{\mathbf{x}},\tilde{\mathbf{y}})$ denote the solutions of equation (B.21) with T and $\tilde{T}$. Then, we have

$$\begin{aligned} x_{n+1} &= f_T(x_n, y_n), \quad x_0 = \eta; \\ \tilde{x}_{n+1} &= f_{\tilde{T}}(\tilde{x}_n, \tilde{y}_n), \quad \tilde{x}_0 = \eta. \end{aligned}$$

Let $w_n = x_n - \tilde{x}_n$. Then

$$w_{n+1} = D_x f_T(\tau x_n + (1-\tau)\tilde{x}_n, y_n)w_n + f_T(\tilde{x}_n, y_n) - f_T(\tilde{x}_n, \tilde{y}_n) + f_T(\tilde{x}_n, \tilde{y}_n) - f_{\tilde{T}}(\tilde{x}_n, \tilde{y}_n).$$

Using Lemma B.2 and B.3, we have that

$$|w_n| \le e^{\alpha_1}|w_{n+1}| + e^{\alpha_1}\big(\text{Lip }_y f_T|y_n - \tilde{y}_n| + K_5(1 + |\tilde{x}_n| + |\tilde{y}_n|)|T - \tilde{T}|\big).$$

As (B.24), we have

$$|w_n| \le \frac{\text{Lip }_y f_T}{1-e^{\alpha_1-\gamma}}e^{-\gamma n}|\mathbf{y} - \tilde{\mathbf{y}}|_{C_\gamma} + \frac{K_5}{1-e^{\alpha_1-\gamma}}e^{-\gamma n}(1 + |\tilde{\mathbf{x}}|_{C_\gamma} + |\tilde{\mathbf{y}}|_{C_\gamma})|T - \tilde{T}|. \tag{B.31}$$

From the second equation of (B.21), we have

$$y_n - \tilde{y}_n = \sum_{i=-\infty}^{n-1}\Big\{e^{(n-1-i)T\tilde{A}^s}\Big(g_T(x_i, y_i) - g_{\tilde{T}}(\tilde{x}_i, \tilde{y}_i)\Big) + \Big(e^{(n-1-i)T\tilde{A}^s} - e^{(n-1-i)\tilde{T}\tilde{A}^s}\Big)g_{\tilde{T}}(\tilde{x}_i, \tilde{y}_i)\Big\}.$$

Then using (B.31) and Lemma B.3, we obtain

$$
\begin{aligned}
|y_n-\tilde{y}_n| \le & \sum_{i=-\infty}^{n-1} e^{-(n-1-i)\beta_1-\gamma i}\mathrm{Lip}(g_T)\big(1+\frac{\mathrm{Lip}\ _y f_T}{1-e^{\alpha_1-\gamma}}\big)|\mathbf{y}-\tilde{\mathbf{y}}|_{C_\gamma} \\
& + \sum_{i=-\infty}^{n-1} e^{-(n-1-i)\beta_1-\gamma i}K_5 e^{-\frac{c}{2}T}\big(\frac{K_5}{1-e^{\alpha_1-\gamma}}+1\big)(1+|\tilde{\mathbf{x}}|_{C_\gamma}+|\tilde{\mathbf{y}}|_{C_\gamma})|T-\tilde{T}| \\
& + C^*|T-\tilde{T}|e^{-\frac{c}{2}T}
\end{aligned}
\tag{B.32}
$$

which gives

$$
\begin{aligned}
|\mathbf{y}-\tilde{\mathbf{y}}|_{C_\gamma(Y)} &\le 2K_5(2K_5+1)e^{-\frac{c}{2}T}(1+|\mathbf{x}|_{C_\gamma}+|\mathbf{y}|_{C_\gamma})|T-\tilde{T}|+C^*|T-\tilde{T}|e^{-\frac{c}{2}T} \\
&\le \big(4K_5(2K_5+1)(1+|\eta|+K_5+|f_{\tilde{T}}(0,0)|)+C^*\big)e^{-\frac{c}{2}T}|T-\tilde{T}|
\end{aligned}
\tag{B.33}
$$

provided that μ_0 is small enough, T_0 is sufficiently large and $\tilde{T}$ is close to T, where C^* is a constant independent of μ and T. Here, we used (B.28).

