Nonuniformly Expanding 1D Maps With Logarithmic Singularity

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Let $f_{a,L} : \mathbb{R} \to \mathbb{R}$ be such that

$$f_{a,L} \colon \theta \mapsto \theta + a + L \ln |\Phi(\theta)| \tag{1}$$

where $a \in [0,1]$, $L \in \mathbb{R}$ are real parameters and $\Phi(\theta)$ is such that $\Phi(\theta+1) = \Phi(\theta)$. We assume that $\Phi(\theta)$ is a Morse function and the graph of $y = \Phi(\theta)$ is transversal to the θ-axis. The functions $f_{a,L}$ induce a two parameter family of 1D maps from S^1 to S^1 where $S^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle. In this paper we prove that there exists an $L_0 > 0$ sufficiently large, so that for every L satisfying $|L| > L_0$, there exists a set $\Delta(L)$ of positive measure for a, such that for $a \in \Delta(L)$, $f_{a.L}$ admits an invariant measure that is absolutely continuous with respect to the Lebesgue measure. We also prove that $|\Delta(L)| \to 1$ as $|L| \to \infty$. If $\Phi(\theta) \neq 0$ for all $\theta \in S^1$, then this result (minus the asymptotic estimate on $|\Delta(L)|$) follows from combining the theory of [WY1] on multi-modal 1D maps with the proof of [WY2] on the existence of multi-modal Misiuriewicz maps. When $\Phi(\theta) = 0$ is allowed, however, no previous theory on 1D maps apply and the result of this paper is new.

Our study of $f_{a,L}$ in (1) is motivated by the recent studies of [WO], [WOk] and [W] on homoclinic tangles and strange attractor in periodically perturbed differential equations. When a homoclinic solution of a dissipative saddle is periodically perturbed, the perturbation either pulls the stable and the unstable manifold of the saddle fix point completely apart, or it creates chaos through homoclinic intersections. In both cases, the separatrix map induced by the solutions of the perturbed equation in the extended phase space is a family of 2D maps with a singular 1D limit in the form of (1) (with the absolute value sign around $\Phi(\theta)$ removed). Let μ be a small parameter representing the magnitude of the perturbation and ω be the forcing frequency. We have $a \sim \omega \ln \mu^{-1}$ $mod(1)$, $L \sim \omega$; and $\Phi(\theta)$ is the classical Melnikov function (See [WO] and [WOk]).

When we start with *two* unperturbed homoclinic loops and assume symmetry, then the separatrix maps are a family of 2D maps, the 1D singular limit of which is precisely $f_{a,L}$ in (1) (See [W]). If the stable and unstable manifolds of the perturbed saddle are pulled completely apart by the forcing function, then $\Phi(\theta) \neq 0$ for all θ. In this case we obtain strange attractors, to which the theory of rank one maps developed in [WY3] apply. If the stable and unstable manifold intersect, then $\Phi(\theta) = 0$ is allowed and the strange attractors are associated to homoclinic intersections. For the modern theory of chaos and dynamical systems, this is a case of historical and practical importance; see [GM], [SSTC1], [SSTC2]. To this case, unfortunately, the theory of rank one maps in [WY3] does not apply because of the existence of logarithmal singularities in $f_{a,L}$. Our ultimate goal is to develop a theory that can be applied to the separatrix maps allowing $\Phi(\theta) = 0$. This paper is the first step, in which we develop a 1D theory.

We now present our main result in precise terms. For $f = f_{a,L}$, let $C(f) = \{f'(\theta) = 0\}$ be the set of critical points and $S(f) = \{\Phi(\theta) = 0\}$ be the set of singular points. The distances from $\theta \in S^1$ to $C(f)$ and $S(f)$ are denoted as $d_C(\theta)$ and $d_S(\theta)$ respectively.

Assumptions on $\Phi(\theta)$*:* We assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is C^2 satisfying (i) $\Phi(\theta) = \Phi(\theta+1)$; (ii) $\Phi'(\theta) \neq 0$ on $\{\Phi(\theta) = 0\}$; and (iii) $\Phi''(\theta) \neq 0$ on $\{\Phi'(\theta) = 0\}$.

We also need a few constructive parameters and they are λ, α, N_0, σ and δ:

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- λ is the targeted exponent for derivative growth (see Theorem B(a));
- α is a small exponent that confines the returns of critical orbits to the critical and the singular set (see (D1) and (D2) in Theorem A(b));
- N_0 and $\sigma = |L|^{-\frac{1}{6}}$ are for the initial set ups: all critical orbits are kept out of the σ-neighborhood of the critical and the singular set for the first N_0 iterations;
- $\delta = |L|^{-\alpha N_0}$ is the size of the critical interval, to which the accumulative effect of returns of critical orbits are counted (see (D1) in Theorem A(b)).

The choices for λ and α are quite arbitrary as far as $0 \ll \alpha \ll \lambda \ll 1$. A valid choice for λ and α is $\lambda = 10^{-3}$ and $\alpha = 10^{-6}$. N_0 is a large integer independent of L. It must satisfy (36) located at the end of Sect. 2.2C. For $c \in C(f)$, let $v_0 = f(c)$ be the critical value and $\{v_i = f^i v_0,\ i \in \mathbb{Z}^+\}$ be the critical orbit.

Theorem A. *There exists an L_0 sufficiently large so that for every L satisfying $|L| > L_0$, there exists $\Delta(L) \subset [0,1)$ of positive Lebesgue measure for the parameter a, such that for $a \in \Delta(L)$ and for all $c \in C(f)$, the following hold for $f = f_{a,L}$:*
(a) for $0 \le n \le N_0$, $d_C(v_n),\ d_S(v_n) > \sigma$;
(b) for all $n > N_0$,
(D1) $\prod_{N_0 < i \le n:\ d_C(v_i) < \delta} d_C(v_i) \ge |L|^{-\alpha n}$;
(D2) $d_S(v_n) > |L|^{-4\alpha n}$.
In addition, we have $|\Delta(L)| \to 1$ as $|L| \to \infty$.

We also have

Theorem B. *If $f = f_{a,L}$ satisfies the conclusions (a) and (b) of Theorem A, then*
(a) $|(f^n)' v_0| > |L|^{\lambda n}$ for all $c \in C(f)$ and for all $n > 0$; and
(b) f admits an invariant measure that is absolute continuous with respect to the Lebesgue measure.

A dynamics theory for uni-modal 1D maps was developed in a series of papers in the late 1970s and 1980s ([J], [M], [CE], [BC1], [R], [BC2], [NS] [T1] and others). For multi-modal 1D maps, two approaches, both originated from [BC1] and [BC2], are later developed separately in [T2] and [WY1]. In this paper we elect to follow the approach of [T1] and [T2] because the extension of this approach is more transparent in our dealing with the issues related to large distortion and abnormal derivative growth caused by the logarithmal singularity of $f_{a,L}$. We note that the rule of distance exclusion (D1) is different from the ones used in [WY1] and [BC1]. It implies more readily the exponential growth of derivatives along all critical orbits as asserted in Theorem B(a).

1. Phase space analysis

In this section we study the dynamics of maps of good critical behavior. Elementary facts on $f_{a,L}$ are introduced in Sect. 1.1. In Sect. 1.2 we first discuss the initial set ups then introduce conditions $(\mathrm{G1})_n$-$(\mathrm{G3})_n$ for maps of good critical behavior. In Sects. 1.3 and 1.4 we study the dynamics of maps satisfying $(\mathrm{G1})_n$-$(\mathrm{G3})_n$.

1.1. **Elementary facts.** From this point on we use L for both L and $|L|$. For $f = f_{a,L}$, let $C(f) = \{f'(\theta) = 0\}$ be the set of critical points and $S(f) = \{\Phi(\theta) = 0\}$ be the set of singular points. For $\varepsilon > 0$, we use C_ε and S_ε to denote the ε-neighborhoods of C and S respectively. The distances from $\theta \in S^1$ to $C(f)$ and $S(f)$ are denoted as $d_C(\theta)$ and $d_S(\theta)$ respectively. We denote $v_0 = f(c)$ for $c \in C(f)$, and $v_i = f^i v_0$. We also use $log(\cdot)$ for the logarithmal function of base L. This is to say that $log(\cdot) = log_L(\cdot)$. Finally, the letter K is used for generic constants independent of λ, α, N_0 and L. The values of K may vary from expression to expression.

Lemma 1.1. *There exists $K_0 > 1$ and $\varepsilon_0 > 0$, such that for all L sufficiently large and $f = f_{a,L}$,*
(a) for all $\theta \in S^1$,

$$K_0^{-1} L \frac{d_C(\theta)}{d_S(\theta)} \le |f'\theta| \le K_0 L \frac{d_C(\theta)}{d_S(\theta)}, \qquad |f''\theta| \le \frac{K_0 L}{d_S^2(\theta)};$$

(b) for all $\varepsilon > 0$ and $\theta \notin C_\varepsilon$, $|f'\theta| \ge K_0^{-1} L\varepsilon$; and
(c) for all $\theta \in C_{\varepsilon_0}$, $K_0^{-1} L < |f''\theta| < K_0 L$.

Proof: This lemma follows immediately from

$$f'\theta = 1 + L \cdot \frac{\Phi'(\theta)}{\Phi(\theta)}; \qquad f''\theta = L \cdot \frac{\Phi''(\theta)\Phi(\theta) - (\Phi'(\theta))^2}{\Phi^2(\theta)}$$

and our assumptions on $\Phi(\theta)$ in previous section. □

1.2. **Maps of good critical behavior.** In this subsection we start with a rather straight forward measure estimate on the set of parameters satisfying Theorem A(a). We then introduce properties $(\mathrm{G1})_n$-$(\mathrm{G3})_n$ for maps in $f_{a,L}$. Let L be fixed and denote $f_a = f_{a,L}$. Observe that the set of critical points C of f_a is independent of a.

A. Initial good critical behavior Consider the one parameter family $f_a = f_{a,L}$. We start with maps satisfying Theorem A(a), of which all critical orbits are kept out of $C_\sigma \cup S_\sigma$ for an initial stretch of N_0 iterations. One way to find these maps is to first look for Misiuriewicz maps, of which all critical orbits stay out of $C_\sigma \cup S_\sigma$ for all forward times. We would then confine ourselves in a small parameter interval of a Misiuriewicz map, and we would eventually prove that the Misiuriewicz parameter is a Lebesgue density points of the good parameter set $\Delta(L)$. This approach for initial set ups, however, is with some drawbacks. First, for a one parameter family of multi-modal maps, the Misiuriewicz maps are relatively hard to come around because of the need of controlling many critical orbits by one parameter. Though the argument in [WY2] is readily extended to cover the 1D family $f_a = f_{a,L}$, we are nevertheless up to a hard start. Second, with the rest of the study confined in a small neighborhood of a Misiuriewicz map, it is not clear how we could prove the global asymptotic measure estimate ($\Delta(L) \to 1$ as $|L| \to \infty$) of Theorem A.

