THE RESTRICTED THREE-BODY PROBLEM AS A PERTURBED DUFFING EQUATION

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ABSTRACT. In this paper, we first write the equation of the restricted three-body problem as a perturbed Duffing equation. We then adopt what was recently introduced in [2] to derive integral equations for the primary stable and unstable solutions. These integral equations are used to prove analytic dependency of the splitting distance of the primary stable and unstable manifold on the mass ratios of primaries and on Jacobi constant. We then use the homoclinic solution of the unperturbed Duffing equation to evaluate Poincare-Melnikov integral. We conclude that, for every negative Jacobi constant of sufficiently large magnitude, the surface of unperturbed parabolic solutions breaks to induce homoclinic tangle for all but at most finitely many mass ratios of primaries. This result is slightly weaker than that of [4]. Our proof, however, is much less elaborated. This paper is also self-contained: we do not rely on McGehee's analysis in [7] to justify the applicability of Poincare-Melnikov method.

The planar three-body problem comprises a set of six second order equations that describes the motions of three Newtonian gravitational particles $m_1, m_2, m_3 > 0$ in a two-dimensional Euclidean space $\mathbb{R}^2$. In contrast to the two-body problem, which is completely integrable, the three-body problem is very difficult to work with partly because it is a system of relatively high phase dimension.

The *restricted* planar three-body problem is a problem of much lower phase dimension that is derived from the planar three-body problem by letting $m_3 \to 0$. In limit, the equations of the planar three-body problem are decomposed into a subset of equations of two-body problem for m_1 and m_2 that is independent of the motions of m_3, and a subset of non-trivial equations for the motions of m_3 obtained by a straight forward extension from the case of $m_3 > 0$ to $m_3 = 0$. The subset of equations for the motions of m_3 is the equations of the restricted planar three-body problem.

Following tradition [12], we call m_1, m_2 the primary masses, and m_3 the *infinitesimal* mass. The restricted planar three-body problem describes the motion of an infinitesimal mass in the gravitational field induced by the primary masses m_1, m_2 in $\mathbb{R}^2$. From the solutions of the two-body problem, we know that the trajectories of the primary masses in $\mathbb{R}^2$ are *conic curves*, each specific type of which would induces a corresponding restricted planar three-body problem. We have then, respectively, equations for the restricted *circular* planar three-body problem, for which the primary masses move in circles in $\mathbb{R}^2$; the restricted *elliptic* planar three-body problem, for which the primary masses move in ellipse; and so on. It has been a convention in celestial mechanics to refer the *restricted circular planar three-body problem* as the restricted three-body problem unless it is otherwise stated. This restricted three-body problem admits a first integral, which is commonly referred to as the Jacobi integral.

Restricted three-body problem featured prominently in Poincare's ingenious development of his geometric analysis on ordinary differential equations. Celestial mechanics was an academic subject of ultimate practical and theoretic importance in Poincare's time. In the course of his study of the restricted three-body problem, Poincare gradually came to the realization that, regarding the mass ratio of the primaries as a small parameter of perturbation, the invariant surface formed by the unperturbed parabolic solution of the two-body problem in phase space is unlikely to persist under small perturbation, and the break of this invariant surface would induce exceedingly complicated dynamic objects he names as *homoclinic tangles* [9, 10, 11].

Poincare's study went way beyond a bare realization. He introduced a computational method aimed in verifying the existence of homoclinic tangle in Hamiltonian equations, and he applied this method to an explicit example to illustrate that homoclinic tangles as a dynamical phenomenon do exist. This computational method was, in essence, reformulated to cover periodically perturbed Hamiltonian equations by Melnikov [8] at a much later time.

However, Poincare *did not* apply this computational scheme to prove that, for the restricted three-body problem, the unperturbed invariant surface formed by parabolic solutions of the two-body problem indeed breaks to form homoclinic tangles. It appeared that there existed a list of technical hurdles in applying Poincare's computational scheme (commonly named as the Poincare-Melnikov method in current literature) to the restricted three-body problem: First, the fixed point at infinity, to which the parabolic solutions of the two-body problem approach, is a highly degenerate fixed point, the local solutions structure of which is not easily determined. Second, with this noted degeneracy, the size of the neighborhood around the fixed point, on which the local dynamics could be fully understood by using techniques in local analysis tends to be much smaller than that of a non-degenerate saddle, introducing an uncertainty on the validity in applying the computational scheme Poincare introduced to the restricted three-body problem. Finally, assuming these two hurdles are somewhat removed, it would remain a challenging computational task to explicitly evaluate the Poincare-Melnikov integral for the restricted three-body problem.

The first hurdle was removed, at a much later time, by a paper of McGehee on local stable and unstable manifold around certain degenerate fixed point. McGehee proved in [7] that, for the restricted three-body problem, the local stable and unstable manifold of the fixed point at infinity are real analytic in phase space. This study is then followed by a paper of Llibre and Simo [5] regarding McGehee's result as a proper justification for Poincare's computation scheme to apply. They calculated the corresponding Poincare-Melnikov integral, and affirmed that, assuming first the Jacobi constant is sufficiently large then the ratio of the masses of the two primaries is sufficiently small, homoclinic tangle exists in Poincare's original setting,

In [13], Xia also regarded McGehee's result as a proper justification for Poincare-Melnikov method to apply in this case. He went one step further to acclaim that McGehee's method can also be extended to prove the analytic dependency of the splitting distance of the stable and unstable manifold on the ratio of primary masses and on the Jacobi constant. Xia argued, in addition, that because of the existence of singularity of binary collision, the Melnikov function automatically possess non-tangential zero if the Jacobi constant is close to what is allowed for binary collision. He then concluded that, excluding at most finitely many mass ratio of primaries, homoclinic tangle exists in Poincare's original setting.

There is also a relatively recent paper on this matter [4], in which the authors adopted an elaborated theory, gradually developed in the last forty plus years by a list of authors (See

the reference list of [1]) on exponentially small splitting in the study of equations of high frequency perturbations, to study the restricted three-body problem. They concluded that for all negative Jacobi constant of sufficiently large magnitude, and for all ratio of masses of the primary bodies, homoclinic tangles exist.

We note that the scope of review we adopt here is very narrow: we only cited results exclusively on the planar circular restricted three-body problem. We refer the reader to [3] for the exciting history of inter-plays in between the studies of the N-body problem and the development of the modern theory of dynamical systems.

In this paper, we first rewrite the equation of the restricted three-body problem literally as a perturbed Duffing equation. We then adopt what was recently introduced in [2] to derive integral equations for the primary stable and unstable solutions. These integral equations are then used to prove analytic dependency of the splitting distance on the ratios of primary masses and on Jacobi constant. We also use the homoclinic solution of the Duffing equation to evaluate the Melnikov integral. The main conclusion is as follows.

Theorem 1. *Assume the Jacobi constant $J < 0$ is such that $|J|$ is sufficiently large. Then for all but at most finitely many mass ratios of the primaries, the surface of unperturbed parabolic solutions breaks, inducing homoclinic tangles as originally anticipated by Henry Poincare.*

This theorem is slightly weaker than that of [4]. Our proof, however, is much less elaborated. This paper is also self-contained: we do not rely on McGehee's analysis in [7] to justify the applicability of Poincare-Melnikov method.

1. Derivation of Equations Around Parabolic Solution

In Sect. 1.1, we derive the equations of the planar three-body problem by using the polar form of the Jacobi coordinates through the Lagrangian formulation in classical mechanics. We then induce the equations of the restricted three-body problem by taking $m_1 + m_2 = 1, m_3 \to 0$ in Sect. 1.2. We apply McGehee's change of coordinates to the equations of the restricted three-body problem in Sect. 1.3. The contents of these three subsections are mostly elementary. At the end of Sect. 1.3, a new set of coordinates is introduced to re-write the equations of the restricted three-body problem as a *perturbed Duffing equation*. We then apply a simple shift of coordinates to direct our attention to the vicinity of the homiclinic solution of the unperturbed Duffing equation in Sect. 1.4. The equations obtained at the end are summarized in details in Sect. 1.5.

1.1. Equations for the Three-body Problem.

For the three bodies m_1, m_2, m_3 in $\mathbb{R}^2$, let $z_1 = (x_1, y_1)$ be the vector from m_1 to m_2 and $z_2 = (x_2, y_2)$ be the vector from the center of masses of m_1 and m_2 to m_3. Denote $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$. The variables z_1, z_2 are the Jacobi coordinates for the three-body problem. Following the convention in classical mechanics, we use dots on top to represent derivatives with respect to t. One dot is for velocity and two dots are for acceleration.

Let

$$T = \frac{1}{2}\mu_1|\dot{z}_1|^2 + \frac{1}{2}\mu_2|\dot{z}_2|^2$$

be the kinetic energy where

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}, \qquad \mu_2 = \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}.$$

We use polar coordinates for z_1 and z_2 by letting

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}. \tag{1}$$

We have, for the kinetic energy,

$$T = \frac{1}{2}\mu_1 \left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \frac{1}{2}\mu_2 \left(\dot{r}_2^2 + r_2^2\dot{\theta_2}^2\right)$$

and for the potential energy Also let

$$U = \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \tag{2}$$

be the potential energy where r_{ij} are the distances from m_i to m_j. Let

$$\alpha_1 = \frac{m_2}{m_1 + m_2}, \qquad \alpha_2 = \frac{m_1}{m_1 + m_2}. \tag{3}$$

We have

$$\begin{aligned} r_{12} =& |z_1| = r_1; \\ r_{13} =& |z_2 + \alpha_1 z_1| = \sqrt{r_2^2 + \alpha_1^2 r_1^2 + 2\alpha_1 r_1 r_2 \cos(\theta_2 - \theta_1)}; \\ r_{23} =& |z_2 - \alpha_2 z_1| = \sqrt{r_2^2 + \alpha_2^2 r_1^2 - 2\alpha_2 r_1 r_2 \cos(\theta_2 - \theta_1)}. \end{aligned} \tag{4}$$

The equation of motion for thr planar three-body problem is

$$\begin{aligned} \ddot{r}_1 &= r_1\dot{\theta}_1^2 + \mu_1^{-1}\partial_{r_1}U; \qquad \ddot{\theta}_1 = -\frac{2\dot{r}_1\dot{\theta}_1}{r_1} + \mu_1^{-1}\frac{1}{r_1^2}\partial_{\theta_1}U; \\ \ddot{r}_2 &= r_2\dot{\theta}_2^2 + \mu_2^{-1}\partial_{r_2}U; \qquad \ddot{\theta}_2 = -\frac{2\dot{r}_2\dot{\theta}_2}{r_2} + \mu_2^{-1}\frac{1}{r_2^2}\partial_{\theta_2}U \end{aligned} \tag{5}$$

The Lagrangian for the planar three-body problem is

$$L = T + U = \frac{1}{2}\mu_1 \left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \frac{1}{2}\mu_2 \left(\dot{r}_2^2 + r_2^2\dot{\theta_2}^2\right) + U$$

and the Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\partial_{\dot{r}_1}L &= \partial_{r_1}L; \qquad \frac{d}{dt}\partial_{\dot{\theta}_1}L = \partial_{\theta_1}L; \\ \frac{d}{dt}\partial_{\dot{r}_2}L &= \partial_{r_2}L; \qquad \frac{d}{dt}\partial_{\dot{\theta}_2}L = \partial_{\theta_2}L. \end{aligned}$$