We now show that $\mathbf{y}(T,\eta)$ is differentiable at T. We consider the following equations with $\bar{\mathbf{U}}\in C_\gamma(X)$ and $\bar{\mathbf{V}}\in C_\gamma(Y)$

$$
\begin{aligned}
\bar{U}_{n+1} = & D_x f_T\big(x_n(T,\eta),y_n(T,\eta))\big)\bar{U}_n + D_y f_T(x_n(T,\eta),y_n(T,\eta))\bar{V}_n \\
& + D_T f_T\big(x_n(T,\eta),y_n(T,\eta))\big) \quad U_0=0;
\end{aligned}
\tag{B.34}
$$

$$
\begin{aligned}
\bar{V}_n = & \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s}\Big(D_x g_T(x_i(T,\eta),y_i(T,\eta))\bar{U}_i + D_y g_T(x_i(T,\eta),y_i(T,\eta))\bar{V}_i\Big) \\
& + \sum_{i=-\infty}^{n-1}(n-1-i)\tilde{A}^s e^{(n-1-i)T\tilde{A}^s} g_T\big(x_i(T,\eta),y_i(T,\eta)\big) \\
& + \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s} D_T g_T\big(x_i(T,\eta),y_i(T,\eta)\big).
\end{aligned}
\tag{B.35}
$$

First, by Theorem 1.4.3 in Henry [H], there is a constant C_0^* independent of t such that for $1\le k\le 3$,

$$
\big|(\tilde{A}^s)^k e^{t\tilde{A}^s}w\big| \le C_0^* t^{-k}e^{-\beta_0 t}|w|, \quad t>0.
$$

Then, for $\mathbf{x}(T,\eta)\in C_\gamma(X)$ and $\mathbf{y}(T,\eta)\in C_\gamma(Y)$, using Lemma B.3, the above infinite series are well-defined. Again, using the same argument as we used in Step 1, we have that the above equation has a unique solution $(\bar{\mathbf{U}}(T,\eta),\bar{\mathbf{V}}(T,\eta))\in C_{\gamma-\sigma}(X)\times C_{\gamma-\sigma}(Y)$ for all $\sigma\in[-\sigma_0,\sigma_0]$. Next, we show that $\bar{\mathbf{V}}(T,\eta)$ is the derivative of $\mathbf{y}(T,\eta)$ in T. We want to show that

$$
\mathbf{y}(\tilde{T},\eta)-\mathbf{y}(T,\eta)-\bar{\mathbf{V}}(T,\eta)(\tilde{T}-T)=o(|\tilde{T}-T|), \quad \text{as } \tilde{T}\to T.
$$

As we estimated (B.24), we have

$$
\begin{aligned}
& |x_n(\tilde{T},\eta)-x_n(T,\eta)-U_n(T,\eta)(\tilde{T}-T)| \\
& \le e^{-n(\gamma-\sigma_0/2)}\Big(o(|\xi-\eta|)\big(1+|\bar{\mathbf{U}}|_{C_{\gamma-\sigma_0}(X)}+|\bar{\mathbf{V}}|_{C_{\gamma-\sigma_0}(Y)}\big) \\
& \quad + \frac{\mathrm{Lip}\ _y(f_L)}{1-e^{\alpha_1-\gamma}}\big|\mathbf{y}(\tilde{T},\eta)-\mathbf{y}(T,\eta)-\mathbf{V}(T,\eta)(\tilde{T}-T)\big|_{C_{\gamma-\sigma_0/2}(Y)}\Big)
\end{aligned}
\tag{B.36}
$$

Now, estimating $\big|y_n(\tilde{T},\eta)-y_n(T,\eta)-\bar{V}_n(T,\eta)(\tilde{T}-T)\big|$ we get

$$
|\mathbf{y}(\tilde{T},\eta)-\mathbf{y}(T,\eta)-\bar{\mathbf{V}}(T,\eta)(\tilde{T}-T)\big|_{C_{\gamma-\sigma_0/2}}=o(|\tilde{T}-T|)\big(1+|\mathbf{U}|_{C_{\gamma-\sigma_0}}+|\mathbf{V}|_{C_{\gamma-\sigma_0}}\big)=o(|T-\tilde{T}|),
$$