An alternative route that is made possible by the approach of this paper is for us to start with a rather straight forward but relatively weak estimate on the measure of parameters satisfying Theorem A(a). This is a much easier initial set up. To be more precise, we let

$$\Delta_n(L) = \{a \in [0,1) \colon f_a^{i+1}(C) \cap (C_\sigma \cup S_\sigma) = \emptyset \text{ for every } 0 \le i \le n\},$$

and start with the following measure estimate on $\Delta_{N_0}(L)$.

Lemma 1.2. *There exists an $L_0 = L_0(N_0)$ sufficiently large, such that for any given L satisfying $|L| \ge L_0$,*

$$|\Delta_{N_0}(L)| \ge 1 - N_0 L^{-\frac{1}{9}}.$$

Lemma 1.2 is sufficient for us to move forward. This approach for initial setups is easier, new, and it leads to the desired asymptotic measure estimate on $\Delta(L)$.

We move to the expanding property of the maps in $\Delta_{N_0}(L)$. Recall that λ is the targeted exponent for derivative growth; $\delta = L^{-\alpha N_0}$. We have

Lemma 1.3. *Assume that $L > L_0(N_0)$ is sufficiently large and let $f = f_{a,L}$ be such that $a \in \Delta_{N_0}(L)$. Then, (a) if $n \ge 1$ and $\theta, f\theta, \cdots, f^{n-1}\theta \notin C_\delta$, then $|(f^n)'\theta| \ge \delta L^{2\lambda n}$; and (b) if moreover $f^n\theta \in C_\delta$, then $|(f^n)'\theta| \ge L^{2\lambda n}$.*

Lemma 1.3 is about the dynamics of f_a outside of C_δ for all a in $\Delta_{N_0}(L)$. It states that, for orbit segments that are out of C_δ, we have exponential growth of derivatives. Lemmas 1.2 and 1.3 are proved in the Appendix.

Standing assumption: In the rest of this section we assume $a \in \Delta_{N_0}(L)$.

B. Good critical behavior Let L be fixed and $f_a = f_{a,L}$. For $f = f_a$ and $c \in C$, we say that the orbit of $v_0 = f(c)$ has good critical behavior up to $n > N_0$ if,

(G1)$_n$ $|(f^{j-i})'v_i| \geq L \min\{\sigma, L^{-\alpha i}\}$ for every $0 \leq i < j \leq n+1$;
(G2)$_n$ $|(f^i)'v_0| \geq L^{\lambda i}$ for every $0 < i \leq n+1$;
(G3)$_n$ $d_S(v_i) \geq L^{-4\alpha i}$ for every $N_0 \leq i \leq n$.

We say that f satisfies (G1)-(G3) up to time $n > N_0$ if (G1)$_n$-(G3)$_n$ hold for all $c \in C$. Item (G3) is (D2) in Theorem A(b). Item (G2) is a straight forward condition on exponential derivative growth. Item (G1) is also a condition on derivatives. It requires that from any time i to $j > i$, the derivatives never drop accumulatively to lower than $L^{-\alpha i}$, which is independent of j. Items (G1) and (G2) are obviously *independent conditions*, and as we will see momentarily in Sect. 1.4A, they both follow from (D1) and (D2) in Theorem A(b). In between (G1) and (G2), (G1) is less intuitive than (G2). We prefer (G1) and (G2) over (D1) because conditions on derivatives are easier to use in phase space analysis.

1.3. **Dynamics of maps of good critical behavior.** In this subsection we first prove a statement on distortion. We then define the bound period and prove a standard result on bound recovery of the derivatives. We note that there is a major combinatorial difference in the ways the bound periods are defined in this paper and in [WY1]. For $\theta \in C_\delta$ let $d_C(\theta) = |\theta - c|$ where $c \in C$. The bound period of θ in this paper is defined explicitly by the properties of the critical orbit of $v_0 = f(c)$ and the initial distance $|\theta - c|$ (see Definition 1.1), not by the future distances between $f^p(\theta)$ and $f^p(c)$, as was used in [WY1] following [BC1].

A. Distortion and bound period For $\theta_* \in S^1$, $n \geq 1$, let

$$D_n(\theta_*) = \frac{1}{\sqrt{L}} \cdot \left[\sum_{0 \leq i \leq n-1} d_i^{-1}(\theta_*) \right]^{-1} \quad \text{where} \quad d_i(\theta_*) = \frac{d_C(f^i\theta_*) \cdot d_S(f^i\theta_*)}{|(f^i)'\theta_*|}. \tag{2}$$

In particular, we adopt the convention that $|(f^i)'\theta| = 1$ for $i = 0$, so $d_0(\theta_*) = d_C(\theta_*) \cdot d_S(\theta_*)$.

Lemma 1.4 (Bound Distortion)**.** *For $\theta \in [\theta_* - D_n(\theta_*), \theta_* + D_n(\theta_*)]$ we have*

$$\frac{1}{2} \leq \frac{|(f^n)'\theta|}{|(f^n)'\theta_*|} \leq 2$$

provided that $\{f^i\theta_*\}_{i=0}^{n-1} \cap (C \cup S) = \emptyset$.

Proof: Denote $I = [\theta_* - D_n(\theta_*), \theta_* + D_n(\theta_*)]$. For $\theta \in I$,

$$\log \frac{|(f^n)'\theta|}{|(f^n)'\theta_*|} \leq \sum_{0 \leq j \leq n-1} \log \left| 1 + \frac{f'(f^j\theta) - f'(f^j\theta_*)}{f'(f^j\theta_*)} \right| \leq \sum_{0 \leq j \leq n-1} |f^j I| \sup_{\phi, \psi \in f^j I} \frac{|f''\phi|}{|f'\psi|}.$$

Lemma 1.4 would hold if for all $j \leq n-1$ we have $f^j I \cap (S \cup C) = \emptyset$ and

$$|f^j I| \sup_{\phi, \psi \in f^j I} \frac{|f''\phi|}{|f'\psi|} \leq \log 2 \cdot d_j^{-1}(\theta_*) \left[\sum_{0 \leq i \leq n-1} d_i^{-1}(\theta_*) \right]^{-1}. \tag{3}$$

We prove (3) by induction on j. Assume (3) holds for all $j < k$. Summing (3) over $j = 0, 1, \cdots, k-1$ implies

$$\frac{1}{2} \leq \frac{|(f^k)'\theta|}{|(f^k)'\theta_*|} \leq 2 \tag{4}$$

for all $\theta \in I$. We have

$$|f^k I| \leq 2|(f^k)'\theta_*|D_n = 2d_k^{-1}(\theta_*) \cdot d_C(f^k\theta_*) \cdot d_S(f^k\theta_*)D_n \leq \frac{K}{\sqrt{L}} \min\{d_C(f^k\theta_*), d_S(f^k\theta_*)\} \tag{5}$$

where the first inequality follows from the mean value theorem and (4), and (2) for $d_k(\theta_*)$ is used for the equality in the middle. We have $f^k I \cap (C \cup S) = \emptyset$ from (5); and for $\phi, \psi \in f^k I$,

$$\begin{aligned} |f^k I|\frac{|f''\phi|}{|f'\psi|} &\leq K|(f^k)'\theta_*|D_n \cdot \frac{d_S(\psi)}{d_C(\psi)d_S^2(\phi)} \leq Kd_k^{-1}(\theta_*)d_C(f^k\theta_*)d_S(f^k\theta_*)D_n \cdot \frac{d_S(\psi)}{d_C(\psi)d_S^2(\phi)} \\ &= Kd_k^{-1}(\theta_*)D_n \cdot \frac{d_C(f^k\theta_*)d_S(f^k\theta_*)d_S(\psi)}{d_C(\psi)d_S^2(\phi)} \leq \frac{K}{\sqrt{L}} \cdot d_k^{-1}(\theta_*)\left(\sum d_i^{-1}(\theta_*)\right)^{-1} \end{aligned}$$

where we used $|f^k I| < 2|(f^k)'\theta_*|D_n$ (the first inequality in (5)) and Lemma 1.1(a) for $\frac{|f''\phi|}{|f'\psi|}$ for the first inequality; (2) for $d_k(\theta_*)$ for the second inequality. For the last inequality we observe that for $\phi, \psi \in f^k I$,

$$\frac{d_C(f^k\theta_*)d_S(f^k\theta_*)d_S(\psi)}{d_C(\psi)d_S^2(\phi)} < 2$$

by (5). □

An interpretation for $D_n(\theta_*)$ and $d_i(\theta_*)$: $D_n(\theta_*)$ is the radius of the interval around θ_*, in which the derivatives of all orbits copy that of θ_* up to time n. As we iterate forward, the size of this interval declines and $d_i(\theta_*)$ represents the contribution of the i-th iteration to the shrinkage of this interval. We note that $D_n(\theta_*)$ and $d_i(\theta_*)$ are quantities defined explicitly by the orbit of θ_*.

We now define bound periods by using Lemma 1.4. Assume that $c \in C$ is such that $\{f^i v_0\}_{i=0}^{n-1} \cap (C \cup S) = \emptyset$ where $v_0 = f(c)$. For $p \in [2, n]$, let

$$I_p(c) = \left(c + \sqrt{K_0^{-1}L^{-1}D_p(v_0)}, c + \sqrt{K_0^{-1}L^{-1}D_{p-1}(v_0)}\right].$$

Let $I_{-p}(c)$ be the mirror image of $I_p(c)$ with respect to c.

Definition 1.1 (Bound period)**.** *Let θ and p be such that $\theta \in I_p(c) \cup I_{-p}(c)$. We regard the orbit of θ as been bound to the critical orbit of c up to time p; and we call p the* **bound period** *of θ to c.*

Observe that for $\theta \in I_p(c) \cup I_{-p}(c)$, we have

$$|f\theta - v_0| < D_{p-1}(v_0).$$

According to Lemma 1.4, the derivatives along the orbit of $f\theta$ shadow that of the orbit of v_0 for $p-1$ iterates.

B. Derivative recovery We start with

Lemma 1.5. *Assume that a critical point $c \in C$ satisfies (G1)$_n$-(G3)$_n$. Then, for $i \leq n$, we have (a) $d_C(v_i) \geq K^{-1}\sigma L^{-\alpha i}$; and (b) $|(f^i)'v_0|D_{i+1}(v_0) \geq L^{-1-7\alpha i}$ where $v_0 = f(c)$.*

Proof: For (a): (G1) implies $|f'v_i| \geq L\min\{\sigma, L^{-\alpha i}\}$; item (a) then follows by using Lemma 1.1(a). For (b): Let $j \in [0, i]$. By definition we have

$$|(f^i)'v_0|d_j(v_0) = \frac{|(f^i)'v_0|}{|(f^j)'v_0|}d_C(v_j)d_S(v_j).$$

From (G1) we have $\frac{|(f^i)'v_0|}{|(f^j)'v_0|} \geq L^{-\alpha j}\sigma$; from (a) we have $d_C(v_j) \geq K^{-1}\sigma L^{-\alpha j}$; and from (G3) we have $d_S(v_j) \geq \sigma L^{-4\alpha j}$. Hence, $|(f^i)'v_0|d_j(v_0) \geq K^{-1}\sigma^3 L^{-6\alpha j}$. It then follows that

$$\sum_{j=0}^{i} |(f^i)'v_0|^{-1} d_j^{-1}(v_0) \leq \sigma^{-3} L^{7\alpha i},$$

from which (b) follows. □

Our next lemma is on bound period and bound recovery of derivatives.