Written explicitly, the equations of motion of the planar three-body problem are

$$\begin{aligned} \ddot{r}_1 &= r_1\dot{\theta}_1^2 + \mu_1^{-1}\partial_{r_1}U; \qquad \ddot{\theta}_1 = -\frac{2\dot{r}_1\dot{\theta}_1}{r_1} + \mu_1^{-1}\frac{1}{r_1^2}\partial_{\theta_1}U; \\ \ddot{r}_2 &= r_2\dot{\theta}_2^2 + \mu_2^{-1}\partial_{r_2}U; \qquad \ddot{\theta}_2 = -\frac{2\dot{r}_2\dot{\theta}_2}{r_2} + \mu_2^{-1}\frac{1}{r_2^2}\partial_{\theta_2}U \end{aligned} \tag{6}$$

where

$$\begin{aligned}
\partial_{r_1}U = & -\frac{m_1m_2}{r_1^2}-\frac{m_1m_3\alpha_1\left(\alpha_1r_1+r_2\cos(\theta_2-\theta_1)\right)}{r_{13}^3}-\frac{m_2m_3\alpha_2\left(\alpha_2r_1-r_2\cos(\theta_2-\theta_1)\right)}{r_{23}^3};\\
\partial_{r_2}U = & -\frac{m_1m_3\left(r_2+\alpha_1r_1\cos(\theta_2-\theta_1)\right)}{r_{13}^3}-\frac{m_2m_3\left(r_2-\alpha_2r_1\cos(\theta_2-\theta_1)\right)}{r_{23}^3};\\
\partial_{\theta_1}U = & -\frac{m_1m_3\alpha_1r_1r_2\sin(\theta_2-\theta_1)}{r_{13}^3}+\frac{m_2m_3\alpha_2r_1r_2\sin(\theta_2-\theta_1)}{r_{23}^3};\\
\partial_{\theta_2}U = & \frac{m_1m_3\alpha_1r_1r_2\sin(\theta_2-\theta_1)}{r_{13}^3}-\frac{m_2m_3\alpha_2r_1r_2\sin(\theta_2-\theta_1)}{r_{23}^3}.
\end{aligned}$$

1.2. **Equations for the Restricted Three-body Problem.** To obtain the equations for the restricted three-body problem, we let

$$m_3=0,\qquad m_1+m_2=1.$$

The equations for r_1, θ_1, in this case, become

$$\ddot{r}_1=r_1\dot{\theta}_1^2-\frac{1}{r_1^2};\qquad \ddot{\theta}_1=-\frac{2\dot{r}_1\dot{\theta}_1}{r_1}$$

and this set of equations allows a specific solution

$$r_1=1,\qquad \theta_1=t.$$

We adopt this specific solution to write the equations for r_2,θ_2 as

$$\ddot{r}_2=r_2\dot{\theta}_2^2+f;\qquad \ddot{\theta}_2=-\frac{2\dot{r}_2\dot{\theta}_2}{r_2}+g \tag{7}$$

where

$$\begin{aligned}
f = & -\frac{m_1\left(r_2+m_2\cos(\theta_2-t)\right)}{r_{13}^3}-\frac{m_2\left(r_2-m_1\cos(\theta_2-t)\right)}{r_{23}^3};\\
g = & \frac{m_1m_2\sin(\theta_2-t)}{r_2}\left(\frac{1}{r_{13}^3}-\frac{1}{r_{23}^3}\right);
\end{aligned}$$

and

$$r_{13}=\sqrt{r_2^2+m_2^2+2m_2r_2\cos(\theta_2-t)};\quad r_{23}=\sqrt{r_2^2+m_1^2-2m_1r_2\cos(\theta_2-t)}.$$

With equation (7), we study the motions of a particle of mass zero in the gravitational field induced by two finite masses making circular motion in the two-dimensional physical space.

Lemma 1.1. *We have, for all solution of equation (7), the Jacobi integral*

$$J=\frac{1}{2}\left(\dot{r}_2^2+r_2^2(\dot{\theta}_2-1)^2\right)-\frac{1}{2}r_2^2-m_1r_{13}^{-1}-m_2r_{23}^{-1}$$

where J is an integral constant, commonly referred to as the Jacobi constant.

Proof. Verification by direct calculation. We leave it to the reader as an exercise. □

1.3. **Equations in McGehee's Coordinates.** The phase variables for equation (7) are $(r_2, \dot{r}_2, \theta_2, \dot{\theta}_2)$. We introduce new phase variables (u, v, θ, w) and a new time τ, following McGehee, by letting

$$u = r_2^{-1}, \quad v = u^{-1/2}\dot{r}_2, \quad \theta = \theta_2 - t, \quad w = u^{-3/2}\dot{\theta}_2; \quad d\tau = \frac{1}{\sqrt{2}}u^{3/2}dt. \tag{8}$$

Equations for u, v, θ, w in τ are

$$\begin{aligned} \frac{du}{d\tau} =& -\sqrt{2}vu; \\ \frac{d\theta}{d\tau} =& \sqrt{2}\left(w - u^{-3/2}\right); \\ \frac{dw}{d\tau} =& -\frac{1}{\sqrt{2}}vw + \sqrt{2}uG; \\ \frac{dv}{d\tau} =& \frac{1}{\sqrt{2}}v^2 + \sqrt{2}w^2 - \sqrt{2} + \sqrt{2}F \end{aligned} \tag{9}$$

where

$$\begin{aligned} F =& 1 - m_1\left(1 + m_2 u\cos\theta\right)R_{13}^{-3} - m_2\left(1 - m_1 u\cos\theta\right)R_{23}^{-3}; \\ G =& m_1 m_2 \sin\theta\left(R_{13}^{-3} - R_{23}^{-3}\right) \end{aligned}$$

in which

$$R_{13} = \sqrt{1 + m_2^2 u^2 + 2m_2 u\cos\theta}; \qquad R_{23} = \sqrt{1 + m_1^2 u^2 - 2m_1 u\cos\theta}.$$

We further introduce new variables X, Y by letting

$$X = w, \quad Y = -\frac{1}{\sqrt{2}}vw. \tag{10}$$

The equations for θ, X, Y in τ are

$$\begin{aligned} \frac{d\theta}{d\tau} =& \sqrt{2}\left(X - u^{-3/2}\right); \\ \frac{dX}{d\tau} =& Y + \sqrt{2}uG; \\ \frac{dY}{d\tau} =& X - X^3 - XF + \sqrt{2}u\frac{Y}{X}G. \end{aligned} \tag{11}$$

Here we dropped the equation for u despite a clear occurrence of u on the right hand of the equations for θ, X, Y. We, however, can solve u for θ, X, Y by using the Jacobi integral

$$u^{1/2}J = \frac{1}{2}u^{3/2}v^2 + \frac{1}{2}u^{3/2}w^2 - w - m_1 u^{3/2}R_{13}^{-1} - m_2 u^{3/2}R_{23}^{-1}.$$

In this paper, we only consider the case of $J < 0$ and we also assume $|J| >> 1$. We now replace u with a new variable U by letting

$$U = |J|u^{1/2}X^{-1}, \tag{12}$$

and in reverse,

$$u = |J|^{-2}U^2X^2.$$

We rewrite the equation (11) for the restricted three-body problem as

$$\begin{aligned}\frac{d\theta}{d\tau} =&\sqrt{2}\left(X-|J|^3U^{-3}X^{-3}\right);\\ \frac{dX}{d\tau} =&Y+\sqrt{2}|J|^{-2}U^2X^2G;\\ \frac{dY}{d\tau} =&X-X^3-XF+\sqrt{2}|J|^{-2}U^2XYG\end{aligned}\tag{13}$$

where

$$\begin{aligned}F =&1-m_1\left(1+m_2|J|^{-2}U^2X^2\cos\theta\right)R_{13}^{-3}-m_2\left(1-m_1|J|^{-2}U^2X^2\cos\theta\right)R_{23}^{-3};\\ G =&m_1m_2\sin\theta\left(R_{13}^{-3}-R_{23}^{-3}\right)\end{aligned}$$

and

$$\begin{aligned}R_{13} =&\sqrt{1+m_2^2|J|^{-4}U^4X^4+2m_2|J|^{-2}U^2X^2\cos\theta};\\ R_{23} =&\sqrt{1+m_1^2|J|^{-4}U^4X^4-2m_1|J|^{-2}U^2X^2\cos\theta}.\end{aligned}$$

We also note that U as a function of X, Y, θ is implicitly defined by the Jacobi integral

$$U=1-|J|^{-3}U^3Y^2-\frac{1}{2}|J|^{-3}U^3X^4+m_1|J|^{-3}U^3X^2R_{13}^{-1}+m_2|J|^{-3}U^3X^2R_{23}^{-1}.$$

1.4. **Equations Around Unperturbed Parabolic Solution.** In what follows, we denote $|J|^{-1}$ as ε and m_2 as ρ to have

$$|J|^{-1}=\varepsilon;\quad m_2=\rho;\quad m_1=1-\rho.\tag{14}$$

We rewrite equation (13) as

$$\begin{aligned}\frac{d\theta}{d\tau} =&\sqrt{2}\left(X-\frac{1}{\varepsilon^3U^3X^3}\right);\\ \frac{dX}{d\tau} =&Y+\sqrt{2}\varepsilon^2U^2X^2G;\\ \frac{dY}{d\tau} =&X-X^3-XF+\sqrt{2}\varepsilon^2U^2XYG.\end{aligned}\tag{15}$$

We also have

$$\begin{aligned}F =&1-(1-\rho)\left(1+\rho\varepsilon^2U^2X^2\cos\theta\right)R_{13}^{-3}-\rho\left(1-(1-\rho)\varepsilon^2U^2X^2\cos\theta\right)R_{23}^{-3};\\ G =&\rho(1-\rho)\sin\theta\left(R_{13}^{-3}-R_{23}^{-3}\right)\end{aligned}$$

where

$$\begin{aligned}R_{13} =&\sqrt{1+\rho^2\varepsilon^4U^4X^4+2\rho\varepsilon^2U^2X^2\cos\theta};\\ R_{23} =&\sqrt{1+(1-\rho)^2\varepsilon^4U^4X^4-2(1-\rho)\varepsilon^2U^2X^2\cos\theta}\end{aligned}$$

and U, as a function of X, Y, θ, is determined by

$$U=1-\varepsilon^3U^3Y^2-\frac{1}{2}\varepsilon^3U^3X^4+(1-\rho)\varepsilon^3U^3X^2R_{13}^{-1}+\rho\varepsilon^3U^3X^2R_{23}^{-1}.\tag{16}$$

Let $(X(\tau), Y(\tau), \theta(\tau))$ be the solution of of equation (15) satisfying the initial condition

$$(X(0), Y(0), \theta(0))=(X_0, 0, \theta_0).$$

The solution $(X(\tau), Y(\tau), \theta(\tau))$ is initiated on the X-axis. In what follows, *we also regard* θ_0 *as a fixed parameter*. We change θ to $\theta + \theta_0$ to rewrite equation (15) as

$$\begin{aligned} \frac{d\theta}{d\tau} =& \sqrt{2}\left(X - \frac{1}{\varepsilon^3 U_{\theta_0}^3 X^3}\right); \\ \frac{dX}{d\tau} =& Y + \sqrt{2}\varepsilon^2 U_{\theta_0}^2 X^2 G_{\theta_0}; \\ \frac{dY}{d\tau} =& X - X^3 - X F_{\theta_0} + \sqrt{2}\varepsilon^2 U_{\theta_0}^2 XY G_{\theta_0} \end{aligned} \tag{17}$$

where $F_{\theta_0}, G_{\theta_0}$ are obtained by changing θ in F, G to $\theta + \theta_0$, and U_{θ_0} is obtained by solving U from the Jacobi integral in which θ is changed to $\theta + \theta_0$. We then study the solution of equation (17) satisfies the initial condition

$$(X(0), Y(0), \theta(0)) = (X_0, 0, 0). \tag{18}$$

Regarding ε as a parameter of perturbation, the set of equations of (15) for X, Y is a perturbed Duffing equation. Let

$$a(\tau) = \frac{2\sqrt{2}}{e^{\tau} + e^{-\tau}}; \quad b(\tau) = \frac{2\sqrt{2}\,(e^{-\tau} - e^{\tau})}{(e^{\tau} + e^{-\tau})^2}.$$

The saddle fixed point $(X, Y) = (0, 0)$ of the unperturbed Duffing equation takes $(a(\tau), b(\tau))$ as a homoclinic solution. Observe that, by letting $\varepsilon = 0$ in Jacobi integral, we obtain $U = 1$. The equation for θ, however, is singular at $\varepsilon = 0$. We substitute $U = 1$, $X = a(\tau)$ into the equation for θ to obtain

$$\frac{d\theta}{d\tau} = \sqrt{2}\left(a - \frac{1}{\varepsilon^3 a^3}\right).$$

By a direct integration, we obtain $\theta(\tau) = \theta(0) + \psi(\tau)$ where

$$\psi(\tau) = 4\tan^{-1} e^{\tau} - \pi - \frac{1}{48\varepsilon^3}\left(e^{3\tau} - e^{-3\tau}\right) - \frac{3}{16\varepsilon^3}\left(e^{\tau} - e^{-\tau}\right).$$

We can also rewrite $\psi(\tau)$ as

$$\psi(\tau) = 2\tan^{-1}\frac{1}{2}(e^{\tau} - e^{-\tau}) - \frac{1}{48\varepsilon^3}\left(e^{3\tau} - e^{-3\tau}\right) - \frac{3}{16\varepsilon^3}\left(e^{\tau} - e^{-\tau}\right).$$

Let

$$\ell^+ = \{(a(\tau), b(\tau)) : \quad \tau \in [0,\ +\infty)\}$$

be the positive part of the homoclinic solution $(a(\tau), b(\tau))$ in (x, y)-plane and D_ℓ^+ be a small neighborhood of $\ell^+ \cup (0, 0)$. We also use I to denote a small segment of the X-axis centered at $a(0) = \sqrt{2}$ and let $X_0 \in I$ to study the solution $(X(\tau), Y(\tau), \theta(\tau))$ of equation (17) satisfies the initial condition $(X(0), Y(0), \theta(0)) = (X_0, 0, 0)$.