by choosing μ_0 small enough and T_0 large enough. Hence, $\mathbf{y}(T,\eta)$ is differentiable in T and $D_T\mathbf{y}(T,\eta) = \tilde{\mathbf{V}}(T,\eta)$ and

$$\begin{aligned} |D_T\mathbf{y}(T,\cdot)|_{C_\gamma(Y)} &\le \big(4K_5(2K_5+1)(1+|\eta|+K_5+|f_T(0,0)|)+C^*\big)e^{-\frac{c}{2}T} \\ &\le K_6 e^{-\frac{c}{3}T} \end{aligned} \tag{B.37}$$

for some constant K_6 independent of T provided that T_0 is large enough. Here we use the fact $|f_T(0,0)| = O(T)$. Similarly, we may show that the derivatives of first order are continuous in T and η.

Step 3. We show that $\mathbf{y}(T,\eta)$ is C^3.

We claim for each fixed $T > T_0$

$$\mathbf{y}(T,\cdot) : X \to C_{j\gamma}(Y) \text{ is } C^j, \quad D^j_\eta \mathbf{y}(T,\eta) \in L^j(X, C_{j\gamma-\sigma_0})(Y), \quad 1 \le j \le 3$$

and

$$||D^j_\eta \mathbf{y}(T,\eta)||_{L^j(X,C_{j\gamma}(Y))} \le K_6 e^{-\frac{c}{2}T}. \tag{B.38}$$

Here $L^j(X, C_{\gamma j}(Y))$ is the usual space of bounded j linear forms. In the case there is no confusion, we simply use $||\cdot||$ to denote the operator norm.

We prove it by induction. In Step 1 and Step 2, we proved that this is true for $j = 1$. Let $2 \le m \le k-1$. By the induction hypothesis, we have that $\mathbf{y}(T,\cdot)$ is C^j from X to $C_{\gamma j}(Y)$ for all $1 \le j \le m-1$ and $||D^j_\eta \mathbf{y}(T,\eta)||_{L^j(X,C_{\gamma j}(Y))} \le K_j$. We want to show that $\mathbf{y}(T,\cdot)$ is C^m from X to $C_{m\gamma}(Y)$. Note that $D^{m-1}_\eta \mathbf{y}(T,\eta)$ satisfies the following equation

$$\begin{aligned} D^{m-1}x_{n+1}(T,\eta) &= D_x f_T(x_n(T,\eta), y_n(T,\eta))(D^{m-1}x_n(T,\eta) \\ &\quad + D_y f_T(x_n(T,\eta), y_n(T,\eta))D^{m-1}y_n(T,\eta) + R_n(T,\eta); \\ D^{m-1}y_n(T,\eta) &= \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s}\Big(D_x g_T(x_i(T,\eta), y_i(T,\eta))D^{m-1}x_i(T,\eta) \\ &\quad + D_y g_T(x_i(T,\eta), y_i(T,\eta))D^{m-1}y_i(T,\eta)\Big) + \sum_{i=-\infty}^{n-1} e^{(n-1-i)T\tilde{A}^s} S_i(T,\eta). \end{aligned}$$

where

$$\begin{aligned} R_n(T,\eta) &= \sum_{l=0}^{m-3}\binom{m-2}{l} D^{m-2-l}_\eta\big(D_x f_T(x_n(T,\eta), y_n(T,\eta))\big)D^{l+1}x_n(T,\eta) \\ &\quad - \sum_{l=0}^{m-3}\binom{m-2}{l} D^{m-2-l}_\eta\big(D_y f_T(x_n(T,\eta), y_n(T,\eta))\big)D^{l+1}y_n(T,\eta), \\ S_i(T,\eta) &= \sum_{l=0}^{m-3}\binom{m-2}{l} D^{m-2-l}_\eta\big(D_x g_T(x_i(T,\eta), y_i(T,\eta))\big)D^{l+1}x_i(T,\eta) \\ &\quad + \sum_{l=0}^{m-3}\binom{m-2}{l} D^{m-2-l}_\eta\big(D_y g_T(x_i(T,\eta), y_i(T,\eta))\big)D^{l+1}y_i(T,\eta). \end{aligned}$$