Lemma 1.6. *Assume that a critical point $c \in C$ satisfies (G1)$_n$-(G3)$_n$. Then for $p \in [2, n]$ and $\theta \in I_p(c) \cup I_{-p}(c)$ we have*

(a) $p \leq \log |c-\theta|^{-\frac{2}{\lambda}}$;

(b) if $\theta \in C_\delta$, then $|(f^p)'\theta| \geq |c-\theta|^{-1+\frac{16\alpha}{\lambda}}$;

(c) if $\theta \in C_\delta$, then $|(f^p)'\theta| \geq L^{\frac{\lambda}{3}p}$.

Proof: By definition we have

$$|c-\theta|^2 \leq D_{p-1}(v_0) \leq L^{-\frac{1}{2}} d_{p-2}(v_0) < L^{-\frac{1}{2}} |(f^{p-2})'(v_0)|^{-1};$$

then by (G2),

$$|c-\theta|^2 \leq L^{-\frac{1}{2}-\lambda(p-2)} \leq L^{-\lambda p}, \tag{6}$$

from which (a) follows. Item (c) follows from (b) and (6).

For Lemma 1.6(b), we have

$$|(f^p)'\theta| = |(f^{p-1})'f\theta| \cdot |f'\theta| \geq K^{-1}|(f^{p-1})'v_0| \cdot L|c-\theta| \geq K^{-1}|(f^{p-1})'v_0| \cdot |c-\theta|^{-1} D_p(v_0),$$

where for the first inequality we use Lemma 1.4 and Lemma 1.1(c), and for the last inequality we use $D_p(v_0) < KL|\theta - c|^2$ by Definition 1.1. We then obtain

$$|(f^p)'\theta| \geq K^{-1} L^{-1-7\alpha p} |c-\theta|^{-1}$$

by using Lemma 1.5(b) for $i = p-1$. Substituting into this the upper estimate of p in (a) we obtain

$$|(f^p)'\theta| \geq K^{-1} L^{-1} |c-\theta|^{-1+\frac{15\alpha}{\lambda}} \geq |c-\theta|^{-1+\frac{16\alpha}{\lambda}}.$$

We also used $|c-\theta| \leq \delta = L^{-\alpha N_0}$ for the last inequality. □

We note that (G3) is necessary here. It is used for Lemma 1.5(b). Lemma 1.6(b) would fail without (G3).

1.4. **Global dynamical properties.** In this subsection we prove that (D1) and (D2) imply (G1) and (G2). We then define *deep free returns*, a concept that is critically important for the measure estimate in Section 2. This concept has no immediate correspondences in [WY1], [BC2] and [BC2].

A. Deriving (G1)-(G3) from (D1)-(D2) Let $f = f_{a,L}$ be such that $a \in \Delta_{N_0}(L)$. For $n \geq N_0$, assume that (G1)$_n$-(G3)$_n$ hold for all $c \in C$. We also assume that $\theta \in S^1$ is such that

(i) $f^i\theta \notin C \cup S$ for all i; and

(ii) for the orbit of θ, the bound period initiated at all returns to C_δ is $< n$.

Bound/free structure: We divide the orbit of θ into alternative bound/free segments as follows. Let n_1 be the smallest $j \geq 0$ such that $f^j\theta \in C_\delta$. For $k > 1$, we define free return times n_k inductively as follows. Let p_{k-1} be the bound period of $f^{n_{k-1}}\theta$, and let n_k be the smallest $j \geq n_{k-1} + p_{k-1}$ such that $f^j\theta \in C_\delta$. We decompose the orbit of θ into bound segments corresponding to time intervals $(n_k, n_k + p_k)$ and free segments corresponding to time intervals $[n_k + p_k, n_{k+1}]$. The times n_k are the *free return* times. We have

$$n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots.$$

Lemma 1.7. *If (D1)-(D2) hold up to time n for all $c \in C(f)$, then (G1)-(G3) hold up to time n for all $c \in C(f)$ as well.*

Proof: Let f be such that (D1)-(D2) hold up to time $n-1$. We assume inductively that $(\mathrm{G1})_{n-1}$-$(\mathrm{G3})_{n-1}$ hold for f. Let $c \in C(f)$. It suffices for us to prove $(\mathrm{G1})_n$ and $(\mathrm{G2})_n$ assuming $(\mathrm{D1})_n$ for $v_0 = f(c)$. First we observe that the bound periods for all returns to C_δ for the critical orbit of c up to time n must $< \frac{2\alpha}{\lambda} n << n-1$. This follows from $(\mathrm{D1})_n$ and Lemma 1.6(a).

We now prove $(\mathrm{G1})_n$ for $v_0 = f(c)$. Let $0 \le i < j \le n+1$. We introduce the bound-free structure starting from v_i to v_j. We consider the following two cases separately.

Case I: j is free. For free segments we use Lemma 1.3, and for bound segments we use Lemma 1.6(c). We obtain exponential growth of derivatives from time i to j, which is much better than what is asserted by (G1).

Case II: j is bound. Let $\hat{j}$ denote the free return with a bound period p such that $j \in [\hat{j}+1, \hat{j}+p]$. We have $\hat{j} \le n$, for otherwise $j > n+1$. Consequently,

$$\frac{|(f^j)'v_0|}{|(f^i)'v_0|} \ge \frac{|(f^{\hat{j}})'v_0|}{|(f^i)'v_0|} \cdot \frac{|(f^{\hat{j}+1})'v_0|}{|(f^{\hat{j}})'v_0|} \cdot \frac{|(f^j)'v_0|}{|(f^{\hat{j}+1})'v_0|} > L^{\frac{1}{3}\lambda(\hat{j}-i)} \cdot K^{-1} L d_C(v_{\hat{j}}) \cdot K^{-1} L^{\lambda(j-\hat{j}-1)}$$

where for the last inequality, we use Lemma 1.6(c) combined with Lemma 1.3 for the first factor. For the third factor we use bound distortion and the inductively assumed $(\mathrm{G2})_{n-1}$ for the binding critical orbit. It then follows that

$$\frac{|(f^j)'v_0|}{|(f^i)'v_0|} \ge L^{\alpha(j-i)-\alpha\hat{j}} \ge L^{-\alpha i}.$$

Property $(\mathrm{G2})_n$ follows directly from $(\mathrm{D1})_n$: By the chain rule we have $(f^{n+1})'v_0 = \prod_{i=0}^n f'v_i$. Use $|f'v_i| > K^{-1} L d_C(v_i)$ if v_i is a return to C_δ, and use Lemma 1.3 for segments in between two consecutive returns, we have

$$|(f^{n+1})'v_0| \ge \delta L^{2\lambda n} \prod_{N_0 < i \le n;\ v_i \in C_\delta} d_C(v_i) > L^{\lambda n}$$

where $(\mathrm{D1})_n$ is used for the last inequality. This use of $(\mathrm{D1})_n$ is what this rule of exclusion is designed for. □

B. Deep returns Assume that f satisfies $(\mathrm{G1})_n$-$(\mathrm{G3})_n$ for all $c \in C$. For $c \in C$, let $v_0 = f(c)$ and assume that $v_\nu = f^\nu(v_0)$ makes a free return to C_δ at time $\nu \in [0, n+1]$.

Definition 1.2. *We say ν is a* **deep free return** *if for every free return $i \in [0, \nu-1]$;*

$$\sum_{\substack{j \in [i+1,\nu] \\ \text{free return}}} 2 \log d_C^{-1}(v_j) \ge \log d_C^{-1}(v_i). \tag{7}$$

The concept of deep free return is critical for the combinatorics of [T1], [T2]. It is introduced to deal with a potential problem in connection with using (D1) for parameter deletion. Observe that with (D1) we have to sometimes delete due to free returns that are close to the boundary of C_δ, which we regard as *shallow returns*. For example, if up to time $n-1$ we already have

$$\sum_{i \le n-1;\ d_C(v_i) < \delta} \log d_C(v_i) = -\alpha(n-1) + 1.$$

We would then have to delete parameters due to possibly a very shallow return to C_δ afterwards to maintain (D1). Unfortunately, it is not possible for us to control the measure of the parameters deleted due to shallow returns, and this is *a major problem* for the use of (D1).

As an initial step towards resolving this problem, we first distinguish the shallow returns that could cause trouble and the deep returns that would not for measure estimates; and it has turned out that Definition 1.2 is technically the most convenient. Our next lemma provides a geometric interpretation.

Lemma 1.8. *Assume that f satisfy (G1)$_n$-(G3)$_n$ for all $c \in C$. Also assume that, for a given $c \in C$, $v_\nu = f^\nu v_0$ makes a deep free return to C_δ at $\nu \in [0, n+1]$. Then*

$$|(f^\nu)' v_0| \cdot D_\nu(v_0) \geq \sqrt{d_C(v_\nu)}.$$

A geometric interpretation from Lemma 1.8 is as follows. For $c \in C$, $I_\nu = [v_0 - D_\nu(v_0), v_0 + D_\nu(v_0)]$ is an interval around $v_0 = f(c)$, the orbits of which copy the derivative growth of the critical orbit of $v_0 = f(c)$ for the first ν iterates. The size of the image of I_ν at time ν, this is to say, $|f^\nu(I_\nu)|$, is used as a reference in determining if the return v_ν is deep: if it is a deep return then $|f^\nu(I_\nu)| > K^{-1}\sqrt{d_C(v_\nu)} \gg d_C(v_\nu)$.

Proof of Lemma 1.8: The technical correspondence of the proof of this lemma is the global distortion estimate (P3) in [WY1]. Let $0 < n_1 < \cdots < n_t < \nu$ denote all free returns in the first ν iterates of v_0, with $p_1, \cdots, p_t$ the corresponding bound periods. Recall that

$$D_\nu(v_0) = \frac{1}{\sqrt{L}} \cdot \left[\sum_{0 \leq i \leq \nu - 1} d_i^{-1}(v_0) \right]^{-1}, \quad \text{where} \quad d_i(v_0) = \frac{d_C(v_i) \cdot d_S(v_i)}{|(f^i)' v_0|}$$

and $d_i(v_0)$ represents the contribution of the i-th iteration on the critical orbit to the total distortion. Let

$$\Theta_{n_k} = \sum_{i=n_k}^{n_k+p_k-1} d_i^{-1}(v_0) \quad \text{and} \quad \Theta_0 = \sum_{i=0}^{\nu-1} d_i^{-1}(v_0) - \sum_{k=1}^{t} \Theta_{n_k}.$$

The quantity Θ_{n_k} is the contribution of the bound segment from n_k to $n_k + p_k - 1$ to the total distortion and Θ_0 is the contribution of all free segments to the total distortion.