Finally, we let

$$x = X - a(\tau), \qquad y = Y - b(\tau), \qquad \Theta = \theta - \psi(\tau) \tag{19}$$

to rewrite equation (17) in (x, y, Θ) as

$$
\begin{aligned}
\frac{d\Theta}{d\tau} =& \sqrt{2}\left(x + \frac{x^3 + 3ax^2 + 3a^2x + \left(U_{\theta_0,\psi}^3 - 1\right)(x+a)^3}{\varepsilon^3 U_{\theta_0,\psi}^3 (x+a)^3 a^3}\right);\\
\frac{dx}{d\tau} =& y + \sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2 (x+a)^2 G_{\theta_0,\psi};\\
\frac{dy}{d\tau} =& (1-3a^2)x - x^3 - 3ax^2 - (x+a)F_{\theta_0,\psi} + \sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2 (x+a)(y+b)G_{\theta_0,\psi}
\end{aligned} \tag{20}
$$

where $F_{\theta_0,\psi}, G_{\theta_0,\psi}$ are obtained by changing θ in F, G to $\Theta + \psi + \theta_0$, and $U_{\theta_0,\psi}$ is obtained by solving U from a new version of the Jacobi integral in which θ is changed to $\Theta + \psi + \theta_0$. We also need to substitute X, Y by using $x + a, y + b$ in all three to write them as functions in x, y and Θ. We study the solution $(x(\tau), y(\tau), \Theta(\tau))$ of equation (20) satisfying the initial condition

$$(x(0), y(0), \Theta(0)) = (X_0 - a(0), 0, 0).$$

1.5. **A brief summary of equations.** In what follows, we denote

$$
\begin{aligned}
F =& 1 - (1-\rho)\left(1 + \rho\varepsilon^2 U^2 X^2 \cos\theta\right) R_{13}^{-3} - \rho\left(1 - (1-\rho)\varepsilon^2 U^2 X^2 \cos\theta\right) R_{23}^{-3};\\
G =& \rho(1-\rho)\sin\theta\left(R_{13}^{-3} - R_{23}^{-3}\right)
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
R_{13} =& \sqrt{1 + \rho^2\varepsilon^4 U^4 X^4 + 2\rho\varepsilon^2 U^2 X^2 \cos\theta};\\
R_{23} =& \sqrt{1 + (1-\rho)^2\varepsilon^4 U^4 X^4 - 2(1-\rho)\varepsilon^2 U^2 X^2 \cos\theta};
\end{aligned} \tag{22}
$$

and U, as a function of X, Y, θ, is defined by the Jacobi integral

$$U = 1 - \varepsilon^3 U^3 Y^2 - \frac{1}{2}\varepsilon^3 U^3 X^4 + (1-\rho)\varepsilon^3 U^3 X^2 R_{13}^{-1} + \rho\varepsilon^3 U^3 X^2 R_{23}^{-1}. \tag{23}$$

We write the equations of motion of the restricted three-body problem in variables (x, y, Θ) around the homoclinic loop $\ell = \{(a(\tau), b(\tau)) : \tau \in \mathbb{R}\}$ as

$$\frac{d\Theta}{d\tau} = \frac{\sqrt{2}S}{\varepsilon^3 U_{\theta_0,\psi}^3 (x+a)^3 a^3}; \quad \frac{dx}{d\tau} = y + P; \quad \frac{dy}{d\tau} = (1-3a^2)x + Q \tag{24}$$

where

$$
\begin{aligned}
S =& x^3 + 3ax^2 + 3a^2x + \left(U_{\theta_0,\psi}^3 - 1\right)(x+a)^3 + \varepsilon^3 U_{\theta_0,\psi}^3 x(x+a)^3 a^3;\\
P =& \sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2 (x+a)^2 G_{\theta_0,\psi};\\
Q =& -x^3 - 3ax^2 - (x+a)F_{\theta_0,\psi} + \sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2 (x+a)(y+b)G_{\theta_0,\psi}.
\end{aligned} \tag{25}
$$

In the above,

(i) $U_{\theta_0,\psi}$ is defined by the Jacobi integral (23) in which we let $X = x + a, Y = y + b, \theta = \theta_0 + \psi + \Theta$;
(ii) $F_{\theta_0,\psi}, G_{\theta_0,\psi}$ are obtained by letting $X = x + a, Y = y + a, \theta = \Theta + \theta_0 + \psi$, and by using $U_{\theta_0,\psi}$ for U in F, G;

(iii) we have, for a, b and ψ,

$$\begin{aligned} a(\tau) =& \frac{2\sqrt{2}}{e^{\tau}+e^{-\tau}}; \qquad b(\tau) = \frac{2\sqrt{2}\,(e^{-\tau}-e^{\tau})}{(e^{\tau}+e^{-\tau})^2}; \\ \psi(\tau) =& 2\tan^{-1}\frac{1}{2}(e^{\tau}-e^{-\tau}) - \frac{1}{48\varepsilon^3}\left(e^{3\tau}-e^{-3\tau}\right) - \frac{3}{16\varepsilon^3}\left(e^{\tau}-e^{-\tau}\right). \end{aligned} \tag{26}$$

In the rest of this paper, we study the solution $(x(\tau), y(\tau), \Theta(\tau))$ of equation (24) satisfying the initial condition

$$(x(0), y(0), \Theta(0)) = (X_0 - a(0), 0, 0). \tag{27}$$

2. Integral Equations for Primary Stable Solutions

Let ε_0 be a small positive number. In this section, we assume

$$(\rho, \varepsilon) \in D_{\rho,\varepsilon} := \left[-\varepsilon_0, \frac{1}{2}+\varepsilon_0\right] \times (0, \varepsilon_0).$$

This is to say that we impose no restriction on the masses of the primary bodies but assume the Jacobi constant $J < 0$ is such that $|J|$ is large. *We regard ε as the parameter for perturbation.*

Definition 2.1. *We say that a solution $(x(\tau), y(\tau), \Theta(\tau))$, $\tau \in [0, +\infty)$ of equation (24) satisfying (27), that is, $(x(0), y(0), \Theta(0)) = (X_0 - a(0), 0, 0)$, is a primary stable solution if (i) $(x(\tau), y(\tau)) \in \mathcal{D}_\ell^+$ for all $\tau \in [0, +\infty)$; and (ii) $(x(\tau), y(\tau)) \to (0, 0)$ as $\tau \to +\infty$.*

In this section, we prove

Proposition 2.1. *There exists an $\varepsilon_0 > 0$ so that for any given $(\theta_0, \rho, \varepsilon) \in \mathcal{D}_{\theta_0,\rho,\varepsilon}$ where*

$$\mathcal{D}_{\theta_0,\rho,\varepsilon} = \mathbb{R} \times (-\varepsilon_0, \varepsilon_0 + 1/2) \times (0, \varepsilon_0),$$

equations (24) admits a unique primary stable solution. In addition, $x(0)$ of this solution as a function of $\theta_0, \rho, \varepsilon$ are real analytic on $\mathcal{D}_{\theta_0,\rho,\varepsilon}$.

In Sect. 2.1, we apply a recent theory on perturbed Duffing equations ([2]) to derive integral equations for primary stable solutions. We then get into a rather detailed study of the functions of perturbation treating the restricted three-body problem as a perturbed Duffing equation in Sect. 2.2. In Sect. 2.3, we prove Proposition 2.1 through an iteration scheme. Certain computational proofs are postponed to Sect. 2.4.

2.1. Canonical Coordinates for Perturbed Duffing Equations.

We follow a recent design of [2] to derive integral equations for primaries stable solutions. These integral equations are the main technical vehicle upon which we rely to move our study forward. In what follows, we let

$$h(\tau) = \frac{3(e^{2\tau} - e^{-2\tau} + 4\tau)}{2(e^{\tau}+e^{-\tau})^2}; \tag{28}$$

and

$$H(\tau) = \frac{1}{a(\tau)}\left[b(\tau)h(\tau) + a(\tau)\right]; \qquad \tilde{H}(\tau) = \frac{1}{a(\tau)}\left[b'(\tau)h(\tau) + 2b(\tau)\right]. \tag{29}$$

We note that $h(\tau), H(\tau), \tilde{H}(\tau)$ are uniformly bounded functions for all real τ, $h(\tau)$ and $\tilde{H}(\tau)$ are odd, but $H(\tau)$ is even in τ. In addition, $h(\tau)$ is such that

$$h' - \frac{2b}{a}h - 3 = 0. \tag{30}$$

Following the design of [2], we let M, W be such that

$$M = \frac{1}{a}\left(b'x - by\right); \quad W = \left(\tilde{H}x - Hy\right). \tag{31}$$

We have, in reverse,

$$x = \frac{1}{a}\left(bW - aHM\right); \quad y = \frac{1}{a}\left(b'W - a\tilde{H}M\right). \tag{32}$$

New variables M, W are designed to transform the equations of the first variations of the unperturbed Duffing equation, that is,

$$\frac{dx}{d\tau} = y, \qquad \frac{dy}{d\tau} = (1 - 3a^2(\tau))x$$

to

$$\frac{dM}{d\tau} = -\frac{b(\tau)}{a(\tau)}M, \qquad \frac{d\eta}{d\tau} = \frac{b(\tau)}{a(\tau)}W.$$

Lemma 2.1. *Equation (24) in M, W, Θ are transformed to*

$$\begin{aligned} \frac{d\Theta}{d\tau} =& \frac{\sqrt{2}S}{\varepsilon^3 U^3_{\theta_0,\psi}(x+a)^3 a^3}; \\ \frac{dM}{d\tau} =& -\frac{b}{a}M + \frac{1}{a}\left(b'P + bQ\right); \\ \frac{dW}{d\tau} =& \frac{b}{a}W + \tilde{H}P - HQ \end{aligned} \tag{33}$$

where on the right hand side, S, P, Q are as in (25), in which we need to further change from x, y to M, W by using (32).

Proof. The equation on Θ is the same as before. For M we have

$$\begin{aligned} \frac{dM}{d\tau} &= -\frac{b}{a}M + \frac{1}{a}\left(b'x' + b''x - b'y - by'\right) \\ &= -\frac{b}{a}M + \frac{1}{a}\left(b'(y+P) + b''x - b'y - b((1-3a^2)x + Q)\right) \\ &= -\frac{b}{a}M + \frac{1}{a}\left(b'P - bQ\right) \end{aligned}$$

where for the last equality we used $b'' = (a - a^3)' = (1 - 3a^2)b$.