Using the induction hypothesis, we have that $R_n(T,\eta)$ and $S_j(T,\eta)$ are C^1 and

$$||DR_n(T,\eta)|| \le Ke^{-(m-1)\gamma n} \quad \text{and} \quad ||DS_j(T,\eta)|| \le Ke^{-(m-1)\gamma j}e^{-\frac{c}{2}T}$$

for some positive constant K independent of n, j. Then, using the same argument as in Step 2, we show that $D^{m-1}\mathbf{y}(T,\cdot)$ is differentiable from X to $L^{m-1}(X, C_{m\gamma-\sigma_0}(Y))$ and $D^m\mathbf{y}(T,\cdot)$ is continuous from X to $L^m(X, C_{m\gamma}(Y))$. Furthermore,

$$||D^m\mathbf{y}(T,\cdot)||_{L^m(X,C_{m\gamma}(Y))} \le K_6 e^{-\frac{c}{2}T},$$

where K_6 is a constant independent of T. Similarly, we have $\mathbf{y}(T,\eta)$ is C^3 in T and η and for $2 \le k \le 3$

$$||D_{T,\eta}\mathbf{y}|| \le K_6 e^{-\frac{c}{3}T}.$$

This completes the proof of Lemma B.5. □

Proof of Proposition B.2 Let $\mathbf{W}(T,\eta) = y_0(T,\eta)$. Then, from Lemma B.5, $\mathbf{W}$ is C^3 from $(T_0,\infty)\times X$ to Y and $|D^j\mathbf{W}(T,\eta)| \le K_6 e^{-\frac{c}{3}T}$, for $j = 1,2,3$. Let

$$\mathcal{W}(T) = \big\{(\eta, \mathbf{W}(T,\eta) \mid \eta \in X\big\}.$$

We claim that $\mathcal{W}$ is an invariant manifold. First, from Lemma B.4 and B.5 we have that for each $\eta \in X$ the following system

$$x_n = f_T(x_{n-1}, y_{n-1}), x_0 = \eta$$
$$y_n = e^{\tilde{A}^s T} y_{n-1} + g_T(x_{n-1}, y_{n-1})$$

has a unique solution $\{(x_n, y_n)\}_{n\le 0}$ such that $\{y_n\}$ is bounded. Set

$$\eta_1 = f_T(\eta, \mathbf{W}(T,\eta)),$$
$$\zeta_1 = e^{\tilde{A}^s T}\mathbf{W}(T,\eta) + g_T(\eta, \mathbf{W}(T,\eta)).$$

By Lemma B.4 and B.5, there exists a unique solution $\{(x_n(T,\eta_1), y_n(T,\eta_1)\}_{n\le 0}$ for equation (B.21) with $x_0 = \eta_1$. Clearly, $y_0(T,\eta_1) = \mathbf{W}(T,\eta_1)$ from the definition of $\mathbf{W}$. On the other hand, $\{(\tilde{x}_n, \tilde{y}_n)\}_{n\le 0}$ with $(\tilde{x}_n, \tilde{y}_n) = (x_{n+1}(T,\eta), y_{n+1}(T,\eta))$ for $n \le -1$ and

$$(\tilde{x}_0, \tilde{y}_0) = \big(f_T(\eta, \mathbf{W}(T,\eta)), e^{\tilde{A}^s T}\mathbf{W}(T,\eta) + g_T(\eta, \mathbf{W}(T,\eta))\big)$$

is also a solution for equation (B.21) with $x_0 = \eta_1$. The uniqueness implies that $(x_n(T,\eta_1), y_n(T,\eta_1)) = (\tilde{x}_n, \tilde{y}_n)$. Thus, $\zeta_1 = \mathbf{W}(T,\eta_1)$. Hence, $\mathcal{W}$ is invariant. Clearly, $\mathbf{W}(T,\eta) = \mathbf{W}(T,\xi,\Theta)$ is periodic in Θ since the system (B.21) is periodic in Θ. □

Restricting to $\mathbb{A}$ and using Proposition B.2, we have Proposition 6.3

Appendix C. Proofs of Proposition 7.1 and Lemma 7.1

We assume at the outset that $L > 10^5$. [4] The following is from direct computation:

Lemma C.1. *Let $f = f_a$, any a. Then*
(a) $|f'| < \frac{3}{2}L$; $|f''| < 5L$;
(b) in $C_{\frac{1}{3}}$: $|f''| > \frac{2}{5}L$, and $|f''x|/|f''y| < \frac{5}{2}$ for all x, y;
(c) outside of $C_{\frac{1}{3}}$: $|f'| > \frac{2}{5}L \cdot \frac{1}{3} = \frac{2}{15}L$.