Step 1 (Estimate for bound segments): We prove that for the bound segment from n_k to $n_k + p_k - 1$,

$$|(f^{n_k+p_k})' v_0|^{-1} \Theta_{n_k} \leq |d_C(v_{n_k})|^{-\frac{18\alpha}{\lambda}}. \tag{8}$$

To prove (8), we start from

$$|(f^{n_k+p_k})' v_0|^{-1} \Theta_{n_k} = \sum_{i=n_k}^{n_k+p_k-1} |(f^{n_k+p_k})' v_0|^{-1} d_i^{-1}(v_0) = \sum_{i=n_k}^{n_k+p_k-1} (|(f^{n_k+p_k-i})' v_i| d_C(v_i) d_S(v_i))^{-1}$$

$$= (|(f^{p_k})' v_{n_k}| d_C(v_{n_k}) d_S(v_{n_k}))^{-1} + \sum_{i=n_k+1}^{n_k+p_k-1} (|(f^{n_k+p_k-i})' v_i| d_C(v_i) d_S(v_i))^{-1}.$$

To estimate the first term we use Lemma 1.6(b) to obtain

$$(|(f^{p_k})' v_{n_k}| d_C(v_{n_k}) d_S(v_{n_k}))^{-1} < \sigma^{-1} (d_C(v_{n_k}))^{-\frac{16\alpha}{\lambda}}. \tag{9}$$

To estimate the second term we let $\tilde{c}$ be the critical point to which v_{n_k} is bound. By using Lemma 1.4 and (5) in the proof of Lemma 1.4, which implies $d_C(v_i) > \frac{1}{2} d_C(\tilde{v}_{i-n_k-1})$ and $d_S(v_i) > \frac{1}{2} d_S(\tilde{v}_{i-n_k-1})$ for $i \in [n_k + 1, n_k + p_k - 1]$, we have

$$\frac{|(f^{n_k+p_k})' v_0|}{|(f^i)' v_0|} d_C(v_i) d_S(v_i) \geq K^{-1} \frac{|(f^{p_k})' \tilde{v}_0|}{|(f^{i-n_k})' \tilde{v}_0|} d_C(\tilde{v}_{i-n_k-1}) d_S(\tilde{v}_{i-n_k-1}) \geq \sigma^2 L^{-5\alpha(i-n_k-1)}$$

where the last inequality is obtained by using (G1), Lemma 1.5(a) and (G3) for $\tilde{c}$. From this estimate and (9), it follows that

$$|(f^{n_k+p_k})'v_0|^{-1}\Theta_{n_k} \le \sigma^{-1}(d_C(v_{n_k}))^{-\frac{16\alpha}{\lambda}} + \sigma^{-2}L^{6\alpha p_k} < (d_C(v_{n_k}))^{-\frac{18\alpha}{\lambda}}.$$

Here for the last inequality we use $\sigma \gg \delta$ and $L^{6\alpha p_k} \le |d_C(v_{n_k})|^{-\frac{12\alpha}{\lambda}}$ from Lemma 1.6(a).

Step 2 (Estimate for free segments): We prove that

$$|(f^{\nu})'v_0|^{-1}\Theta_0 \le \frac{2}{\delta^{\frac{1}{3}}}. \tag{10}$$

By definition,

$$|(f^{\nu})'v_0|^{-1}\Theta_0 = \sum_{i\in[0,\nu-1]\setminus(\cup[n_k,n_k+p-1])} (|(f^{\nu-i})'(v_i)|d_C(v_i)d_S(v_i))^{-1}.$$

Here we can not simply use (G3) for $d_S(v_i)$ in proving (10). We observe, instead, that either we have $v_i \notin S_\sigma$, for which $d_S(v_i) > \sigma$; or $v_i \in S_\sigma$ for which we have

$$|(f^{\nu-i})'(v_i)| = (f^{\nu-i+1})'v_{i+1}\cdot f'v_i > K^{-1}L^{\frac{\lambda}{3}(\nu-i+1)}(d_S(v_i))^{-1}.$$

It then follows, by using $d_C(v_i) > \delta$, that

$$|(f^{\nu})'v_0|^{-1}\Theta_0 \le \sum_{i\in[0,\nu-1]\setminus(\cup[n_k,n_k+p-1])} KL^{-\frac{\lambda}{3}(\nu-i)}(\sigma\delta)^{-1} \le \frac{1}{\sigma\delta}. \tag{11}$$

This estimate is unfortunately not good enough for (10). To obtained (10), we need to use $C_{\delta^{\frac{1}{20}}}$ in the place of C_δ to define a new bound/free structure for each free segment out of C_δ. For the new free segments, we can now replace δ by $\delta^{\frac{1}{20}}$ in (11); for the bound segments, we use (8) with $d_C > \delta$. We then obtain

$$|(f^{\nu})'v_0|^{-1}\Theta_0 < \frac{1}{\sigma\delta^{\frac{1}{20}}} + \sum_{i\in[0,\nu-1]\setminus(\cup[n_k,n_k+p-1])} KL^{-\frac{1}{3}\lambda(\nu-i)}\delta^{-\frac{18\alpha}{\lambda}} < \frac{1}{\delta^{\frac{1}{3}}}.$$

Step 3 (Proof of the Lemma): From the assumption that ν is a deep free return, we have

$$|d_C(v_{n_k})|^{-1} \le |d_C(v_\nu)|^{-2} \prod_{j:\, n_j\in(n_k,\nu)} |d_C(v_{n_j})|^{-2}.$$

Substituting this into (8) gives

$$|(f^{n_k+p_k})'v_0|^{-1}\Theta_{n_k} \le |d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j:\, n_j\in(n_k,\nu)} |d_C(v_{n_j})|^{-\frac{36\alpha}{\lambda}}. \tag{12}$$

Meanwhile, splitting the orbit from time n_k+p_k+1 to ν into bound and free segments and we have

$$|(f^{\nu-n_k-p_k})'v_{n_k+p_k}|^{-1} \le \left(\prod_{j:\, n_j\in(n_k,\nu)} |(f^{p_j})'v_{n_j}|\right)^{-1}. \tag{13}$$

Multiplying (12) with (13) gives

$$|(f^{\nu})'v_0|^{-1}\Theta_{n_k} \le |d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j:\, n_j\in(n_k,\nu)} \left(|(f^{p_j})'v_{n_j}|\cdot|d_C(v_{n_j})|^{\frac{36\alpha}{\lambda}}\right)^{-1}$$

$$\le |d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}} \prod_{j:\, n_j\in(n_k,\nu)} (d_C(n_j))^{\frac{1}{2}} \;\le\; \delta^{(t-k)/2}|d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}}.$$

where for the second inequality we use Lemma 1.6(b) for $|(f^{p_j})'v_{n_j}|$, and for the last we use $d_C(v_{n_j}) < \delta$. Thus

$$\sum_{n_k\in[0,\nu-1]} |(f^\nu)'v_0|^{-1}\Theta_{n_k} \le |d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}} \sum_{k=1}^{t} \delta^{(t-k)/2} \le 2|d_C(v_\nu)|^{-\frac{36\alpha}{\lambda}}.$$

Combining this with (10) we obtain

$$|(f^\nu)'v_0|^{-1}D_\nu^{-1} = \sqrt{L}\left(\sum_{1\le k\le t} |(f^\nu)'v_0|^{-1}\Theta_{n_k} + |(f^\nu)'v_0|^{-1}\Theta_0\right) \le \frac{1}{\sqrt{d_C(v_\nu)}}.$$

This completes the proof of Lemma 1.8. □

2. Measure of the deleted parameters

Let L be fixed. We regard $f_a := f_{a,L} : S^1 \to S^1,\ a \in [0,1)$ as a 1-parameter family and denote the set of critical points of $f_a = f_{a,L}$ as C. For $c \in C$, we define the critical curves $\gamma_i^{(c)} : \Delta_{N_0}(L) \to S^1$ by letting $\gamma_i^{(c)}(a) = f_a^i(v_0(a))$ where $v_0(a) = f_a(c)$ is the critical value for c. We wish to delete, for all $c \in C$, the part of $\gamma_i^{(c)}$ that violates either (D1) or (D2) for $i = N_0, N_0+1, \cdots$, and to estimate each step along the way the measure of the parameters deleted. Condition (D1), however, can not be used directly as a rule for deletion because we could not control the measure of parameters deleted due to shallow returns. See the discussion in Sect. 1.4C.

We use, instead, a new rule

$$\text{(R1)} \qquad \prod_{i\in[0,n] \text{ is a deep free return}} d_C(v_i) \ge L^{-\frac{1}{20}\lambda\alpha n}$$

for deletion where the concept of deep free returns and its implication have been studied in length in Sect. 1.4C. We prove in Sect. 2.1 that (R1) is stronger than (D1) in the sense that a map satisfies (R1) and (D2) must also satisfy (D1) and (D2).

The rest of the proof of Theorem A goes as follows. For $n > N_0$, let

$$\Delta_n = \{a \in \Delta_{N_0}(L) : \quad \text{(R1) and (D2) hold for all } c \in C,\ N_0 < i \le n\}.$$

To estimate the measure of parameters deleted due to (R1) at time n, we divide the critical orbit $\{v_i,\ i \in [0,n]\}$ into free/bound segments for $a \in \Delta_{n-1}$, $c \in C$, and let $t_1 < t_2 < \cdots < t_q \le n$ be the consecutive times for *deep free returns* to C_δ. Let $c^{(i)}$ be the corresponding binding critical point at time t_i, $r^{(i)} = (r_i, s)$ where $r_i = [\log d_C^{-1}(v_{t_i})]$ is the integer part of $\log d_C^{-1}(v_{t_i})$ and $s \in \{+,-\}$ is used to indicate if v_{t_i} is on the right or on the left side of $c^{(i)}$. We call

$$\mathbf{i} := (t_1, r^{(1)}, c^{(1)};\ t_2, r^{(2)}, c^{(2)};\ \cdots;\ t_q, r^{(q)}, c^{(q)})$$

the *itinerary* of $v_0 = f_a(c)$ up to time n.

We are interested in itineraries $\mathbf{i}$ for parameters that are in Δ_{n-1} but not in Δ_n due to the failure of (R1) at time n. For these itineraries we must have $t_q = n$ and

$$\text{(14)} \qquad \sum_{t_k\le m} r_k < \frac{1}{20}\lambda\alpha m \ \text{ for all } \ m \le n-1 \quad \text{ but } \quad \sum_{t_k\le n} r_k \ge \frac{1}{20}\lambda\alpha n.$$

For a $c \in C$ and an itinerary $\mathbf{i}$ satisfying (14), let $\mathbb{D}_n(c,\mathbf{i})$ be the set of parameters a in Δ_{n-1}, for which the itinerary for $v_0(a) = f_a(c)$ up to time n is $\mathbf{i}$. We prove in Sect. 2.2 that the measure of $\mathbb{D}_n(c,\mathbf{i})$ decreases exponentially in n. We then count all possible itineraries for all $c \in C$ to conclude that the total measure deleted from Δ_{n-1} for Δ_n due to $(\text{R1})_n$ decreases exponentially in n. Estimates for deletions due to (D2) at time n are similar but only simpler.

We remark that the strategy on the estimation of the measure of deleted parameters outlined above is combinatorially different from the ones adopted in [WY1], [BC1] and [BC2]. The current strategy does not rely on a structure of full $I_{\mu,j}$ intervals for each of the critical curves maintained through inductively defined canonical sub-divisions. In addition, there is no longer the need for a large deviation argument, introduced originally in [BC2] as an independent step of parameter deletion to maintain (G2) for all critical orbits.