Now for W, we have

$$\begin{aligned} \frac{dW}{d\tau} &= \left(\frac{1}{a}\left[b'h + 2b\right]\right)' x - \left(\frac{1}{a}\left[bh + a\right]\right)' y + \left(\frac{1}{a}\left[b'h + 2b\right]\right) x' - \left(\frac{1}{a}\left[bh + a\right]\right) y' \\ &= -\frac{b}{a}W + \frac{1}{a}\left[b''h + b'h' + 2b'\right] x - \frac{1}{a}\left[b'h + bh' + b\right] y + \frac{1}{a}\left[b'h + 2b\right](y + P) \\ &\quad -\frac{1}{a}\left[bh + a\right]((1 - 3a^2)x + Q). \end{aligned}$$

To continue, we use $h' = \frac{2b}{a}h + 3$ to obtain

$$\begin{aligned}\frac{dW}{d\tau} &= -\frac{b}{a}W + \frac{1}{a}\left[b''h + b'\left(\frac{2b}{a}h + 3\right) + 2b'\right]x - \frac{1}{a}\left[b'h + b\left(\frac{2b}{a}h + 3\right) + b\right]y \\ &\quad + \frac{1}{a}\left[b'h + 2b\right]y + \frac{1}{a}\left[b'h + 2b\right]P - \frac{1}{a}\left[bh + a\right](1 - 3a^2)x - \frac{1}{a}\left[bh + a\right]Q \\ &= \frac{b}{a}W + \tilde{H}P - HQ.\end{aligned}$$

Here, we used

$$b' = a - a^3, \quad b'' = (1 - 3a^2)b, \quad b^2 = a^2 - \frac{1}{2}a^4$$

for the last equality. □

Lemma 2.2. *The primary stable solutions satisfying $Y(0) = 0, \Theta(0) = 0$ is a solution of the integral equations*

$$\begin{aligned}\Theta(\tau) &= \int_0^\tau \frac{\sqrt{2}S}{\varepsilon^3 U_{\theta_0,\psi}^3 (x + a)^3 a^3} d\tau; \\ M(\tau) &= -\frac{1}{a}\int_\tau^{+\infty} \left(b'P + bQ\right) d\tau; \\ W(\tau) &= a\int_0^\tau \frac{1}{a}(\tilde{H}P - HQ) d\tau.\end{aligned} \tag{34}$$

Proof. With a fixed θ_0, we are interested in solutions of equation (33) satisfying

$$\Theta(0) = 0, \quad M(0) = -(X_0 - a(0)), \quad W(0) = 0.$$

Consequently, the solution we are seeking satisfies the integral equations

$$\begin{aligned}\Theta(\tau) &= \int_0^\tau \frac{\sqrt{2}S}{\varepsilon^3 U_{\theta_0,\psi}^3 (x + a)^3 a^3} d\tau; \\ M(\tau) &= \frac{1}{a}\left(-\sqrt{2}(X_0 - a(0)) + \int_0^\tau \left(b'P + bQ\right) d\tau\right); \\ W(\tau) &= a\int_0^\tau \frac{1}{a}(\tilde{H}P - HQ) d\tau.\end{aligned}$$

For a primary stable solution, we have $\lim_{\tau \to +\infty} a(\tau)M(\tau) = 0$, which implies

$$\sqrt{2}(X_0 - a(0)) = \int_0^{+\infty} \left(b'P + bQ\right) d\tau.$$

Substitute to the integral equation for $M(\tau)$, we obtain

$$M(\tau) = -\frac{1}{a}\int_\tau^{+\infty} \left(b'P + bQ\right) d\tau.$$

□

2.2. **Functions of perturbations for small ε.** All conclusions on P, Q, S for the restricted three-body problem presented in this subsection are obtained by straight forward uses of the binomial series expansion

$$(1+x)^\alpha = 1 + \alpha x + \sum_{n=2}^{\infty} C_{\alpha,n} x^n \tag{35}$$

where

$$C_{\alpha,n} = \frac{1}{n!}\alpha(\alpha-1)\cdots(\alpha-(n-1))$$

provided that $|x|<1$. We note that, by the assumption that ε_0 is sufficiently small, the condition $|x|<1$ is automatically fulfilled in all occasions this expansion is used throughout. We stay away from collision singularity in this paper.

Before getting into the details of P, Q, S, we introduce yet one more change of coordinates by re-scaling. Let

$$\mathbb{M} = \frac{1}{\varepsilon^3\sqrt{\varepsilon}a} M, \qquad \mathbb{W} = \frac{1}{\varepsilon^3\sqrt{\varepsilon}a} W. \tag{36}$$

We certainly need to further rewrite everything as functions of $\mathbb{M}, \mathbb{W}$ and Θ.

In what follows, we let

$$(\mathbb{M}, \mathbb{W}, \Theta) \in \mathbb{D}_\ell := [-1,1]\times[-1,1]\times\mathbb{R} \tag{37}$$

and we work exclusively on $\mathbb{D}_\ell$ for $(\mathbb{M}, \mathbb{W}, \Theta)$. The domain for parameters (ρ, ε) remains to be $D_{\rho,\varepsilon} = (-\varepsilon_0, \varepsilon_0 + 1/2)\times(0, \varepsilon_0)$.

Notation: In what follows, K is used to represent a generic constant independent of $\theta_0, \rho, \varepsilon$, the exact values of which are allowed to vary from line to line. We also use $\mathbb{O}(1)$ to represent a generic function that is real analytic in $\mathbb{M}, \mathbb{W}, \Theta, \rho, \varepsilon$ on $\mathbb{D}_\ell \times D_{\rho,\varepsilon}$ with a uniformly bounded C^1-norm on $\mathbb{D}_\ell \times D_{\rho,\varepsilon}$.

Proposition 2.2. *We have, for primary stable solutions,*

$$\begin{aligned}
\Theta(\tau) = \mathcal{F}_\Theta :=& \sqrt{2}\sqrt{\varepsilon}\int_0^\tau \frac{\mathbb{S}_1(\mathbb{X},\mathbb{Y})}{(\varepsilon^4\mathbb{X}+1)^3 a^3} d\tau + \sqrt{2}\sqrt{\varepsilon}\int_0^\tau \frac{a^4\mathbb{S}(\mathbb{X},\mathbb{Y},\Theta)}{(\varepsilon^4\mathbb{X}+1)^3 a^3} d\tau;\\
\mathbb{M}(\tau) = \mathcal{F}_\mathbb{M} :=& \frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} ba^3\left(\varepsilon^{10}\sqrt{\varepsilon}\mathbb{X}^3 + 3\varepsilon^7\mathbb{X}^2\right) d\tau\\
&- \frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} a^4\left(b'\mathbb{P}(\mathbb{X},\mathbb{Y},\Theta) + b\mathbb{Q}(\mathbb{X},\mathbb{Y},\Theta)\right) d\tau;\\
\mathbb{W}(\tau) = \mathcal{F}_\mathbb{W} :=& \sqrt{\varepsilon}\int_0^\tau Ha^2(\varepsilon^{10}\sqrt{\varepsilon}\mathbb{X}^3 + 3\varepsilon^7\mathbb{X}^2) d\tau\\
&+ \sqrt{\varepsilon}\int_0^\tau a^3(\tilde{H}\mathbb{P}(\mathbb{X},\mathbb{Y},\Theta) - H\mathbb{Q}(\mathbb{X},\mathbb{Y},\Theta)) d\tau
\end{aligned} \tag{38}$$

where $\mathbb{S}_1(\mathbb{X},\mathbb{Y})$ is a polynomial in $\mathbb{X},\mathbb{Y}$ of uniformly bounded coefficient. It is independent of Θ. For $\mathbb{S} = \mathbb{S}(\mathbb{X},\mathbb{Y},\Theta)$, $\mathbb{P} = \mathbb{P}(\mathbb{X},\mathbb{Y},\Theta)$, $\mathbb{Q} = \mathbb{Q}(\mathbb{X},\mathbb{Y},\Theta)$, we have $\mathbb{S} = \varepsilon^2\sqrt{\varepsilon}\mathbb{O}(1)$; and

$$\begin{aligned}
\mathbb{P} =& -\frac{3\sqrt{2}}{2}\rho(1-\rho)(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4\sin 2(\Theta+\psi+\theta_0) + \varepsilon^2 a^2\mathbb{O}(1);\\
\mathbb{Q} =& -\frac{3\sqrt{2}}{2}\rho(1-\rho)(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^3(\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+ba^{-1})\sin 2(\Theta+\psi+\theta_0) + a\mathbb{O}(1)
\end{aligned} \tag{39}$$

where

$$\mathbb{X} = \frac{1}{a}\left(b\mathbb{W} - aH\mathbb{M}\right); \qquad \mathbb{Y} = \frac{1}{a}\left(b'\mathbb{W} - a\tilde{H}\mathbb{M}\right). \tag{40}$$

A remark on the order of perturbation: We note that, when a solution moves into the vicinity of the saddle fixed point $(X, Y) = (0, 0)$, the order of singularity of the function on the right hand side of the equation for Θ is $\sim \varepsilon^{-3}a^{-3}(\tau)$ as $\tau \to \infty$ (See (34)). To balance this singularity, the perturbation functions in equations for (X, Y) must have a high enough order of dependency on $a(\tau)$. This requirement on perturbation functions are *stricter* than what was required in McGehee's paper [7] for the stable and unstable manifold to be real analytic. Fortunately, this extra requirement is met by the restricted three-body problem.

Proposition 2.2 and its proof are a little tedious because (i) we need to maintain, in precise detail, the functions of P, Q, S as they went through not so short of a list of change of variables introduced thus far; (ii) we need to keep the track of two orders instead of one for all terms involved: the first is the order in ε, and the second is the order in $a(\tau)$; (iii) we also need to distinguish, to certain degree, the part of P, Q, S that is Θ dependent from the part that is not. Nevertheless, the proof of this proposition is conceptually as simple as can be: it is a task of expanding a few given functions by using the binomial series.

We also note that the right hand side of each of the three integral equations in Proposition 2.2 has two integrals, the first of which is independent of Θ. We emphasize that the order of the integral functions in a is always higher for the second integral than that of the first integral. In particular, it is four degrees higher for the equation in Θ. As we will see momentarily, the factor a^4b' and a^4b in the second integral for $\mathbb{M}$, and the factor a^3 in the second integral for $\mathbb{W}$ is the bare minimum for the singularity of Θ as $\tau \to \infty$ to reach a desirable balance.

We postpone the proof of Proposition 2.2 to Sect. 2.4 so not to let a purely computational proof to interrupt our flow of presentation.

2.3. **Analytical dependency on $\theta_0, \rho, \varepsilon$.** We denote $\mathcal{V} = (\mathbb{M}(\tau), \mathbb{W}(\tau), \Theta(\tau))$ and let

$$\|\mathcal{V}\| = \sup_{\tau\in[0,+\infty)} |\mathbb{M}(\tau)| + \sup_{\tau\in[0,+\infty)} |\mathbb{W}(\tau)| + \sup_{\tau\in[0,+\infty)} a^3(\tau)|\Theta(\tau)|. \tag{41}$$

Let $\mathcal{F}$ be such that

$$\mathcal{F}(\mathcal{V}) := \left(\mathcal{F}_\Theta(\mathcal{V}),\ \mathcal{F}_{\mathbb{M}}(\mathcal{V}),\ \ \mathcal{F}_{\mathbb{W}}(\mathcal{V})\right). \tag{42}$$

where $\mathcal{F}_\Theta,\ \mathcal{F}_{\mathbb{M}},\ \ \mathcal{F}_{\mathbb{W}}$ are as in (38). We define $\mathcal{V}_n = (\mathbb{M}_n(\tau), \mathbb{W}_n(\tau), \Theta_n(\tau))$ inductively by letting

$$\mathcal{V}_{n+1}(\tau) = \mathcal{F}(\mathcal{V}_n(\tau)) \tag{43}$$

and we initiate this iteration by letting $\Theta_0(\tau) = 0$. The initial functions $\mathbb{M}_0(\tau), \mathbb{W}_0(\tau)$ are obtained by setting $\Theta = 0$, $\varepsilon = 0$ in $\mathbb{P}, \mathbb{Q}$ in the integrals for $\mathbb{M}, \mathbb{W}$ in (38). We have

$$\begin{aligned}
\mathbb{M}_0(\tau) =& \frac{3\sqrt{2}}{2}\rho(1-\rho)\frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} a^4(b' + b^2a^{-1})\sin 2(\theta_0 + \psi))d\tau;\\
\mathbb{W}_0(\tau) =& -\frac{3\sqrt{2}}{2}\rho(1-\rho)\sqrt{\varepsilon}\int_0^\tau a^3\left(\tilde{H} - ba^{-1}H\right)\sin 2(\theta_0 + \psi)d\tau.
\end{aligned} \tag{44}$$

Lemma 2.3. *We have, for all $n \geq 1$,*

$$\|\mathcal{V}_{n+1}(\tau) - \mathcal{V}_n(\tau)\| \ \leq\ K\sqrt{\varepsilon}\ \|\mathcal{V}_n(\tau) - \mathcal{V}_{n-1}(\tau)\|.$$