Let $\sigma = L^{-\frac{1}{4}}$. First we note that from Lemma C.1(b)(c) we have, for $x \notin C_\sigma$,

$$|f_a'(x)| > \frac{2}{5}L^{\frac{3}{4}}. \tag{C.1}$$

This is because if $x \in C_{\frac{1}{3}}$, then $|f_a'(x)| > (\min_{C_\sigma}|f''|)\cdot\sigma > \frac{2}{5}L^{\frac{3}{4}}$, and if $x \notin C_{\frac{1}{3}}$, then $f_a'(x) > \frac{2}{15}L > \frac{2}{5}L^{\frac{3}{4}}$. Now for a in an interval Δ and $k = 1,2$, we define $\gamma_0^{(k)}(a) \equiv c_k$, and let $\gamma_i^{(k)}(a) := f_a^i(\gamma^{(k)}(a))$ for $i \ge 1$. Then

$$\frac{d}{da}\gamma_1^{(k)}(a) \equiv 1, \qquad \text{and} \qquad \frac{d}{da}\gamma_{i+1}^{(k)}(a) = f_a'(\gamma_i^{(k)}(a))\frac{d}{da}\gamma_i^{(k)}(a) + 1.$$

If we stipulate that $\gamma_j^{(k)}(a) \notin C_\sigma$ for all $j \le i$, then we have

$$|\frac{d}{da}\gamma_{i+1}^{(k)}(a)|/|\frac{d}{da}\gamma_i^{(k)}(a)| \approx f_a'(\gamma_i^{(k)}(a)) > \frac{2}{5}L^{\frac{3}{4}} > 400. \tag{C.2}$$

[4] We focus here on explaining the ideas of the proofs; the constants are very far from optimal due to the repeated use of crude worst-case-scenario estimates.

Lemma C.2. *Let Δ be such that the following hold for $k = 1, 2$ and $n > 1$:*
(i) $\gamma_i^{(k)}(\Delta) \cap C_\sigma = \emptyset$ for $0 < i < n$;
(ii) $|\gamma_i^{(k)}(\Delta)| \le \frac{1}{4}\sigma$ for $0 < i < n$.
Then for all $a, a' \in \Delta$,

$$\left| \frac{\frac{d}{da}\gamma_n^{(k)}(a)}{\frac{d}{da}\gamma_n^{(k)}(a')} \right| < 2.$$

Proof: Let $\Theta_i = \gamma_i^{(k)}(a)$ and $\Theta_i' = \gamma_i^{(k)}(a')$. Then

$$\frac{\frac{d}{da}\Theta_n}{\frac{d}{da}\Theta_n'} = \frac{f_a'(\Theta_{n-1})\frac{d}{da}\Theta_{n-1} + 1}{f_{a'}'(\Theta_{n-1}')\frac{d}{da}\Theta_{n-1}' + 1} \approx \frac{f_a'(\Theta_{n-1})\frac{d}{da}\Theta_{n-1}}{f_{a'}'(\Theta_{n-1}')\frac{d}{da}\Theta_{n-1}'},$$

so

$$\log \frac{\frac{d}{da}\Theta_n}{\frac{d}{da}\Theta_n'} \approx \sum_{i=1}^{n-1} \log \frac{f_a'(\Theta_i)}{f_{a'}'(\Theta_i')} \le \sum_{i=1}^{n-1} \frac{|f_a'(\Theta_i) - f_{a'}'(\Theta_i'))|}{|f_{a'}'(\Theta_i')|} = \sum_{i=1}^{n-1} \frac{|f_a''(\hat{\Theta}_i)|}{|f_a'(\Theta_i')|}|\Theta_i - \Theta_i'|$$