2.1. **Deriving (D1) from (R1).** In this subsection we prove

Proposition 2.1. *If $a \in \Delta_n$, then (D1) holds up to time n for f_a for all $c \in C$.*

Proof: We prove by induction. Assume that for $f = f_a$, (D1) and (D2) holds up to time $k-1$ for all $c \in C(f)$ for $k \le n$. It suffices for us to prove that if $(\mathrm{R1})_k$ holds for a given $c \in C(f)$, then $(\mathrm{D1})_k$ holds for this c. Note that by inductively assuming that (D1) and (D2) hold up to time $k-1$ for f, $(\mathrm{G1})_{k-1}$-$(\mathrm{G3})_{k-1}$ hold for all $c \in C$ as well by Lemma 1.7.

We start with an estimate on bound returns. Let $v_\nu, \nu < k$ be a free return and p be the corresponding bound period. Observe that from Lemma 1.6(a) we have $p \le \frac{2}{\lambda} \log d_C^{-1}(v_\nu) << k$. We claim that

$$\sum_{i \in [\nu+1, \nu+p] : v_i \in C_\delta} \log d_C^{-1}(v_i) \le \frac{6}{\lambda} \log d_C^{-1}(v_\nu). \tag{15}$$

To prove (15), we let $\tilde{c}$ be the critical point to which v_ν is bound. Let $I = \{i \in [\nu+1, \nu+p] \colon v_i \in C_\delta\}$. Let $I_1 = \{i \in I \colon \tilde{v}_{i-\nu-1} \in C_\delta\}$ and $I_2 = \{i \in I \colon \tilde{v}_{i-\nu-1} \notin C_\delta\}$. (5) in Sect. 1.3A implies $d_C(v_i) \ge d_C(\tilde{v}_{i-\nu-1})/2$, from which it follows that

$$\sum_{i \in I_1} \log d_C^{-1}(v_i) \le \sum_{0 \le j \le p \colon \tilde{v}_j \in C_\delta} \log(2 d_C^{-1}(\tilde{v}_j)) \le 2\alpha p \tag{16}$$

where the last inequality is from the inductive assumption that $(\mathrm{D1})_{k-1}$ holds for all $c \in C$.

Let us also observe that two consecutive returns to C_δ are separated by at least $\frac{1}{2}\alpha N_0$ iterates for $a \in \Delta_{N_0}$ (see Lemma 2.3 in Sect. 2.2C). It then follows that $\sharp I_2 \le 2p/\alpha N_0$, and $d_C(v_i) \ge \delta/2$ for $i \in I_2$. Hence

$$\sum_{i \in I_2} \log d_C^{-1}(v_i) \le \frac{2p}{\alpha N_0} \cdot \log(2\delta^{-1}) \le 2.5p. \tag{17}$$

(15) then follows from (16), (17) and $p \le \frac{2}{\lambda} \log d_C^{-1}(v_\nu)$ (Lemma 1.6(a)).

We call a free return *shallow* if it is not a deep free return. Let $\mu \in (0, k)$ be a shallow free return time, and $i(\mu)$ be the largest deep free return time $< \mu$. We claim that

$$\sum_{\substack{i(\mu)+1 \le j \le \mu \\ \text{free return}}} \log d_C^{-1}(v_j) \le \log d_C^{-1}(v_{i(\mu)}). \tag{18}$$

We first prove $(\mathrm{D1})_k$ assuming (18). Let μ_1 be the largest free shallow return time in $(0, k]$, and i_1 be the largest deep free return time $< \mu_1$. We then let μ_2 be the largest shallow free return time $< i_1$, and i_2 be the largest deep free return time $< \mu_2$, and so on. We obtain a sequence of deep free return times $i_1 > i_2 \cdots > i_q$, and we have

$$\sum_{\substack{0 \le j \le k \\ \text{shallow free return}}} \log d_C^{-1}(v_j) \le \sum_{j=1}^{q} \log d_C^{-1}(v_{i_j}) \le \sum_{\substack{0 \le i \le k \\ \text{deep free return}}} \log d_C^{-1}(v_i) \tag{19}$$

where the first inequality is from (18). We then have

$$\sum_{0\le j\le k,\ v_j\in C_\delta} \log d_C^{-1}(v_j) \le (6\lambda^{-1}+1)\left(\sum_{\substack{0\le j\le k\\ \text{shallow free return}}} \log d_C^{-1}(v_j) + \sum_{\substack{0\le j\le k\\ \text{deep free return}}} \log d_C^{-1}(v_j)\right)$$
$$\le 2(6\lambda^{-1}+1) \sum_{\substack{0\le j\le k\\ \text{deep free return}}} \log d_C^{-1}(v_j) \ < \ \alpha k$$

where the first inequality is from (15); the second from (19); and the last from $(\mathrm{R1})_k$. This proves $(\mathrm{D1})_k$.

To prove (18), we let β_1 be the smallest free return time $\le \mu-1$ such that

$$\sum_{\substack{\beta_1+1\le j\le \mu\\ \text{free return}}} 2\log d_C^{-1}(v_j) < \log d_C^{-1}(v_{\beta_1}). \tag{20}$$

By Definition 1.2, no deep free return occurs during the period $[\beta_1+1,\mu]$. This is because if $i'\in[\beta_1+1,\mu]$ is a deep return, then we must have

$$\sum_{\substack{\beta_1+1\le j\le \mu\\ \text{free return}}} 2\log d_C^{-1}(v_j) \ge \sum_{\substack{\beta_1+1\le j\le i'\\ \text{free return}}} 2\log d_C^{-1}(v_j) \ge \log d_C^{-1}(v_{\beta_1}),$$

contradicting (20). If β_1 is a deep return, we are done. Otherwise we find a $\beta_2<\beta_1$ so that

$$\sum_{\substack{\beta_2+1\le j\le \beta_1\\ \text{free return}}} 2\log d_C^{-1}(v_j) < \log d_C^{-1}(v_{\beta_2}), \tag{21}$$

and so on. This process will end at a deep free return time, which we denote as $\beta_q := i(\mu)$. (18) follows from adding (20), (21) and so on up to the time for $\beta_q = i(\mu)$. □

2.2. **Parameters deleted from Δ_{n-1} for Δ_n.** For $n>N_0$, let Δ_n be the set of parameters for which (R1) and (D2) hold for all $c\in C$ up to time n. Let $\mathbb{D}_n=\Delta_{n-1}\setminus\Delta_n$ be the set of parameters deleted at time n, and write $\mathbb{D}_n=\mathbb{D}_n(R1)\cup\mathbb{D}_n(D2)$ where $\mathbb{D}_n(R1)$ is the set of parameters deleted for (R1) and $\mathbb{D}_n(D2)$ for (D2) at time n.

In paragraph A we prove two lemmas, one on the equivalence of phase-space and parameter-space derivatives and one on the distortion in parameter space. In paragraph B we estimate the measure of $\mathbb{D}_n(c,\mathbf{i})$ and in paragraph C we count all possible itineraries to estimate the measure of $\mathbb{D}_n(R1)$. The measure of $\mathbb{D}_n(D2)$ is estimated in paragraph D.

A. Distortion in parameter space For $c\in C$, let $v_0(a)=f_a(c)$ be the critical value and $v_i(a)=f_a^i(v_0(a))$. In this subsection we denote $\tau_i(a)=\frac{dv_i(a)}{da}$. We begin with

Lemma 2.1. *Assume that f_a satisfy $(G2)_n$ for a given $c\in C$. Then, for all $k\le n+1$, we have*

$$\frac{1}{2}\le \frac{|\tau_k(a)|}{|(f_a^k)'(v_0(a))|}\le 2.$$

Proof: Inductively using

$$\tau_k(a)=1+f_a'(v_{k-1}(a))\tau_{k-1}(a) \tag{22}$$

we obtain

$$\frac{\tau_k(a)}{(f_a^k)'(v_0(a))}=1+\sum_{i=1}^{k}\frac{1}{(f^i)'(v_0(a))}.$$

This lemma then follows from applying $(\mathrm{G2})_n$. □

For $a_* \in [0,1)$ and $c \in C$, define

$$\mathbb{I}_n(a_*, c) = [a_* - D_n(a_*, c), a_* + D_n(a_*, c)]$$

where

$$D_n(a_*, c) = \frac{1}{\sqrt{L}} \cdot \left[\sum_{0 \le i \le n-1} d_i^{-1}(a_*, c) \right]^{-1}, \quad d_i(a_*, c) = \frac{d_C(v_i(a_*)) \cdot d_S(v_i(a_*))}{|(f_{a_*}^i)'(v_0(a_*))|}$$

are the same as in (2) with $f = f_{a_*}$, $\theta_* = v_0(a_*)$. We have

Lemma 2.2. *Assume that (G1), (G2) hold up to time $n > N_0$ for f_{a_*} for a given $c \in C$. Then, for all $a \in \mathbb{I}_{n+1}(a_*, c)$ and $k \le n+1$, we have*

$$\frac{1}{2} < \frac{|\tau_k(a)|}{|\tau_k(a_*)|} \le 2.$$

Proof: Let $k \le n+1$. To prove this lemma we inductively assume that, for all $j < k$,

$$\frac{|\tau_j(a)|}{|\tau_j(a_*)|} \le 2, \quad \text{for all } a \in \mathbb{I}_{n+1}(a_*, c). \tag{23}$$

We then prove the same estimate for $j = k$. Use $d_i, \mathbb{I}_{n+1}$ for $d_i(a_*, c), \mathbb{I}_{n+1}(a_*, c)$ respectively. We have for all $a \in \mathbb{I}_{n+1}$,

$$\left| \log \frac{|\tau_{j+1}(a)|}{|\tau_{j+1}(a_*)|} - \log \frac{|\tau_j(a)|}{|\tau_j(a_*)|} \right| \le \left| \log \frac{|\tau_{j+1}(a)|}{|\tau_j(a)|} - \log |(f_{a_*})' v_j(a_*)| \right|$$
$$+ \left| \log \frac{|\tau_{j+1}(a_*)|}{|\tau_j(a_*)|} - \log |(f_{a_*})' v_j(a_*)| \right| \le (I)(a) + (I)(a_*) + (II)$$

where

$$(I)(a) = \left| \log \frac{\tau_{j+1}(a)}{\tau_j(a)} - \log |(f_a)' v_j(a)| \right|, \quad (II) = \left| \log \frac{(f_a)' v_j(a)}{(f_{a_*})' v_j(a_*)} \right|.$$

For (II) we claim that

$$|II| \le 2L^{-\frac{1}{3}} \cdot d_j^{-1} \left[\sum_{0 \le i \le n} d_i^{-1} \right]^{-1}. \tag{24}$$

To prove (24), first we use (23) and Lemma 2.1 to obtain,

$$|v_j(\mathbb{I}_{n+1})| \le 2|\tau_j(a_*)||\mathbb{I}_{n+1}| \le 4|(f_{a_*}^j)' v_0(a_*)||\mathbb{I}_{n+1}| \le 2L^{-\frac{1}{2}} d_C(v_j(a_*)) d_S(v_j(a_*)). \tag{25}$$

This implies $d_S(\theta) \ge \frac{1}{2} d_S(v_j(a_*))$ for all $\theta \in v_j(\mathbb{I}_{n+1})$. Thus from Lemma 1.1(b),

$$|f''| \le \frac{KL}{d_S(v_j(a_*))^2} \quad \text{on } v_j(\mathbb{I}_{n+1}).$$

It then follows that

$$\left| f_a' v_j(a) - f_{a_*}' v_j(a_*) \right| = \left| f_{a_*}' v_j(a) - f_{a_*}' v_j(a_*) \right| \le \frac{KL}{d_S(v_j(a_*))^2} |v_j(\mathbb{I}_{n+1})| \le \frac{KL}{d_S(v_j(a_*))^2} |\tau_j(a_*)| \cdot |\mathbb{I}_{n+1}|$$

where (23) is again used for the last inequality. We have then

$$\begin{aligned}\left|f_a' v_j(a) - f_{a_*}' v_j(a_*)\right| &\le \frac{KL^{\frac{1}{2}} d_C(v_j(a_*))}{d_S(v_j(a_*))} d_j^{-1} \left(\sum_{1\le i\le n} d_i^{-1}\right)^{-1} \\ &\le L^{-\frac{1}{3}} |(f_{a_*}')v_j(a_*)| d_j^{-1} \left(\sum_{1\le i\le n} d_i^{-1}\right)^{-1}\end{aligned} \tag{26}$$

where for the last inequality we used Lemma 1.1(a). (24) follows directly from the last estimate.