Proof. We start with the Θ component. We have

$$\begin{aligned}
a^3|\Theta_{n+1}(\tau)-\Theta_n(\tau)| &\leq \sqrt{2}\sqrt{\varepsilon}a^3(\tau)\int_0^\tau \left|\frac{\mathbb{S}_1(\mathbb{X}_n,\mathbb{Y}_n)}{(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}_n+1)^3a^3}-\frac{\mathbb{S}_1(\mathbb{X}_{n-1},\mathbb{Y}_{n-1})}{(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}_{n-1}+1)^3a^3}\right|d\tau\\
&\quad+\sqrt{2}\sqrt{\varepsilon}a^3(\tau)\int_0^\tau \left|\frac{a\mathbb{S}(\mathbb{X}_n,\mathbb{Y}_n,\Theta_n)}{(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}_n+1)^3}-\frac{a\mathbb{S}(\mathbb{X}_{n-1},\mathbb{Y}_{n-1},\Theta_{n-1})}{(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}_{n-1}+1)^3}\right|d\tau\\
&\leq K\sqrt{\varepsilon}a^3(\tau)\int_0^\tau a^{-3}d\tau\cdot\left(\sup_{\tau\in[0,+\infty)}|\mathbb{X}_n-\mathbb{X}_{n-1}|+\sup_{\tau\in[0,+\infty)}|\mathbb{Y}_n-\mathbb{Y}_{n-1}|\right)\\
&\quad+K\sqrt{\varepsilon}a^3(\tau)\int_0^\tau a^{-2}d\tau\cdot\left(\sup_{\tau\in[0,+\infty)}a^3|\Theta_n-\Theta_{n-1}|\right)\\
&\leq K\sqrt{\varepsilon}\|\mathcal{V}_n-\mathcal{V}_{n-1}\|.
\end{aligned}$$

Here, it is critically important that we separate the terms depending on Θ from the ones that do not. Our estimate is hinged on the factor a^4 in front of $\mathbb{S}(\mathbb{X},\mathbb{Y},\Theta)$.

We turn to $|M_{n+1}-M_n|$. We have

$$\begin{aligned}
M_{n+1}(\tau)-M_n(\tau) &= \frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} ba^3\left(\varepsilon^{10}\sqrt{\varepsilon}(\mathbb{X}_n^3-\mathbb{X}_{n-1}^3)+3\varepsilon^7(\mathbb{X}_n^2-\mathbb{X}_{n-1}^2)\right)d\tau\\
&\quad-\frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} b'a^4\left(\mathbb{P}_n-\mathbb{P}_{n-1}\right)d\tau-\frac{\sqrt{\varepsilon}}{a^2}\int_\tau^{+\infty} ba^4\left(\mathbb{Q}_n-\mathbb{Q}_{n-1}\right)d\tau
\end{aligned}$$

where

$$\mathbb{P}_n=\mathbb{P}(\mathbb{X}_n,\mathbb{Y}_n,\Theta_n),\quad \mathbb{Q}_n=\mathbb{Q}(\mathbb{X}_n,\mathbb{Y}_n,\Theta_n).$$

We now estimate $|\mathbb{P}_n-\mathbb{P}_{n-1}|$ and $|\mathbb{Q}_n-\mathbb{Q}_{n-1}|$. Note that our main concern here is the factor a^{-2} that goes to infinite as $\tau\to\infty$. In order to balance this blow up factor, we need a factor a^2 from $b'a^4|\mathbb{P}_n-\mathbb{P}_{n-1}|$ and $ba^4|\mathbb{Q}_n-\mathbb{Q}_{n-1}|$ to balance the outside factor a^{-2}. Recall that

$$\mathbb{P}(\mathbb{X},\mathbb{Y},\Theta)=-\frac{3\sqrt{2}}{2}\rho(1-\rho)\left[(\varepsilon^4\mathbb{X}+1)^4\sin(\Theta+\psi+\theta_0)+\varepsilon^2a^2\mathbb{O}(1)\right]. \tag{45}$$

We have, for $|\mathbb{P}_n-\mathbb{P}_{n-1}|$,

$$|\mathbb{P}_n-\mathbb{P}_{n-1}|=\int_0^1\frac{d}{ds}\mathbb{P}(s\mathcal{V}_n+(1-s)\mathcal{V}_{n-1})ds=(I)+(II)+(III)$$

where

$$\begin{aligned}
(I) &= \int_0^1\partial_{\mathbb{M}}\mathbb{P}(s\mathcal{V}_n+(1-s)\mathcal{V}_{n-1})(\mathbb{M}_n-\mathbb{M}_{n-1})ds\\
(II) &= \int_0^1\partial_{\mathbb{W}}\mathbb{P}(s\mathcal{V}_n+(1-s)\mathcal{V}_{n-1})(\mathbb{W}_n-\mathbb{W}_{n-1})ds\\
(III) &= \int_0^1\partial_{\Theta}\mathbb{P}(s\mathcal{V}_n+(1-s)\mathcal{V}_{n-1})(\Theta_n-\Theta_{n-1})ds.
\end{aligned}$$

We have

$$\begin{aligned}
|(I)| &\leq \int_0^1|\partial_{\mathbb{M}}\mathbb{P}(s\mathcal{V}_n+(1-s)\mathcal{V}_{n-1})|\,ds\cdot\sup_{\tau\in[0,+\infty)}|\mathbb{M}_n-\mathbb{M}_{n-1}|\\
&\leq K\varepsilon^2\cdot\sup_{\tau\in[0,+\infty)}|\mathbb{M}_n-\mathbb{M}_{n-1}|.
\end{aligned}$$

In parallel,

$$|(II)| \leq K\varepsilon^2 \cdot \sup_{\tau\in[0,+\infty)} |\mathbb{W}_n - \mathbb{W}_{n-1}|.$$

Estimate for (III) is a little different, and it is the place the singularity for Θ is balanced by the high order of a in $\mathbb{P}$. We have

$$\begin{aligned} |(III)| &= \left| \int_0^1 \partial_\Theta \mathbb{P}(s\mathcal{V}_n + (1-s)\mathcal{V}_{n-1})(\Theta_n - \Theta_{n-1})ds \right| \\ &\leq \int_0^1 \left| a^{-3}\partial_\Theta P(s\mathcal{V}_n + (1-s)\mathcal{V}_{n-1}) \right| ds \cdot \sup_{\tau\in[0,+\infty)} a^3|\Theta_n - \Theta_{n-1}| \\ &\leq Ka^{-3} \cdot \sup_{\tau\in[0,+\infty)} a^3|\Theta_n - \Theta_{n-1}|. \end{aligned}$$

We conclude that

$$|\mathbb{P}_n - \mathbb{P}_{n-1}| \leq Ka^{-3}\|\mathcal{V}_n - \mathcal{V}_{n-1}\|. \tag{46}$$

We also conclude in parallel lines that for $|\mathbb{Q}_n - \mathbb{Q}_{n-1}|$,

$$|\mathbb{Q}_n - \mathbb{Q}_{n-1}| \leq Ka^{-3}\|\mathcal{V}_n - \mathcal{V}_{n-1}\|. \tag{47}$$

We again emphasize that these estimations are hinged on the fact that *all* Θ *dependent terms on the right hand side are in order of at least* a^5 (see equation for $\mathbb{M}$ in (38)). We now conclude that

$$\begin{aligned} |\mathbb{M}_{n+1} - \mathbb{M}_n| \leq & K\sqrt{\varepsilon}\left(\frac{1}{a^2}\int_\tau^\infty (|b'a| + |ba|)d\tau\right)\|\mathcal{V}_n - \mathcal{V}_{n-1}\| \\ \leq & K\sqrt{\varepsilon}\|\mathcal{V}_n - \mathcal{V}_{n-1}\|. \end{aligned}$$

Estimate on $|\mathbb{W}_{n+1} - \mathbb{W}_n|$ are similar but only easier. □

Proof of Proposition 2.1: The sequence $\mathcal{V}_n = (\mathbb{M}_n(\tau), \mathbb{W}_n(\tau), \Theta_n(\tau))$ as constructed in the above is a sequence of real analytic functions of $\theta_0, \rho, \varepsilon$ on $\mathcal{D}_{\theta_0,\rho,\varepsilon}$. By Lemma 2.3, this real analytical sequence is a normal family, offering a unique limit that is also real analytic on $\mathcal{D}_{\theta_0,\rho,\varepsilon}$. □

Splitting Distance and Transversal Homoclinic Intersection: Thus far, our study has been exclusively on the primary stable solutions in $\mathcal{D}_\ell^+$. Let $\mathcal{D}_\ell^-$ be a small neighborhood around the negative part ℓ^- of $(a(\tau), b(\tau))$. We can also define *primary unstable solutions* as solutions that stay in $\mathcal{D}_\ell^-$ for all $\tau \in (-\infty, 0]$. Changing $+\infty$ to $-\infty$ all the way, the study for primary stable solutions can be repeated *verbatim* for the primary unstable solutions.

For a fixed pair $(\rho, \varepsilon) \in \mathcal{D}_{\rho,\varepsilon} := (-\varepsilon_0, \varepsilon_0 + 1/2) \times (0, \varepsilon_0)$, the Jacobi integral with $J = -\varepsilon^{-1}$ defines a three-dimensional invariant surface in the original four-dimensional phase space for the restricted three-body problem of primaries masses $m_2 = \rho, m_1 = 1 - \rho$. For a given $\theta_0 \in \mathbb{R}$, we denote the primary stable solution as

$$\mathcal{V}^s(\tau, \theta_0, \rho, \varepsilon) = (\mathbb{M}^s(\tau, \theta_0, \rho, \varepsilon),\ \mathbb{W}^s(\tau, \theta_0, \rho, \varepsilon), \Theta^s(\tau, \theta_0, \rho, \varepsilon));\quad \tau \in [0, +\infty)$$

and the primary unstable solution as

$$\mathcal{V}^u(\tau, \theta_0, \rho, \varepsilon) = (\mathbb{M}^u(\tau, \theta_0, \rho, \varepsilon),\ \mathbb{W}^u(\tau, \theta_0, \rho, \varepsilon), \Theta^u(\tau, \theta_0, \rho, \varepsilon));\quad \tau \in (-\infty, 0].$$

Primary stable solutions from all $\theta_0 \in \mathbb{R}$ form an immersed two-dimensional manifold, which we denote as W^s, in the three-dimensional surface defined by the Jacobi integral. Likewise,

the primary unstable solutions form an immersed two dimensional manifold, which we denote as W^u.

Definition 2.2. *Let*

$$\mathbb{D}(\theta_0, \rho, \varepsilon) = \frac{1}{\rho}\left(\mathbb{M}^s(0) - \mathbb{M}^u(0)\right).$$

We define $\mathbb{D}(\theta_0, \rho, \varepsilon)$ *as the* **splitting distance** *of the stable manifold* W^s *and the unstable manifold* W^u*. Note that we divided a copy of* ρ *for* $\mathbb{D}$ *to avoid* $\mathbb{D} = 0$ *for all* θ_0 *at* $\rho = 0$*.*

If $\theta_0 \in \mathbb{R}$ is such that $\mathbb{D}(\theta_0, \rho, \varepsilon) = 0$, then the corresponding primary stable and primary unstable solution fit together to form a homoclinic solution for the perturbed equation. If for this value of θ_0, we have in addition that $\partial_{\theta_0}\mathbb{D}_0(\theta_0, \rho, \varepsilon) \neq 0$, then W^s and W^u intersect transversally over this specific homoclinic solution. Consequently, to prove the existence of a transversal homoclinic intersection of W^s and W^u, it suffices for us to prove that there exists a θ_0 so that

$$\mathbb{D}(\theta_0, \rho, \varepsilon) = 0, \qquad \partial_{\theta_0}\mathbb{D}(\theta_0, \rho, \varepsilon) \neq 0. \tag{48}$$

2.4. **Proof of Proposition 2.2.** We start by using $O(1)$ to represent a generic real analytic function in $X, Y, \theta, \rho, \varepsilon$ for $(\rho, \varepsilon) \in D_{\rho,\varepsilon} = (-\varepsilon_0, \frac{1}{2} + \varepsilon_0) \times (0, \varepsilon_0)$ and $(X, Y) \in D_\ell^+$, $\theta \in \mathbb{R}$ so that the C^1-norm of this function is bounded by a uniform constant.