where $\hat{\Theta}_i$ lies between Θ_i and Θ_i'. We have used $f_a' \equiv f_{a'}'$ in the last equality. We estimate $(*) := |f_a''(\hat{\Theta}_i)|/|f_a'(\Theta_i')|$ as follows: If $\Theta_i, \Theta_i' \in C_{\frac{1}{3}}$, then $|f_a'(\Theta_i')| > |f_a''(\check{\Theta}_i)| \cdot \sigma$ for some $\check{\Theta}_i$ between Θ_i' and the critical point, and $(*) \le \frac{|f_a''(\hat{\Theta}_i)|}{|f_a''(\check{\Theta}_i)|}\frac{1}{\sigma} < \frac{5}{2}L^{\frac{1}{4}}$ by Lemma C.1(b). If one of Θ_i and Θ_i' is not in $C_{\frac{1}{3}}$, then neither is in C_γ where $\gamma = \frac{1}{3} - \frac{1}{4}\sigma > \frac{1}{3} - \frac{1}{60} = \frac{19}{60}$, and by Lemma C.1(a)(c), $(*) < (5L)/(\frac{2}{5}L \cdot \frac{19}{60}) = \frac{750}{19}$. Let $C = \max\{\frac{5}{2}L^{\frac{1}{4}}, \frac{750}{19}\}$. Then

$$\log \frac{\frac{d}{da}\Theta_n}{\frac{d}{da}\Theta_n'} \le C \sum_{i=1}^{n-1} |\Theta_i - \Theta_i'| \le C \left(\sum_{i=0}^{n-1} (\frac{2}{5}L^{\frac{3}{4}})^{-i} \right) |\Theta_{n-1} - \Theta_{n-1}'| < \log 2.$$

Here (C.2) is used in the second inequality, and $|\Theta_{n-1} - \Theta_{n-1}'| \le \frac{1}{4}L^{-\frac{1}{4}}$ the last. □

Proof of Proposition 7.1: We describe below an algorithm that produces, for $n = 1, 2, \cdots$, a decreasing sequence of parameter intervals $\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \cdots$ and for each critical point $c_k, k = 1, 2$, a sequences of times $i_{k,1} \le i_{k,2} \le \cdots$ with $i_{k,n+1} = i_{k,n}$ or $i_{k,n} + 1$. The sought after parameter a^* will be in $\cap_i \Delta_i$, and the $i_{k,n}$ are used to adjust the number of iterates so as to control the lengths of $\gamma_{i_{k,n}}^{(k)}(\Delta_n)$. If this curve is too short, it may fall into C_σ in the next iterate; and if it winds around $\mathbb{S}^1$ too many times, a subinterval of Δ_n corresponding to $\mathbb{S}^1 \setminus C_\sigma$ may be very short, and that can be problematic for *the other* critical point.

Specifically, we call $(\Delta_n; i_{1,n}, i_{2,n})$ an admissible configuration if the following conditions are satisfied for $k = 1, 2$:

(A1) $\gamma_i^{(k)}(\Delta_n) \cap C_\sigma = \emptyset$ for all $i \le i_{k,n}$.
(A2) For all $a, a' \in \Delta_n$,

$$\left| \frac{\frac{d}{da}\gamma_{i_{k,n}}(a)}{\frac{d}{da}\gamma_{i_{k,n}}(a')} \right| < 2.$$

(A3) (minimum length one iteration later): $\left|\gamma_{i_{k,n}+1}^{(k)}(\Delta_n)\right| \ge 10\sigma$.

To construct an admissible configuration for $n = 1$, we let $i_{k,1} = 1$ for $k = 1, 2$. Since $\frac{d}{da}\gamma_1^{(k)}(a) = 1$, $|\gamma_1^{(k)}(\Delta_0)| = 5\sigma < \frac{1}{2}$, so each $\gamma_1^{(k)}(\Delta_0)$ meets at most one component of C_σ. Now even in the worst-case scenario where the two intervals $(\gamma_1^{(k)})^{-1}C_\sigma, k = 1, 2$, of length 2σ each, are evenly spaced in Δ_0, we can still find an interval $\Delta_1 \subset \Delta_0$ with $|\Delta_1| = \frac{1}{3}\sigma$ such that $\gamma_1^{(k)}(\Delta_1) \cap C_\sigma = \emptyset$ for $k = 1, 2$. (A3) holds since $|\gamma_2^{(k)}(\Delta_1)| > \frac{2}{5}L^{\frac{3}{4}}|\gamma_1^{(k)}(\Delta_1)| = (\frac{2}{5}L^{\frac{3}{4}})(\frac{1}{3}\sigma) > 10\sigma$.