For (I)(a) we have from (22),

$$(I)(a) < \log\left(1 + \frac{1}{|(f_a)' v_j(a)| \cdot |\tau_j(a)|}\right) < \frac{1}{|(f_a)' v_j(a)| \cdot |\tau_j(a)|} < L^{-\frac{\lambda}{3} j} \tag{27}$$

where for the last estimates we use the inductive assumption (23) and Lemma 2.1 for $|\tau_j(a_*)|$. We also use

$$|(f_a)' v_j(a)| \ge \frac{1}{2} |(f_{a_*})' v_j(a_*)| \ge \frac{L}{2} \cdot \min\{\sigma, L^{-\alpha j}\}$$

where the first inequality is from (26) and the second from (G1). That $\frac{|\tau_k(a)|}{|\tau_k(a_*)|} < 2$ now follows from combining (27) and (24) for all $j < k$. □

B. Measure estimate for $\mathbb{D}_n(c, \mathbf{i})$ Let $\mathbb{D}_n(c, \mathbf{i})$ be the set of all $a \in \Delta_{n-1}$ for which the orbit of $v_0 = f_a(c)$ for $c \in C(f)$ follows an itinerary $\mathbf{i} = (t_1, r^{(1)}, c^{(1)}; t_1, r^{(2)}, c^{(2)}; \cdots; t_q, r^{(q)}, c^{(q)})$ for which (R1) fails at time n. For $a \in \mathbb{D}_n(c, \mathbf{i})$, the orbit of $v_0(a) = f_a(c)$ makes deep free return at

$$N_0 < t_1 < t_2 < \cdots t_q = n$$

and associated to t_k is the binding critical point $c^{(k)}$ and a pair $r^{(k)} = (r_k, s)$ where $r_k = [\log |v_{t_k} - c^{(k)}|^{-1}]$ is an integer and $s \in \{+, -\}$. We prove

Proposition 2.2. *Let* $\mathbf{i}$ *be a given itinerary for* $c \in C$*, then* $|\mathbb{D}_n(c, \mathbf{i})| < L^{-\frac{1}{3} r}$ *where* $r = r_1 + r_2 \cdots + r_q$.

Proof: For $a_* \in D_n(c, \mathbf{i})$, denote

$$\mathbb{I}_{t_k}(a_*, c) = (a_* - D_{t_k}(a_*, c), a_* + D_{t_k}(a_*, c)).$$

and let $\gamma_{t_k}^{(c)} : \mathbb{I}_{t_k}(a_*, c) \to S^1$ be such that $\gamma_{t_k}^{(c)}(a) = f_a^{t_k}(v_0(a))$. We claim that, for $1 \le k \le q$,

$$\gamma_{t_k}^{(c)}(\mathbb{I}_{t_k}(a_*, c)) \supset [c^{(k)} \pm \frac{1}{2} L^{-r_k - 1}, c^{(k)} \pm 2L^{-r_k}] \supset \gamma_{t_k}^{(c)}(\mathbb{I}_{t_{k+1}}(a_*, c)) \tag{28}$$

where the $\pm$ is taken as positive if $r^{(k)} = (r_k, +)$ and as negative if $r^{(k)} = (r_k, -)$. Note that, since t_{q+1} is not defined, we only claim for $k = q$ the first inclusion in (28). To prove (28) we observe that

$$|\gamma_{t_k}^{(c)}(\mathbb{I}_{t_k}(a_*, c))| \ge |\tau_{t_k}(a_*)| D_{t_k}(a_*, c) \ge \frac{1}{2} |(f_{a_*}^{t_k})'(v_0(a_*))| D_{t_k}(a_*, c) > K^{-1} \sqrt{d_C(v_{t_k}(a_*))} \tag{29}$$

where the first inequality follows from Lemma 2.2, the second from Lemma 2.1 and the last from Lemma 1.8. The first inclusion of (28) then follows from (29) and the fact that

$$d_C(v_{t_k}(a_*)) \approx |[c^{(k)} \pm \frac{1}{2} L^{-r_k - 1}, c^{(k)} \pm 2L^{-r_k}]|. \tag{30}$$

The second inclusion of (28) also follows from (30) and

$$|\gamma_{t_k}^{(c)}(\mathbb{I}_{t_{k+1}}(a_*, c))| \le 4|\tau_{t_k}(a_*)| D_{t_{k+1}} < 8(f_{a_*}^{t_k})'(v_0(a_*)) D_{t_{k+1}} \le KL^{-\frac{1}{2}} d_C(v_{t_k}(a_*)) d_S(v_{t_k}(a_*)). \tag{31}$$

Let $J_{t_k}(a_*, c)$ be the image of $[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]$ under the inverse of $\gamma_{t_k}^{(c)}$. Denote

$$\mathbb{O}_k = \cup_{a_* \in \mathbb{D}_n(c,\mathbf{i})} \mathbb{I}_{t_k}(a_*, c).$$

for $1 \le k \le q$, and let

$$\mathbb{O}_{q+1} = \cup_{a_* \in \mathbb{D}_n(c,\mathbf{i})} J_{t_q}(a_*, c).$$

The conclusion of this proposition would follow if

$$|\mathbb{O}_{k+1}| < L^{-\frac{1}{3}r_k}|\mathbb{O}_k| \tag{32}$$

for $1 \le k \le q$.

To prove (32) we first write $\mathbb{O}_k$ as the union of a collection of mutually disjoint open intervals, each of which we denote as I_{t_k}. Let $\hat{\gamma}_{t_k}^{(c)} : I_{t_k} \to S^1$ be such that $\hat{\gamma}_{t_k}^{(c)}(a) = f_a^{t_k}(v_0(a))$. Because I_{t_k} is by definition the union of a collection of overlapping $\mathbb{I}_{t_k}(a)$ where $a \in \mathbb{D}_n(c, \mathbf{i})$, $\tau_{t_k} = \frac{d}{da}\hat{\gamma}_{t_k}^{(c)}(a) : I_{t_k} \to \mathbb{R}$ can not be zero on I_{t_k} by Lemma 2.2. It then follows that $\hat{\gamma}_{t_k}^{(c)} : I_{t_k} \to S^1$ is *monotonic*. Note that we use hat on $\gamma_{t_k}^{(c)}$ here to indicate that the domain is now I_{t_k}. We claim that

$$\hat{\gamma}_{t_k}^{(c)}(\mathbb{O}_{k+1} \cap I_{t_k}) \subset [c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]. \tag{33}$$

This is because for any given $a \in \mathbb{D}_n(c, \mathbf{i}) \cap I_k$, we must have

$$\hat{\gamma}_{t_k}^{(c)}(a) \in [c^{(k)} \pm L^{-r_k-1}, c^{(k)} \pm L^{-r_k}] \tag{34}$$

and (33) follows directly from (34) and (31). The rest of the proof is divided into the following two cases.

Case 1: I_{t_k} is such that $|\hat{\gamma}_{t_k}^{(c)}(I_{t_k})| \le \frac{1}{4}$. In this case, the image of $[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]$ under the inverse of $\hat{\gamma}_{t_k}^{(c)}$ consists only one sub-interval in I_{t_k}; and we have by the first inclusion of (28) that,

$$[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}] \subset \hat{\gamma}_{t_k}^{(c)}(\mathbb{I}_{t_k}(a_*, c)) \tag{35}$$

where a_* is taken arbitrarily from $\mathbb{D}_n(c, \mathbf{i}) \cap I_k$. This implies

$$\frac{|\mathbb{O}_{k+1} \cap I_{t_k}|}{|I_{t_k}|} \le \frac{|\mathbb{O}_{k+1} \cap I_{t_k}|}{|\mathbb{I}_{t_k}(a_*, c)|} \le 4\frac{|[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]|}{|\hat{\gamma}_{t_k}^{(c)}(\mathbb{I}_{t_k}(a_*, c))|} < K\sqrt{d_C(v_{t_k}(a_*))} < L^{-\frac{1}{3}r_k}$$

where the second inequality follows from (35), (33) and Lemma 2.2, and the third follows from (30) and Lemma 1.8.