Lemma 2.4. *We expand* F *and* G *to obtain*

$$\begin{aligned} F =& \frac{3}{2}\rho(1-\rho)\varepsilon^4 X^4(1+\cos^2\theta) + \varepsilon^6 X^4(X^2+Y^2)O(1) \\ G =& -3\rho(1-\rho)\varepsilon^2 X^2 \sin\theta\cos\theta + \frac{3}{2}\rho(1-\rho)(1-2\rho)\varepsilon^4 X^4 \sin^3\theta \\ &+ \varepsilon^6 X^4(X^2+Y^2)O(1) \end{aligned} \tag{49}$$

Proof. We start with

$$\begin{aligned} R_{13} =& 1 + \rho\varepsilon^2 U^2 X^2 \cos\theta + \frac{1}{2}\rho^2\varepsilon^4 U^4 X^4 \sin^2\theta + \varepsilon^6 U^6 X^6 O(1) \\ R_{23} =& 1 - (1-\rho)\varepsilon^2 U^2 X^2 \cos\theta + \frac{1}{2}(1-\rho)^2\varepsilon^4 U^4 X^4 \sin^2\theta + \varepsilon^6 U^6 X^6 O(1). \end{aligned} \tag{50}$$

It then follows that

$$\begin{aligned} R_{13}^{-3} =& 1 - 3\rho\varepsilon^2 U^2 X^2 \cos\theta - \frac{3}{2}\rho^2\varepsilon^4 U^4 X^4 \sin^2\theta + \varepsilon^6 U^6 X^6 O(1) \\ R_{23}^{-3} =& 1 + 3(1-\rho)\varepsilon^2 U^2 X^2 \cos\theta - \frac{3}{2}(1-\rho)^2\varepsilon^4 U^4 X^4 \sin^2\theta + \varepsilon^6 U^6 X^6 O(1). \end{aligned} \tag{51}$$

For F we have

$$\begin{aligned} F =& 1 - (1-\rho)\left(1+\rho\varepsilon^2 U^2 X^2\cos\theta\right) R_{13}^{-3} - \rho\left(1-(1-\rho)\varepsilon^2 U^2 X^2\cos\theta\right) R_{23}^{-3} \\ =& 1 - (1-\rho)\left(1+\rho\varepsilon^2 U^2 X^2\cos\theta\right)\left(1 - 3\rho\varepsilon^2 U^2 X^2\cos\theta - \frac{3}{2}\rho^2\varepsilon^4 U^4 X^4 \sin^2\theta\right) + \varepsilon^6 U^6 X^6 O(1) \\ & - \rho\left(1-(1-\rho)\varepsilon^2 U^2 X^2\cos\theta\right)\left(1 + 3(1-\rho)\varepsilon^2 U^2 X^2\cos\theta - \frac{3}{2}(1-\rho)^2\varepsilon^4 U^4 X^4 \sin^2\theta\right). \end{aligned}$$

We have

$$\begin{aligned}F =&1-\left((1-\rho)+\rho(1-\rho)\varepsilon^2U^2X^2\cos\theta\right)\left(1-3\rho\varepsilon^2U^2X^2\cos\theta\right)\\&-\left((1-\rho)+\rho(1-\rho)\varepsilon^2U^2X^2\cos\theta\right)\left(-\frac{3}{2}\rho^2\varepsilon^4U^4X^4\sin^2\theta\right)\\&-\left(\rho-(1-\rho)\rho\varepsilon^2U^2X^2\cos\theta\right)\left(1+3(1-\rho)\varepsilon^2U^2X^2\cos\theta\right)\\&-\left(\rho-(1-\rho)\rho\varepsilon^2U^2X^2\cos\theta\right)\left(-\frac{3}{2}(1-\rho)^2\varepsilon^4U^4X^4\sin^2\theta\right)+\varepsilon^6U^6X^6O(1)\\=&\frac{3}{2}\rho(1-\rho)\varepsilon^4U^4X^4(1+\cos^2\theta)+\varepsilon^6U^6X^6O(1).\end{aligned}$$

For G, we have

$$G=(1-\rho)\rho\sin\theta\left(-3\varepsilon^2U^2X^2\cos\theta+\frac{3}{2}(1-2\rho)\varepsilon^4U^4X^4\sin^2\theta\right)+\varepsilon^6U^6X^6O(1).$$

We now work on U through Jacobi integral. We have from (50),

$$\begin{aligned}R_{13}^{-1}=&1-\rho\varepsilon^2U^2X^2\cos\theta+\varepsilon^4U^4X^4O(1)\\R_{23}^{-1}=&1+(1-\rho)\varepsilon^2U^2X^2\cos\theta+\varepsilon^4U^4X^4O(1)\end{aligned}\tag{52}$$

and

$$\begin{aligned}U=&1-\varepsilon^3U^3Y^2-\frac{1}{2}\varepsilon^3U^3X^4+(1-\rho)\varepsilon^3U^3X^2R_{13}^{-1}+\rho\varepsilon^3U^3X^2R_{23}^{-1}\\=&1-\varepsilon^3U^3Y^2-\frac{1}{2}\varepsilon^3U^3X^4+(1-\rho)\varepsilon^3U^3X^2(1-\rho\varepsilon^2U^2X^2\cos\theta+\varepsilon^4U^4X^4O(1))\\&+\rho\varepsilon^3U^3X^2(1+(1-\rho)\varepsilon^2U^2X^2\cos\theta+\varepsilon^4U^4X^4O(1))\\=&1-\varepsilon^3U^3Y^2-\frac{1}{2}\varepsilon^3U^3X^4+\varepsilon^3U^3X^2+\varepsilon^7U^7X^6O(1).\end{aligned}$$

Substituting into F and G, we obtain what are claimed in this lemma. □

Next we write F, G, U in x, y, Θ to have

Lemma 2.5. *We have*

$$\begin{aligned}F_{\theta_0,\psi}=&\frac{3}{2}\rho(1-\rho)\varepsilon^4(x+a)^4(1+\cos^2(\Theta+\theta_0+\psi))\\&+\varepsilon^6(x+a)^4((x+a)^2+(y+b)^2)O(1)\\G_{\theta_0,\psi}=&-\frac{3}{2}\rho(1-\rho)\varepsilon^2(x+a)^2\sin 2(\Theta+\theta_0+\psi)\\&+\frac{3}{2}\rho(1-\rho)(1-2\rho)\varepsilon^4(x+a)^4\sin^3(\Theta+\theta_0+\psi)\\&+\varepsilon^6(x+a)^4((x+a)^2+(y+b)^2)O(1)\\U_{\theta_0,\psi}=&1-\varepsilon^3(y^2+2by)-\frac{1}{2}\varepsilon^3(x^4+4x^3a+6x^2a^2+4xa)+\varepsilon^3(x^2+2ax)\\&+\varepsilon^6((x+a)^4+(x+a)^2(y+b)^2+(y+b)^4)O(1).\end{aligned}\tag{53}$$

Proof. Only the claim on $U_{\theta_0,\psi}$ needs to be further justified. We have

$$\begin{aligned}U_{\theta_0,\psi} =&1-\varepsilon^3U^3Y^2-\frac{1}{2}\varepsilon^3U^3X^4+(1-\rho)\varepsilon^3U^3X^2R_{13}^{-1}+\rho\varepsilon^3U^3X^2R_{23}^{-1}\\ =&1-\varepsilon^3Y^2-\frac{1}{2}\varepsilon^3X^4+\varepsilon^3X^2+\varepsilon^6(X^4+X^2Y^2+Y^4)O(1)\\ =&1-\varepsilon^3(y+b)^2-\frac{1}{2}\varepsilon^3(x+a)^4+\varepsilon^3(x+a)^2\\ &+\varepsilon^6((x+a)^4+(x+a)^2(y+b)^2+(y+b)^4)O(1)\end{aligned}$$

Using $b^2-a^2+\frac{1}{2}a^4=0$, we obtain

$$\begin{aligned}U_{\theta_0,\psi} =&1-\varepsilon^3(y^2+2by)-\frac{1}{2}\varepsilon^3(x^4+4x^3a+6x^2a^2+4xa)+\varepsilon^3(x^2+2ax)\\ &+\varepsilon^6((x+a)^4+(x+a)^2(y+b)^2+(y+b)^4)O(1).\end{aligned}$$

□

Lemma 2.6. *We have, on $\mathbb{D}_\ell\times D_{\rho,\varepsilon}$ for $\mathbb{M},\mathbb{W},\Theta,\rho,\varepsilon$,*

$$\begin{aligned}F_{\theta_0,\psi} =&\frac{3}{2}\rho(1-\rho)\varepsilon^4a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4(1+\cos^2(\Theta+\theta_0+\psi))+\varepsilon^6a^6\mathbb{O}(1);\\ G_{\theta_0,\psi} =&-3\rho(1-\rho)\varepsilon^2a^2(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^2\sin(\Theta+\theta_0+\psi)\cos(\Theta+\theta_0+\psi)\\ &+\frac{3}{2}\rho(1-\rho)(1-2\rho)\varepsilon^4a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4\sin^3(\Theta+\theta_0+\psi)+\varepsilon^6a^6\mathbb{O}(1);\\ U_{\theta_0,\psi} =&1-\varepsilon^3a^2(\varepsilon^7\mathbb{Y}^2+2ba^{-1}\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+\varepsilon^7\mathbb{X}^2)+\varepsilon^6a^4\mathbb{O}(1).\end{aligned}\tag{54}$$

Proof. All formula are obtained by straight forward substitution using the conclusion of Lemma 2.5. □

We now move to S,P,Q.

Lemma 2.7. *We have*

$$\begin{aligned}S =&a^3\varepsilon^3\sqrt{\varepsilon}\left[\mathbb{S}_1(\mathbb{X},\mathbb{Y})+\varepsilon^2\sqrt{\varepsilon}a^4\mathbb{O}(1)\right];\\ P =&a^4\varepsilon^4\mathbb{P}(\mathbb{X},\mathbb{Y},\Theta);\\ Q =&a^4\varepsilon^4\mathbb{Q}(\mathbb{X},\mathbb{Y},\Theta)-a^3\varepsilon^{10}\sqrt{\varepsilon}\mathbb{X}^3-3a^3\varepsilon^7\mathbb{X}^2.\end{aligned}\tag{55}$$

where $\mathbb{S}_1(\mathbb{X},\mathbb{Y}),\mathbb{P}(\mathbb{X},\mathbb{Y},\Theta),\mathbb{Q}(\mathbb{X},\mathbb{Y},\Theta)$ are as in Proposition 2.2.

Proof. We start with

$$U_{\theta_0,\psi}^3-1=\quad -3\varepsilon^3a^2(\varepsilon^7\mathbb{Y}^2+2ba^{-1}\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+\varepsilon^7\mathbb{X}^2)+\varepsilon^6a^4\mathbb{O}(1).$$

We have

$$\begin{aligned}S =&x^3+3ax^2+3a^2x+\left(U_{\theta_0,\psi}^3-1\right)(x+a)^3+\varepsilon^3U_{\theta_0,\psi}^3x(x+a)^3a^3;\\ =&a^3\varepsilon^{10}\sqrt{\varepsilon}\mathbb{X}^3+3a^3\varepsilon^7\mathbb{X}^2+3a^3\varepsilon^3\sqrt{\varepsilon}\mathbb{X}\\ &+\left(-3\varepsilon^3a^2(\varepsilon^7\mathbb{Y}^2+2ba^{-1}\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+\varepsilon^7\mathbb{X}^2)+\varepsilon^6a^4\mathbb{O}(1)\right)(a\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+a)^3\\ &+\varepsilon^3(1-3\varepsilon^3a^2(\varepsilon^7\mathbb{Y}^2+2ba^{-1}\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+\varepsilon^7\mathbb{X}^2)+\varepsilon^6a^4\mathbb{O}(1))a\varepsilon^3\sqrt{\varepsilon}\mathbb{X}(a\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+a)^3a^3\\ =&a^3\varepsilon^3\sqrt{\varepsilon}\left[3\mathbb{X}+\varepsilon^7\mathbb{X}^3+3\varepsilon^3\sqrt{\varepsilon}\mathbb{X}^2-3\varepsilon^2\sqrt{\varepsilon}a^2(\varepsilon^4\mathbb{Y}^2+2ba^{-1}\sqrt{\varepsilon}\mathbb{Y}+\varepsilon^4\mathbb{X}^2)(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^3\right]\\ &+\varepsilon^6a^7\mathbb{O}(1).\end{aligned}$$

Here we maintained a precise formula for the part of S that is up to order a^5 to emphasize on the fact that all terms dependent of Θ is of order at least a^7.