Assume now we are handed an admissible configuration $(\Delta_n; i_{1,n}, i_{2,n})$. For each k, we iterate forward and set $i_{k,n+1} = i_{k,n} + 1$ if both (A4) and (A5) below are satisfied. If one of these conditions fails, we do not iterate, setting $i_{k,n+1} = i_{k,n}$:

(A4) $\left|\gamma^{(k)}_{i_{k,n}}(\Delta_n)\right| \le \frac{1}{4}\sigma$ (so our distortion estimation holds for the next iteration);

(A5) $\gamma^{(k)}_{i_{k,n}+1}(\Delta_n)$ meets at most one component of C_σ.

Consider first the case where the rule is to iterate forward for both $k = 1, 2$. The parameter interval Δ_{n+1} is chosen as follows: By (A3) and (A5), $10\sigma < |\gamma^{(k)}_{i_{k,n+1}}(\Delta_n)|$, and $\gamma^{(k)}_{i_{k,n+1}}(\Delta_n)$ meets at most one component of C_σ, so the fraction of $\gamma^{(k)}_{i_{k,n+1}}(\Delta_n)$ in C_σ is $\le \frac{1}{5}$. By virtue of (A4) and Lemma C.2, we have

$$|(\gamma^{(k)}_{i_{k,n+1}})^{-1}C_\sigma| < \frac{2}{4\times 1 + 1\times 2}|\Delta_n| = \frac{1}{3}|\Delta_n|. \tag{C.3}$$

(The bound $\frac{1}{3}$ is obtained by taking the derivative of $(\gamma^{(k)}_{i_{k,n+1}})^{-1}$ on C_σ to be twice that outside.) Thus even in the worst-case scenario (as in the $n = 1$ case), there exists a subinterval $\Delta_{n+1} \subset \Delta_n$ of length $\frac{1}{9}|\Delta_n|$ with the property that $\gamma^{(k)}_{i_{k,n+1}}(\Delta_{n+1}) \cap C_\sigma = \emptyset, k = 1, 2$. For this choice of Δ_{n+1}, we have (A1) by design. (A2) is guaranteed by (A4) for step n and Lemma C.2. As for (A3), another application of our distortion estimate together with (A3) from step n gives $|\gamma^{(k)}_{i_{k,n+1}}(\Delta_{n+1})| > \frac{1}{8\times 2+1\times 1}\cdot 10\sigma = \frac{1}{17}\cdot 10\sigma$. One iteration later, such a segment is guaranteed to have a length $> \frac{2}{5}L^{\frac{3}{4}}\cdot\frac{1}{17}10\sigma > 10\sigma$.

Next, consider the case where the rule is to iterate forward one of the critical curves, say $k = 1$, to be definite, but not the other. In this case, even if $(\gamma^{(1)}_{i_{1,n+1}})^{-1}C_\sigma$ is situated exactly in the middle of Δ_n, we may choose Δ_{n+1} with $|\Delta_{n+1}| = \frac{1}{3}|\Delta_n|$ so that $\gamma^{(1)}_{i_{1,n+1}}(\Delta_{n+1}) \cap C_\sigma = \emptyset$. (A1)–(A3) are verified for this critical curve as above. For $k = 2$, (A1) and (A2) are inherited from the previous step, and (A3) is checked as follows: if (A4) fails for $k = 2$, then

$$|\gamma^{(2)}_{i_{2,n+1}}(\Delta_{n+1})| = |\gamma^{(2)}_{i_{2,n}}(\Delta_{n+1})| \ge \frac{1}{5}|\gamma^{(2)}_{i_{2,n}}(\Delta_n)| > \frac{1}{20}\sigma.$$

Notice that the first inequality uses only (A2) from step n. One iteration later, this curve will have length $> \frac{2}{5}L^{\frac{3}{4}}\cdot\frac{1}{20}\sigma > 10\sigma$. If (A4) holds but (A5) fails, then the distortion estimates hold for the next iterate, and

$$|\gamma^{(2)}_{i_{2,n+1}+1}(\Delta_{n+1})| \ge \frac{1}{5}|\gamma^{(2)}_{i_{2,n+1}+1}(\Delta_n)| \ge \frac{1}{5}|\gamma^{(2)}_{i_{2,n}+1}(\Delta_n)| > \frac{1}{5}(2\pi - 2\sigma)$$

which is $> 10\sigma$. This completes the construction for step $n + 1$ in this case.