Case 2: I_{t_k} is such that $|\hat{\gamma}_{t_k}^{(c)}(I_{t_k})| > \frac{1}{4}$. In this case the image of $[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]$ under the inverse of $\hat{\gamma}_{t_k}^{(c)}$ might consist more than one sub-interval of I_{t_k} because $\hat{\gamma}_{t_k}^{(c)}(I_{t_k})$ is allowed to wrap around S^1. Let us assume that we have, say m such sub-intervals where $m > 1$. In this case, there exists $a_1, \cdots, a_m \in \mathbb{D}_n(c.\mathbf{i})$, such that

$$\mathbb{O}_{k+1} \cap I_k \subset \cup_{i=1}^m J_{t_k}(a_i, c).$$

We now define $\hat{\mathbb{I}}_{t_k}(a_i, c)$ for $1 \le i \le m$ as follows: we let $\hat{\mathbb{I}}_{t_k}(a_i, c) = \mathbb{I}_{t_k}(a_i, c)$ if

$$|\gamma_{t_k}^{(c)}(\mathbb{I}_{t_k}(a_i, c))| \le \frac{1}{4};$$

otherwise we obtain $\hat{\mathbb{I}}_{t_k}(a_i,c)$ by cutting $\mathbb{I}_{t_k}(a_i,c)$ shorter around a_i so that (1) $|\gamma_{t_k}^{(c)}(\hat{\mathbb{I}}_{t_k}(a_i,c))| = \frac{1}{4}$, and (2) the size of $\gamma_{t_k}^{(c)}(\hat{\mathbb{I}}_{t_k}(a_i,c))$ on both-side of $\gamma_{t_k}^{(c)}(a_i)$ is as balanced as possible. From this definition and Lemma 1.8, we have $\hat{\mathbb{I}}_{t_k}(a_i,c)) \supset J_{t_k}(a_i,c)$ and

$$\frac{1}{4} \geq |\gamma_{t_k}^{(c)}(\hat{\mathbb{I}}_{t_k}(a_i,c))| \geq K^{-1}\sqrt{d_C(v_{t_k})}.$$

By the monotonicity of $\hat{\gamma}_{t_k}^{(c)}$ on I_k, we then conclude that $\hat{\mathbb{I}}_{t_k}(a_i,c)$, $1 \leq i \leq m$ are mutually disjoint. It then follows that

$$\frac{|\mathbb{O}_{k+1} \cap I_{t_k}|}{|I_{t_k}|} \leq \max_{1\leq i\leq m} \frac{|J_{t_k}(a_i,c)|}{|\hat{\mathbb{I}}_{t_k}(a_i,c)|} \leq 4 \max_{1\leq i\leq m} \frac{[c^{(k)} \pm \frac{1}{2}L^{-r_k-1}, c^{(k)} \pm 2L^{-r_k}]}{|\gamma_{t_k}^{(c)}(\hat{\mathbb{I}}_{t_k}(a_i,c))|} < K\sqrt{d_C(v_{t_k})}$$

where Lemma 1.8 is again used for the last inequality. Inequality (32) then follows directly from the last estimate. □

C. Total measure of deleted parameters for (R1)$_n$ In this paragraph we estimate the measure of the parameter set

$$\mathbb{D}(R1) = \bigcup \mathbb{D}_n(c,\mathbf{i})$$

where the union is over all critical points $c \in C$ and all conceivable itineraries $\mathbf{i}$. We start with

Lemma 2.3. *Assume that f_a is such that $a \in \Delta_{N_0}(L)$. Then the lengths of any given bound period due to a return to C_δ is $\geq \frac{1}{2}\alpha N_0$.*

Proof: From $a \in \Delta_{N_0}$ and Lemma 1.1(a) we have $|(f^{[\frac{1}{2}\alpha N_0]})'v_0| \leq (\sigma^{-1}L)^{\frac{1}{2}\alpha N_0}$ where $\sigma = L^{-\frac{1}{6}}$. It then follows that

$$D_{[\frac{1}{2}\alpha N_0]}(v_0) \geq \frac{L^{-\frac{1}{2}}\sigma^2}{|(f^{[\frac{1}{2}\alpha N_0]})'v_0|} \geq L^{-\frac{1}{2}}\sigma^2(L^{-1}\sigma)^{\frac{1}{2}\alpha N_0} \gg \delta$$

from which the lemma follows. □

We count the number of possible itineraries $\mathbf{i} = (t_1, r^{(1)}, c^{(1)}; t_2, r^{(2)}, c^{(2)}; \cdots; t_q, r^{(q)}, c^{(q)})$. First from Lemma 2.3, we have

$$q \leq \frac{2n}{\alpha N_0}.$$

For itineraries due to deletion at time n, we also have $t_q = n$ and

$$r = r_1 + r_2 + \cdots + r_q > \frac{1}{20}\lambda\alpha n.$$

We now count as follows. First the possible number of choices for the deep free return times is $\binom{n}{q}$. Second, for a fixed $r > \frac{1}{20}\lambda\alpha n$, the number of choices for $(r_1, \cdots r_q)$ satisfying $|r_1| + \cdots + |r_q| = r$ is $\binom{r}{q}$. Finally counting the number of critical points and the fact that each has two sides, we conclude that

$$|\mathbb{D}_n(R1)| \leq \sum_{1\leq q\leq \frac{2n}{\alpha N_0}} (2\#C)^q \binom{n}{q} \sum_{r>\frac{1}{20}\lambda\alpha n} \binom{r}{q} L^{-\frac{1}{3}r}$$

by using Proposition 2.2. By Stirling's formula for factorials, there exists $\beta(N_0) \to 0$ as $N_0 \to \infty$, so that

$$\binom{n}{q} \leq e^{\beta(N_0)n}; \qquad \binom{r}{q} \leq e^{\beta(N_0)r}.$$

Let N_0 be sufficiently large so that

$$\frac{2}{\alpha N_0} + \beta(N_0) < \frac{1}{100}\lambda\alpha. \tag{36}$$

We finally conclude that

$$|\mathbb{D}_n(R1)| < L^{-\frac{1}{100}\lambda\alpha n}. \tag{37}$$

D. Parameters deleted due to (D2) We start with

Lemma 2.4. *For $a_* \in \Delta_{n-1}$, $c \in C$,*

$$|v_n(\mathbb{I}_n(a_*, c))| \geq L^{-3\alpha n}$$

where $\mathbb{I}_n(a_*, c) = [a_* - D_n(a_*, c), a_* + D_n(a_*, c)]$.

Proof: Write $f = f_{a_*}$, $v_0 = f_{a_*}(c)$, $v_i = f^i_{a_*}(v_0)$. Since $a_* \in \Delta_{n-1}$, $(\mathrm{G1})_{n-1}$ and $(\mathrm{G2})_{n-1}$ hold for v_0 by Lemma 1.7. By using Lemma 2.1 and 2.2, we have

$$|v_n(\mathbb{I}_n(a_*, c))| > 4|(f^n)'v_0| \cdot |\mathbb{I}_n(a_*, c)| > \frac{1}{\sqrt{L}} \left(\sum_{0 \leq i \leq n-1} (|(f^n)'v_0| \cdot d_i)^{-1} \right)^{-1}.$$

If $v_i \notin S_\sigma$, then

$$|(f^n)'v_0| d_i = \frac{|(f^n)'v_0|}{|(f^i)'v_0|} d_C(v_i) d_S(v_i) \geq K L^{-2\alpha i} \sigma.$$

If $v_i \in S_\sigma$, then $d_S(v_i) \geq |f'v_i|^{-1}$ from Lemma 1.1(a); and we have

$$|(f^n)'v_0| d_i = \frac{|(f^n)'v_0|}{|(f^i)'v_0|} d_C(v_i) d_S(v_i) \geq \frac{|(f^n)'v_0|}{|(f^{i+1})'v_0|} d_C(v_i) \geq L^{-\alpha(i+1)}\sigma$$

where $(\mathrm{G1})_{n-1}$ is used for the last inequality. Hence we obtain

$$|v_n(\mathbb{I}_n(a_*, c))| > \frac{1}{\sqrt{L}} \left(\sum_{i=0}^{n-1} K L^{2\alpha i} \sigma \right)^{-1} > L^{-3\alpha n}.$$

□

We now estimate the measure deleted from Δ_{n-1} due to $(\mathrm{D2})_n$. For $a_* \in \Delta_{n-1}$, assume that at time n, a_* is deleted because $|v_n(a_*) - s| < L^{-4\alpha n}$ for $v_0 = f_{a_*}(c), c \in C$ and $s \in S$. Fixing the pair (c, s) where $c \in C$, $s \in S$, we denote the set of parameters deleted due to $(\mathrm{D2})_n$ for (c, s) as $\mathbb{D}_n(c, s)$. We now construct two parameter intervals $\tilde{\mathbb{I}}_n(a_*)$ and $\tilde{J}_n(a_*)$ so that

(a) $a_* \in \tilde{J}_n(a_*) \subset \tilde{\mathbb{I}}_n(a_*)$;
(b) $|\tilde{J}_n(a_*)| < L^{-\frac{1}{2}\alpha n} |\tilde{\mathbb{I}}_n(a_*)|$; and
(c) if $a_*, \tilde{a}_* \in \mathbb{D}_n(c, s)$, then either $\tilde{a}_* \in \tilde{J}_n(a_*)$ or $\tilde{\mathbb{I}}_n(a_*) \cap \tilde{\mathbb{I}}_n(\tilde{a}_*) = \emptyset$.

It follows directly from (a)-(c) that

$$|\mathbb{D}_n(c, s)| < L^{-\frac{1}{2}\alpha n},$$

from which it follow that

$$|\mathbb{D}_n(D2)| < \#C \cdot \#S \cdot L^{-\frac{1}{2}\alpha n} < L^{-\frac{1}{3}\alpha n}. \tag{38}$$

To prove (a)-(c), let $\mathbb{I}_n = [a_* - D_n(a_*, c), a_* + D_n(a_*, c)]$ for $a_* \in \Delta_{n-1}$, $c \in C$. We define $\tilde{\mathbb{I}}_n$ as follows. If $|\gamma_n^{(c)}(\mathbb{I}_n)| < \frac{1}{4}$, we let $\tilde{\mathbb{I}}_n = \mathbb{I}_n$. Otherwise we let $\tilde{\mathbb{I}}_n$ be the sub-interval of $\mathbb{I}_n$ so that (1) $|\gamma_n^{(c)}(\tilde{\mathbb{I}}_n)| = \frac{1}{4}$; and (2) $\gamma_n^{(c)}(\tilde{\mathbb{I}}_n)$ is as balanced in length as possible on both side of $\gamma_n^{(c)}(v_n(a_*))$. Let s be the singular point, to which v_n is most close. It is then clear that if $a \in \mathbb{D}_n(c, s) \cap \tilde{\mathbb{I}}_n$, then $v_n(a)$ must be inside of the interval $[s - e^{3\alpha}, s + L^{-3\alpha n}]$. We let $\tilde{J}_n(a_*)$ be the pull back of the interval $[s - e^{-4\alpha n}, s + L^{-4\alpha n}]$ by $\gamma_n^{(c)}$. Item (c) follows by definition; items (a) and (b) follow from $(\mathrm{D2})_n$, Lemma 2.4 and Lemma 2.2.

Proof of Theorem A: Follows directly from (37), (38) and Lemma 1.2. □

Proof of Theorem B: Theorem B(a) is $(G2)_n$ for all $n > 0$. To prove Theorem B(b) we follow the proof presented in Section 3 of [WY1]. The main difference is that, due to the existence of the singular set S, we now need to also divide intervals around S into collections of infinite sub-intervals to control the distortion. The modification needed over the proof of [WY1] is an interesting excise, the details of which we skip here to avoid unnecessary repetitions. □

Appendix A. Proof of Lemmas 1.2 and 1.3

In this appendix we prove Lemmas 1.2 and 1.3. Since various ideas and technical tools developed in the main text are needed, the correct order for the reader is to go over the main text first before getting into the details of these two proofs.