The formula for P and Q as claimed are easier. We note that the lowest order for terms in P and Q that is dependent of θ is a^4. We have for P and Q,

$$\begin{aligned}
P =&\sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2(x+a)^2 G_{\theta_0,\psi};\\
=&\sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2 a^2(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^2[-3\rho(1-\rho)\varepsilon^2 a^2(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^2\sin\theta\cos\theta\\
&+\frac{3}{2}\rho(1-\rho)(1-2\rho)\varepsilon^4 a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4\sin^3\theta+\varepsilon^6 a^6\mathbb{O}(1)]\\
=&-\frac{3\sqrt{2}}{2}\rho(1-\rho)\varepsilon^4 a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4\sin 2(\Theta+\psi+\theta_0)+\varepsilon^4 a^6\mathbb{O}(1);
\end{aligned}$$

$$\begin{aligned}
Q =&-x^3-3ax^2-(x+a)F_{\theta_0,\psi}+\sqrt{2}\varepsilon^2 U_{\theta_0,\psi}^2(x+a)(y+b)G_{\theta_0,\psi}\\
=&-x^3-3ax^2-(x+a)(\frac{3}{2}\rho(1-\rho)\varepsilon^4 a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^4(1+\cos^2\theta)+\varepsilon^6 a^6\mathbb{O}(1))\\
&-\frac{3\sqrt{2}}{2}\rho(1-\rho)\varepsilon^4 a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^3(\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+ba^{-1})\sin 2(\Theta+\psi+\theta_0)+\varepsilon^6 a^6\mathbb{O}(1)\\
=&-\frac{3\sqrt{2}}{2}\rho(1-\rho)\varepsilon^4 a^4(\varepsilon^3\sqrt{\varepsilon}\mathbb{X}+1)^3(\varepsilon^3\sqrt{\varepsilon}\mathbb{Y}+ba^{-1})\sin 2(\Theta+\psi+\theta_0)+\varepsilon^4 a^5\mathbb{O}(1)\\
&-a^3\varepsilon^{10}\sqrt{\varepsilon}\mathbb{X}^3-3a^3\varepsilon^7\mathbb{X}^2.
\end{aligned}$$

□

Proof of Proposition 2.2: Follows directly from Lemma 2.6 and (34). □

3. Existence of Transversal Homoclinic Intersections

In Section 2, we regarded ε as the parameter of perturbation but imposed no restriction on ρ. This view point served us well in our construction of primary stable and unstable solutions. However, splitting distance is **not** real analytic at $\varepsilon = 0$ so we can not expand it as a power series in ε at $\varepsilon = 0$ to approximate $\mathbb{D}(\theta_0, \rho, \varepsilon)$. In this section, we assume

$$0 < \rho << \varepsilon << 1$$

to regard ρ as the parameter of perturbation. We prove

Proposition 3.1. *There exists a positive $\varepsilon_1 << \varepsilon_0$, where ε_0 is as in Proposition 2.1, so that for every fixed $\varepsilon \in (0, \varepsilon_1)$, there exist a $\rho(\varepsilon) > 0$ sufficiently small, so that, for all $\rho \in (0, \rho(\varepsilon))$, the primary stable and unstable manifold admit transversal homoclinic intersections.*

The contents of this section is as follows. In Sect. 3.1 we derive a correspondence of the Poincare-Melnikov integral $\mathbb{D}_0(\theta_0)$ for the restricted three-body problem. In Sect. 3.2, we expand $\mathbb{D}_0(\theta_0)$ into a Fourier series in θ_0. In Sect. 3.3, we evaluate all coefficients of this Fourier expansion. Proposition 3.1 is proved at the end.

3.1. **Splitting Distance and its First Approximation.** We expand $\mathbb{D}(\theta_0,\rho,\varepsilon)$ as a power series in ρ at $\rho=0$ to write

$$\mathbb{D}(\theta_0,\rho,\varepsilon)=\mathbb{D}_0(\theta_0,\varepsilon)+\mathbb{D}_1(\theta_0,\varepsilon)\rho+\mathbb{D}_2(\theta_0,\varepsilon)\rho^2+\cdots.$$

Proposition 3.2. *(Poincare-Melnikov Integral) We have, for the restricted three-body problem regarding ρ as the parameter of perturbation,*

$$\begin{aligned}\mathbb{D}_0(\theta_0) &= -\frac{\sqrt{2}}{2\varepsilon\sqrt{\varepsilon}}\int_{-\infty}^{+\infty}(b'a^2+ab^2)\sin(\theta_0+\psi)\left(1-R_1^{-3}\right)d\tau\\ &\quad+\frac{1}{2\varepsilon^3\sqrt{\varepsilon}}\int_{-\infty}^{+\infty}ab[\varepsilon^2a^2\cos(\theta_0+\psi)(2+R_1^{-3})+(1-R_1^{-3})]d\tau\end{aligned}$$

where

$$R_1=\sqrt{1-2\varepsilon^2a^2\cos(\psi+\theta_0)+\varepsilon^4a^4}.$$

Proof. We take the integral equations (38) for the primary stable solutions as a starting point and replace $\mathbb{M}$, $\mathbb{W}$ by using

$$\tilde{M}=\rho^{-1}\mathbb{M},\qquad \tilde{W}=\rho^{-1}\mathbb{W}, \tag{56}$$

to count in the re-scaling factor ρ^{-1} for the splitting distance $\mathbb{D}$. We rewrite $\tilde{M},\tilde{W}$ back as $\mathbb{M},\mathbb{W}$ respectively to obtain

$$\begin{aligned}\Theta(\tau)&=\int_0^\tau\frac{\sqrt{2}S}{\varepsilon^3U^3_{\theta_0,\psi}(\varepsilon^3\sqrt{\varepsilon}\rho\mathbb{X}+1)^3a^6}d\tau;\\ \mathbb{M}(\tau)&=-\frac{1}{a^2\varepsilon^3\sqrt{\varepsilon}\rho}\int_\tau^{+\infty}\left(b'P+bQ\right)d\tau;\\ \mathbb{W}(\tau)&=\frac{1}{\varepsilon^3\sqrt{\varepsilon}\rho}\int_0^\tau\frac{1}{a}(\tilde{H}P-HQ)d\tau.\end{aligned} \tag{57}$$

At this point, we need to calculate again P,Q,S, but this time we only need to track the order in ρ because the splitting distance is real analytic in ρ at $\rho=0$. We expand P and Q into power series in ρ. To obtain $\mathbb{D}_0(\theta)$, we drop all terms of order ρ^2 and higher. We stat with

$$\begin{aligned}R_{13}=&\sqrt{1+\rho^2\varepsilon^4U^4X^4+2\rho\varepsilon^2U^2X^2\cos\theta};\\ =&1+\rho\varepsilon^2U^2X^2\cos\theta+O(\rho^2)\\ R_{23}=&\sqrt{1+(1-\rho)^2\varepsilon^4U^4X^4-2(1-\rho)\varepsilon^2U^2X^2\cos\theta}\\ =&R\left(1+\frac{-\rho\varepsilon^4U^4X^4+\rho\varepsilon^2U^2X^2\cos\theta}{R^2}\right)+O(\rho^2)\end{aligned}$$

where

$$R=\sqrt{1+\varepsilon^4U^4X^4-2\varepsilon^2U^2X^2\cos\theta}.$$

We have

$$\begin{aligned}R_{13}^{-3}=&1-3\rho\varepsilon^2U^2X^2\cos\theta+O(\rho^2)\\ R_{23}^{-3}=&R^{-3}\left(1+3\frac{\rho\varepsilon^4U^4X^4-\rho\varepsilon^2U^2X^2\cos\theta}{R^2}\right)+O(\rho^2).\end{aligned}$$

This is to imply

$$\begin{aligned} F =&\rho(1-R^3)+\rho\varepsilon^2U^2X^2\cos\theta(2+R^{-3})+O(\rho^2)\\ G =&\rho\sin\theta\left(1-R^{-3}\right)+O(\rho^2)\end{aligned}$$

We now work on U as defined by the Jacobi Integral. First we let $U_1=U_1(X,Y,\theta,\varepsilon)$ be such that

$$U_1=1-\varepsilon^3U_1^3\left(Y^2+\frac{1}{2}X^4-X^2\right).$$

We have

$$U_1(a(\tau),b(\tau),\theta,\varepsilon)=1$$

because

$$b(\tau)^2+\frac{1}{2}a(\tau)^4-a(\tau)^2=0.$$

It also follows, by the Jacobi integral, that

$$U=U_1+O(\rho)$$

Next we calculate S, P and Q. First, recall that

$$S=x^3+3ax^2+3a^2x+\left(U_{\theta_0,\psi}^3-1\right)(x+a)^3+\varepsilon^3U_{\theta_0,\psi}^3x(x+a)^3a^3.$$

The re-scaling of variables (56) introduces a common factor ρ to all terms in S. For P,Q, we have from (3.1),

$$\begin{aligned} P =&\sqrt{2}\varepsilon^2a^2\rho\sin\theta\left(1-R_1^{-3}\right)+O(\rho^2);\\ Q =&-a(\rho(1-R_1^3)+\rho\varepsilon^2a^2\cos\theta(2+R_1^{-3}))+\sqrt{2}\varepsilon^2ab\rho\sin\theta\left(1-R_1^{-3}\right)+O(\rho^2)\end{aligned}$$

where

$$R_1=\sqrt{1+\varepsilon^4a^4-2\varepsilon^2a^2\cos\theta}.$$

We now turn to the equation for $\mathbb{M}$ in (57), dropping all $O(\rho)$ terms, to obtain

$$\begin{aligned}\mathbb{M}^s(0)=&-\frac{\sqrt{2}}{a^2(0)\varepsilon\sqrt{\varepsilon}}\int_0^{+\infty}(b'a^2+ab^2)\sin\theta\left(1-R^{-3}\right)d\tau\\ &+\frac{1}{a^2(0)\varepsilon^3\sqrt{\varepsilon}}\int_0^{+\infty}ab((1-R^3)+\varepsilon^2a^2\cos\theta(2+R^{-3}))d\tau+O(\rho).\end{aligned}$$

The formula for $\mathbb{D}_0$ as claimed in this proposition then directly follow. □

3.2. **Fourier Expansion of $\mathbb{D}_0(\theta_0)$.** We write $\mathbb{D}_0(\theta_0)$ as

$$\mathbb{D}_0(\theta_0) \;=\; -\frac{\sqrt{2}}{2\varepsilon\sqrt{\varepsilon}}(I)+\frac{1}{2\varepsilon^3\sqrt{\varepsilon}}(II)$$

where

$$\begin{aligned}(I) &= \int_{-\infty}^{+\infty}(2a^3-3a^5/2)\sin(\theta_0+\psi)\left(1-R_1^{-3}\right)d\tau\\ (II) &= \int_{-\infty}^{+\infty}ab[\varepsilon^2a^2\cos(\theta_0+\psi)(2+R_1^{-3}).+(1-R_1^{-3})]d\tau\end{aligned}$$

and

$$R_1=\sqrt{1+\varepsilon^4a^4-2\varepsilon^2a^2\cos(\theta_0+\psi)}.$$

We write R_1 as

$$R_1 = (1+\varepsilon^4 a^4)^{1/2}\left(1-\frac{2\varepsilon^2 a^2\cos(\theta_0+\psi)}{(1+\varepsilon^4 a^4)}\right)^{1/2}$$

and use binomial expansion (35) to obtain

$$R_1^{-3} = \frac{1}{(1+\varepsilon^4 a^4)^{3/2}}+\sum_{n=1}^{\infty}\frac{(-1)^n C_{-3/2,n}\varepsilon^{2n}a^{2n}}{(1+\varepsilon^4 a^4)^{n+3/2}}\left(e^{i(\theta_0+\psi)}+e^{-i(\theta_0+\psi)}\right)^n.$$