Finally, suppose neither $k = 1$ or 2 is to move forward. We let Δ_{n+1} be the left half of Δ_n, and observe that the new configuration is again admissible: For each k, and verify (A3) by arguing separately as in the last paragraph the two cases corresponding to (i) the failure of (A4) and (ii) the failure of (A5) but not (A4). Repeat this process until one of the critical curves is short enough to move forward. □

Before proceeding to the proof of Lemma 7.1, we state a distortion lemma similar to Lemma C.2 and easier:

Sublemma C.1. *Let $\Theta, \hat{\Theta}$ and $n \in \mathbb{Z}^+$ be such that ω_i, the segment between $f^i(\Theta)$ and $f^i(\hat{\Theta})$, satisfies $\omega_i \cap C_\sigma = \emptyset$ and $|\omega_i| < \frac{1}{4}\sigma$ for all $0 \le i < n$. Then*

$$\left|\frac{(f^n)'(\Theta)}{(f^n)'(\hat{\Theta})}\right| \le 2.$$

Proof of Lemma 7.1: (a) Following the proof of (C.1), we obtain $|f'| > \frac{2}{5}L^{\frac{1}{4}}$ on $\mathbb{S}^1 \setminus I$.

(b) For $\Theta \in I$, notice first that $n(\Theta) \ge 2$, since $|f(\Theta) - f(c)| \le \frac{1}{2}(\max_I |f''|)\delta_0^2 \le \frac{1}{2}\cdot 5L\cdot(L^{-\frac{3}{4}})^2$, which is $< \frac{1}{4}\sigma$. Writing $n = n(\Theta)$, we estimate $|(f^n)'(\Theta)|$ as follows. Since $|f^n x - f^n c| > \frac{1}{4}\sigma$, it follows from the sublemma above that for some Θ_1 between Θ and c,

$$\frac{1}{2}|f''(\Theta_1)||x - c|^2 \ \cdot 2|(f^{n-1})'(fc)| \ > \ \frac{1}{4}\sigma\ . \tag{C.4}$$

Reversing the inequality at time $n-1$ and using the sublemma again, we have

$$\frac{1}{2}|f''(\Theta_2)||\Theta - c|^2 \cdot \frac{1}{2}|(f^{n-2})'(fc)| \leq \frac{1}{4}\sigma \tag{C.5}$$

for some Θ_2 between Θ and c. Substituting the estimate for $|(f^{n-1})'(fc)|$ from (C.4) into

$$|(f^n)'\Theta| \geq |f''(\Theta_3)||\Theta - c| \cdot \frac{1}{2}\,|(f^{n-1})'(fc)|\,,$$

we obtain

$$|(f^n)'\Theta| \geq \frac{1}{2}\,\frac{|f''(\Theta_3)|}{|f''(\Theta_1)|}\,\sigma\,\frac{1}{|\Theta - c|}\,.$$

Finally, substituting the estimate for $|\Theta - c|$ from (C.5) into the last inequality and use the lower bounds for $|f''(\Theta_2)|$ and $|(f^{n-2})'(fc)|$ from Lemma C.1, we arrive at

$$|(f^n)'\Theta| > \frac{1}{8}\,\frac{|f''(\Theta_3)|}{|f''(\Theta_1)|}\,\sqrt{\sigma}\,|f''(\Theta_2)|^{\frac{1}{2}}\,|(f^{n-2})'(fc)|^{\frac{1}{2}} > A\,L^{\frac{1}{2}-\frac{1}{8}} \cdot \left(\frac{1}{2}L^{\frac{3}{8}}\right)^{n-2}\,.$$

Here A is a constant independent of L; we have used in the last inequality that the f^{n-1}-image of the segment between Θ and c is outside of $C_{\frac{3}{4}\sigma}$. For $n \geq 2$, this is $\geq A'(\frac{1}{2}L^{\frac{3}{16}})^n > L^{\frac{1}{6}n}$ for L is large enough. □

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