A.1. **Proof of Lemma 1.2.** Let L be fixed and $f_a = f_{a,L}$. Let C be the set of critical points and S be the set of singular points for f_a. For $c \in C$, denote $v_0(a) = f_a(c)$, $v_n(a) = f_a^n(v_0)$ and let $\tau_n(a) = \frac{d}{da} v_n(a)$. Also denote

$$\Delta_n(L) = \{a \in [0,1): \quad d_C(v_i(a)),\ d_S(v_i(a)) \geq \sigma \quad \forall c \in C, \quad i \in [0,n]\}.$$

For $a_* \in \Delta_{n-1}$, let $c \in C$ be a critical point and $v_0(a_*) = f_{a_*}(c)$ be the critical value. Denote

$$D_n(a_*, c) = \frac{1}{\sqrt{L}} \cdot \left[\sum_{0 \leq i \leq n-1} d_i^{-1}(v_0(a_*)) \right]^{-1} \qquad \text{where} \qquad d_i(v_0(a_*)) = \frac{d_C(v_i(a_*)) \cdot d_S(v_i(a_*))}{|(f^i)' v_0(a_*)|},$$

and let

$$\mathbb{I}_n(a_*, c) = [a^* - D_n(a_*, c), a^* + D_n(a_*, c)].$$

We start with the following statements on phase-space and parameter-space derivatives.

Lemma A.1. *Assume $a_* \in \Delta_{n-1}$ and let $c \in C$. Denote $f = f_{a_*}$, $v_i = v_i(a_*)$, $D_n = D_n(a_*, c)$. Then we have*

(i) (Derivative growth) $(f^j)'(v_0) > (K^{-1} L\sigma)^{j-i} (f^i)'(v_0)$ *for all* $0 \leq i < j \leq n$;

(ii) (Distortion in phase space) $\frac{1}{2} \leq \frac{|(f^n)'\theta|}{|(f^n)' v_0|} \leq 2$ *for all* $\theta \in [v_0 - D_n, v_0 + D_n]$;

(iii) (Equivalence of space and a-derivatives) $\frac{1}{2} \leq \frac{|\tau_n^{(c)}(a_*)|}{|(f^n)'(v_0)|} \leq 2$;

(iv) (Distortion in parameter space) $\frac{1}{2} \leq \frac{|\tau_n^{(c)}(a)|}{|\tau_n^{(c)}(a_*)|} \leq 2$ *for all* $a \in \mathbb{I}_n(a_*, c)$;

(v) (Length of the image of $\mathbb{I}_n(a_*, c)$) $|v_n(\mathbb{I}_n(a_*, c))| \geq L^{1/7}\sigma$.

Proof: Item (i) follows directly from Lemma 1.1. Proof for (ii) is the same as that of Lemma 1.4, (iii) the same as that of Lemma 2.1 and (iv) the same as that of Lemma 2.2.

Proof for item (v) is as follows. Let $i \in [0, n-1]$. We have from (i) that

$$|(f^n)' v_0(a_*)| d_i(v_0(a_*)) = \frac{|(f^n)' v_0(a_*)|}{|(f^i)' v_0(a_*)|} d_C(v_i(a_*)) d_S(v_i(a_*)) \geq (K_0^{-1} L\sigma)^{n-i} \sigma^2.$$

Taking reciprocals and then summing the result over all $i \in [0, n-1]$ we have

$$\sum_{i=0}^{n-1} |(f^n)' v_0(a_*)|^{-1} d_i(v_0(a_*))^{-1} \leq \frac{K_0}{L\sigma^3}.$$

Recall that $\sigma = L^{-\frac{1}{6}}$. We have

$$|v_n(\mathbb{I}_n(a_*, c))|^{-1} \leq K |(f^n)' v_0(a_*)|^{-1} D_n^{-1}(v_0(a_*)) = K\sqrt{L} \cdot \sum_{i=0}^{n-1} |(f^n)' v_0(a_*)|^{-1} d_i(v_0(a_*))^{-1} \leq \frac{1}{L^{\frac{1}{7}}\sigma},$$

where the first inequality follows from (iv) and (iii). □

Proof of Lemma 1.2: We estimate the size of the parameter set $\Delta_{n-1}(L) \setminus \Delta_n(L)$. For $c \in C$, $s \in C \cup S$, let $E_n(c,s)$ be the set of all $a \in \Delta_{n-1}$ so that $d(v_n(a), s) \le \sigma$. We claim that for every $a_* \in E_n(c,s)$, there exists two parameter intervals, which we call $I_n(a_*)$ and $J_n(a_*)$, so that

(a) $a_* \in J_n(a_*) \subset I_n(a_*)$;
(b) $|J_n(a_*)| \le L^{-1/8}|I_n(a_*)|$;
(c) if $a_*, \tilde{a}_* \in E_n(c,s)$, then either $\tilde{a}_* \in J_n(a_*)$ or $I_n(a_*) \cap I_n(\tilde{a}_*) = \emptyset$.

Postponing the construction of $J_n(a_*)$ and $I_n(a_*)$, we first prove Lemma 1.2 by using (a)-(c): Observe that from (a)-(c) we have $|E_n(c,s)| < L^{-\frac{1}{8}}$, and it follows that

$$|\Delta_{n-1} \setminus \Delta_n| < \#C(\#C + \#S)L^{-\frac{1}{8}},$$

and we obtain

$$|\Delta_{N_0}(L)| \ge 1 - \sum_{n=1}^{N_0} |\Delta_{n-1} \setminus \Delta_n| \ge 1 - N_0 L^{-\frac{1}{9}}.$$

To construct $I_n(a_*)$ and $J_n(a_*)$, we recall that $\mathbb{I}_n(a_*, c) = [a_* - D_n(c, a_*), a_* + D_n(c, a_*)]$. We define $I_n(a_*)$ as follows. If $|v_n(\mathbb{I}_n(a_*, c))| < \frac{1}{4}$, then we let $I_n(a_*) = \mathbb{I}_n(a_*, c)$. Otherwise, take $I_n(a_*)$ to be the interval of length $\frac{1}{10|v_n(\mathbb{I}_n(a_*,c))|}|\mathbb{I}_n(a_*, c)|$ centered at a_*. We note that if $a \in E_n(c,s) \cap \mathbb{I}_n(a_*)$, then $v_n(a)$ must be inside of the interval $[s - \sigma, \hat{s} + \sigma]$, the pull-back of which under v_{-n} we take as $J_n(a_*)$. Item (a) follows from Lemma A.1(v). Item (b) follows from Lemma A.1(iii)-(v). Item (c) follows from the fact that $v_n(I_n(a_*))$ can not wrap around S^1 to intersect the interval $[s - \sigma, \hat{s} + \sigma]$ more than one time because $|v_n(I_n(a_*))| < 1/4$ by definition. □

A.2. **Proof of Lemma 1.3.** For this proof we need the technical materials of Sect. 1.3A and the definition of the bound/free structure of Sect. 1.4A.

Recall that $\delta = L^{-\alpha N_0}$. Let $\{\theta_i\}_{i=0}^{n-1}$ be a given orbit segment that is out of C_δ. Also let $\delta_0 = L^{-\frac{11}{12}}$. For $\{\theta_i\}_{i=0}^{n}$ we first introduce the bound/free structure of Section 1.4A by using C_{δ_0} in the place of C_δ. We then prove a result that is similar to Lemma 1.6, from which Lemma 1.3 would follow directly. To make this idea work, we need to first repeat Lemma 1.4. We then need to show that the bound periods on $C_{\delta_0} \setminus C_\delta$, as defined in Sect. 1.3A, are $\le \alpha N_0$. Let the intervals $I_p(c)$ and the bound period be defined the same as in Sect. 1.3A. We start our proof with

Lemma A.2. *Assume that $f = f_a$ is such that $a \in \Delta_{N_0}(L)$. Then for all $c \in C$, we have $C_{\delta_0}(c) \setminus C_\delta \subset \bigcup_{1 \le |p| \le N_0} I_p(c)$.*

Proof: Let $v_0 = f(c)$. It suffices to show that

$$L^{-1} D_{N_0}(v_0) < \delta^2 < \delta_0^2 < L^{-1} D_1(v_0) \tag{39}$$

where $D_n(v_0)$ is as in (2) with $\theta_* = v_0$. Since $a \in \Delta_{N_0}(L)$, we have $|(f^{N_0-1})' v_0| \ge (KL\sigma)^{N_0-1}$ from Lemma 1.1. It then follows that

$$D_{N_0}(v_0) \le d_{N_0-1}(v_0) \le (KL\sigma)^{-N_0+1} < L\delta^2.$$

For $D_1(v_0)$ we have

$$D_1(v_0) = \frac{1}{\sqrt{L}} \cdot d_C(v_0) d_S(v_0) \ge L^{-\frac{1}{2}} \sigma^2,$$

from which the last inequality of (39) follows directly. □

With the help of Lemma A.2, we know that the bound/free structure for $\{\theta_i\}_{i=1}^{n}$ is well-defined.

Lemma A.3. *Let $p \geq 2$ be the bound period for $\theta \in C_{\delta_0} \setminus C_\delta$. Then:*
(a) for $i \in [1,p]$, we have $d_C(f^i\theta) > \delta_0$ and $|(f^{i-1})'(v_0)|D_i(v_0) > L^{-\frac{1}{2}}\sigma^2$;
(b) $|(f^p)'\theta| \geq K^{-1}L^{-\frac{1}{2}}\sigma^2|c-\theta|^{-1} \geq K^{-1}L^{-\frac{1}{2}}\sigma^2(K_0^{-1}L\sigma)^{p-2}$.

Proof: Item (a) is a correspondence of Lemma 1.5. The estimates are better because here we have $f^{i-j}(v_j) > (KL\sigma)^{i-j}$, and $d_C(v_i), d_S(v_i) > \sigma$. To prove item (b) we start with

$$|(f^p)'\theta| \geq K^{-1}L|c-\theta||(f^{p-1})'v_0| \geq K^{-1}|c-\theta|^{-1}|(f^{p-1})'(v_0)|D_p(v_0)$$

where for the last inequality we use $D_p(v_0) < KL|c-\theta|^2$ by the definition of p. The first inequality of (b) then follows by using (a). To prove the second inequality we start with

$$|c-\theta| \leq D_{p-1}(v_0) \leq \frac{1}{\sqrt{L}}d_{p-2}(v_0).$$

Since $a \in \Delta_{N_0}(L)$ we have $d_{p-2}(v_0) \leq (K_0^{-1}L\sigma)^{-p+2}$ hence $|c-\theta|^{-1} \geq \sqrt{L}(K_0^{-1}L\sigma)^{p-2}$. The second inequality of (b) now follows from the first. □

We now claim that

$$|(f^p)'\theta| \geq L^{\frac{1}{300}p}. \tag{40}$$

Observe that for $p \geq 10$, (40) is much weaker than the second inequality of Lemma A.3(b). For $p < 10$, (40) follows from the first inequality of Lemma A.3(b) and the fact that

$$|c-\theta|^{-1} > \delta_0^{-1} = L^{\frac{11}{12}}.$$

Proof of Lemma 1.3: For $\{\theta_i\}_{i=1}^{n-1}$ that is outside of C_δ, we first using C_{δ_0} to introduce a bound/free structure. If θ_n is free (this includes the case that $\theta_n \in C_\delta$), then we use (40) for bound segments and $|f'(\theta)| > K^{-1}L^{\frac{1}{12}}$ for free segments out of C_{δ_0}. This proves Lemma 1.3(b). If θ_n is bound, then there is a drop of a factor $> \delta$ at the last free return that can not be recovered. In this case we need the factor δ in Lemma 1.3(a). □

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