We divide into odd and even cases for n to obtain

$$\begin{aligned}R_1^{-3} &= \frac{1}{(1+\varepsilon^4 a^4)^{3/2}}+\sum_{k=1}^{\infty}\frac{2C_{-3/2,2k}\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4 a^4)^{2k+3/2}}\sum_{m=0}^{k}C_{2k,k-m}\cos 2m(\theta_0+\psi)\\ &\quad -\sum_{k=0}^{\infty}\frac{2C_{-3/2,2k+1}\varepsilon^{4k+2}a^{4k+2}}{(1+\varepsilon^4 a^4)^{2k+3/2}}\sum_{m=0}^{k}C_{2k+1,k-m}\cos(2m+1)(\theta_0+\psi)\end{aligned}$$

Switch the order of the sums, we have

$$\begin{aligned}R_1^{-3} &= \frac{1}{(1+\varepsilon^4 a^4)^{3/2}}+\sum_{k=1}^{\infty}\frac{2C_{-3/2,2k}\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k,k}\\ &\quad +\sum_{m=1}^{\infty}\sum_{k=m}^{\infty}\frac{2C_{-3/2,2k}\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k,k-m}\cos 2m(\theta_0+\psi)\\ &\quad -\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{2C_{-3/2,2k+1}\varepsilon^{4k+2}a^{4k+2}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k+1,k-m}\cos(2m+1)(\theta_0+\psi).\end{aligned}$$

3.2.1. *Fourier Expansion for (I).* We have

$$\begin{aligned}&\sin(\theta_0+\psi)\left(1-R_1^{-3}\right)\\ =\ &\left(\frac{(1+\varepsilon^4 a^4)^{3/2}-1}{(1+\varepsilon^4 a^4)^{3/2}}-\sum_{k=1}^{\infty}\frac{2C_{-3/2,2k}\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k,k}\right)\sin(\theta_0+\psi)\\ &-\sum_{m=1}^{\infty}\sum_{k=m}^{\infty}\frac{C_{-3/2,2k}\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k,k-m}\left[\sin(2m+1)(\theta_0+\psi)-\sin(2m-1)(\theta_0+\psi)\right]\\ &+\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{C_{-3/2,2k+1}\varepsilon^{4k+2}a^{4k+2}}{(1+\varepsilon^4 a^4)^{2k+3/2}}C_{2k+1,k-m}\left[\sin(2m+2)(\theta_0+\psi)-\sin 2m(\theta_0+\psi)\right].\end{aligned}$$

We then have

$$\begin{aligned}(I) &= \sin\theta_0\int_{-\infty}^{+\infty}f^{(I)}\cos\psi d\tau+\sum_{m=1}^{\infty}\sin 2m\theta_0\int_{-\infty}^{+\infty}\left(\sum_{k=m-1}^{\infty}f_{m,k}^{(I)}\right)\cos(2m)\psi d\tau\\ &\quad +\sum_{m=1}^{\infty}\sin(2m+1)\theta_0\int_{-\infty}^{+\infty}\left(\sum_{k=m}^{\infty}g_{m,k}^{(I)}\right)\cos(2m+1)\psi d\tau\end{aligned}$$

where

$$\begin{aligned} f^{(I)} &= (2a^3-3a^5/2)\left(\frac{(1+\varepsilon^4a^4)^{3/2}-1}{(1+\varepsilon^4a^4)^{3/2}}-\sum_{k=1}^{\infty}\frac{C_{-3/2,2k}(2C_{2k,k}-C_{2k,k-1})\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4a^4)^{2k+3/2}}\right);\\ f^{(I)}_{m,k} &= (2a^3-3a^5/2)\left(\frac{C_{-3/2,2k+1}\varepsilon^{4k+2}a^{4k+2}}{(1+\varepsilon^4a^4)^{2k+3/2}}(C_{2k+1,k-m-1}-C_{2k+1,k-m})\right);\\ g^{(I)}_{m,k} &= -(2a^3-3a^5/2)\left(\frac{C_{-3/2,2k}(C_{2k,k-m}-C_{2k,k-m-1}))\varepsilon^{4k}a^{4k}}{(1+\varepsilon^4a^4)^{2k+3/2}}\right). \end{aligned}$$

Note that all non-trigonometric part of these function are even functions in τ. It then follows that (I) is a summation of integrals in the form of

$$I_{n,m,3}=\varepsilon^{4n}\int_{-\infty}^{+\infty}a^{4n+3}e^{im\psi}d\tau,\quad I_{n,m,5}=\varepsilon^{4n}\int_{-\infty}^{+\infty}a^{4n+5}e^{im\psi}d\tau$$

We emphasize that, for the integral function that defines $I_{n,m}$, the power in a is **odd**. This odd power obviously come from ab^2+a^2b' in front of all functions of integration.

3.2.2. *Fourier expansion for (II).* For (II), we have

$$\begin{aligned} &1-R_1^{-3}+\varepsilon^2a^2\cos(\theta_0+\psi)(2+R_1^{-3})\\ =\;& 1-\frac{1}{(1+\varepsilon^4a^4)^{3/2}}+\left(\frac{-3}{(1+\varepsilon^4a^4)^{5/2}}+\left(2+\frac{1}{(1+\varepsilon^4a^4)^{3/2}}\right)\right)\varepsilon^2a^2\cos(\theta_0+\psi)\\ &+\sum_{k=1}^{\infty}\left(\frac{C_{-3/2,2k-1}}{(1+\varepsilon^4a^4)^{2k+1/2}}-\frac{2C_{-3/2,2k}}{(1+\varepsilon^4a^4)^{2k+3/2}}\right)\varepsilon^{4k}a^{4k}\sum_{m=0}^{k}C_{2k,k-m}\cos 2m(\theta_0+\psi)\\ &-\sum_{k=1}^{\infty}\left(\frac{C_{-3/2,2k}}{(1+\varepsilon^4a^4)^{2k+3/2}}-\frac{2C_{-3/2,2k+1}}{(1+\varepsilon^4a^4)^{2k+5/2}}\right)\varepsilon^{4k+2}a^{4k+2}\sum_{m=0}^{k}C_{2k+1,k-m}\cos(2m+1)(\theta_0+\psi) \end{aligned}$$

We have

$$\begin{aligned} &1-R_1^{-3}+\varepsilon^2a^2\cos(\theta_0+\psi)(2+R_1^{-3})\\ =\;& 1-\frac{1}{(1+\varepsilon^4a^4)^{3/2}}+\sum_{k=1}^{\infty}\left(\frac{C_{-3/2,2k-1}}{(1+\varepsilon^4a^4)^{2k+1/2}}-\frac{2C_{-3/2,2k}}{(1+\varepsilon^4a^4)^{2k+3/2}}\right)\varepsilon^{4k}a^{4k}C_{2k,k}\\ &+\left(\frac{-3}{(1+\varepsilon^4a^4)^{5/2}}+\left(2+\frac{1}{(1+\varepsilon^4a^4)^{3/2}}\right)\right)\varepsilon^2a^2\cos(\theta_0+\psi)\\ &-\sum_{k=1}^{\infty}\left(\frac{C_{-3/2,2k}}{(1+\varepsilon^4a^4)^{2k+3/2}}-\frac{2C_{-3/2,2k+1}}{(1+\varepsilon^4a^4)^{2k+5/2}}\right)\varepsilon^{4k+2}a^{4k+2}C_{2k+1,k}\cos(\theta_0+\psi)\\ &+\sum_{m=1}^{\infty}\left[\sum_{k=m}^{\infty}\left(\frac{C_{-3/2,2k-1}C_{2k,k-m}}{(1+\varepsilon^4a^4)^{2k+1/2}}-\frac{2C_{-3/2,2k}C_{2k,k-m}}{(1+\varepsilon^4a^4)^{2k+3/2}}\right)\varepsilon^{4k}a^{4k}\right]\cos 2m(\theta_0+\psi)\\ &-\sum_{m=1}^{\infty}\left[\sum_{k=m}^{\infty}\left(\frac{C_{-3/2,2k}C_{2k+1,k-m}}{(1+\varepsilon^4a^4)^{2k+3/2}}-\frac{2C_{-3/2,2k+1}C_{2k+1,k-m}}{(1+\varepsilon^4a^4)^{2k+5/2}}\right)\varepsilon^{4k+2}a^{4k+2}\right]\cos(2m+1)(\theta_0+\psi) \end{aligned}$$

It then follows that

$$
\begin{aligned}
(II) =& \sin\theta_0 \int_{-\infty}^{\infty} f^{(II)} \sin\psi d\tau + \sum_{m=1}^{\infty} \sin 2m\theta_0 \left[\sum_{k=m}^{\infty} \int_{-\infty}^{\infty} f_{k,m}^{(II)} \sin 2m\psi d\tau \right] \\
&+ \sum_{m=1}^{\infty} \sin(2m+1)\theta_0 \left[\sum_{k=m}^{\infty} \int_{-\infty}^{\infty} g_{k,m}^{(II)} \sin(2m+1)\psi d\tau \right]
\end{aligned} \tag{58}
$$

where

$$
\begin{aligned}
f^{(II)} =& ab \left(\frac{-3}{(1+\varepsilon^4 a^4)^{5/2}} + \left(2 + \frac{1}{(1+\varepsilon^4 a^4)^{3/2}} \right) \right) \varepsilon^2 a^2 \\
&- \sum_{k=1}^{\infty} ab \left(\frac{C_{-3/2,2k}}{(1+\varepsilon^4 a^4)^{2k+3/2}} - \frac{2C_{-3/2,2k+1}}{(1+\varepsilon^4 a^4)^{2k+5/2}} \right) \varepsilon^{4k+2} a^{4k+2} C_{2k+1,k} \\
f_{k,m}^{(II)} =& - ab \left(\frac{C_{-3/2,2k-1} C_{2k,k-m}}{(1+\varepsilon^4 a^4)^{2k+1/2}} - \frac{2C_{-3/2,2k} C_{2k,k-m}}{(1+\varepsilon^4 a^4)^{2k+3/2}} \right) \varepsilon^{4k} a^{4k} \\
g_{k,m}^{(II)} =& ab \left(\frac{C_{-3/2,2k} C_{2k+1,k-m}}{(1+\varepsilon^4 a^4)^{2k+3/2}} - \frac{2C_{-3/2,2k+1} C_{2k+1,k-m}}{(1+\varepsilon^4 a^4)^{2k+5/2}} \right) \varepsilon^{4k+2} a^{4k+2}
\end{aligned} \tag{59}
$$

From this expansion for (II), it follow that (II) is a summation of integrals in the form of

$$
J_{n,m,1} = \varepsilon^{4n} \int_{-\infty}^{+\infty} a^{4n+1} b e^{im\psi} d\tau; \quad J_{n,m,3} = \varepsilon^{4n+2} \int_{-\infty}^{+\infty} a^{4n+3} b e^{im\psi} d\tau.
$$

Conclusion: Thus far, the task of evaluating $\mathbb{D}_0(\theta_0)$ has basically been reduced to evaluating

$$
I_{n,m,3} = \varepsilon^{4n} \int_{-\infty}^{+\infty} a^{4n+3} e^{im\psi} d\tau, \quad I_{n,m,5} = \varepsilon^{4n} \int_{-\infty}^{+\infty} a^{4n+5} e^{im\psi} d\tau,
$$

and

$$
J_{n,m,1} = \varepsilon^{4n} \int_{-\infty}^{+\infty} a^{4n+1} b e^{im\psi} d\tau \quad J_{n,m,3} = \varepsilon^{4n+2} \int_{-\infty}^{+\infty} a^{4n+3} b e^{im\psi} d\tau.
$$

3.3. **On $I_{n,m,1}$, $I_{n,m,3}$ and $J_{n,m,1}$, $J_{n,m,3}$.** Recall that

$$
\begin{aligned}
a(\tau) =& \frac{2\sqrt{2}}{e^{\tau} + e^{-\tau}}; \qquad b(\tau) = \frac{2\sqrt{2}\,(e^{-\tau} - e^{\tau})}{(e^{\tau} + e^{-\tau})^2}; \\
\psi(\tau) =& 2\tan^{-1} \frac{1}{2}(e^{\tau} - e^{-\tau}) - \frac{1}{48\varepsilon^3} \left(e^{3\tau} - e^{-3\tau} \right) - \frac{3}{16\varepsilon^3} \left(e^{\tau} - e^{-\tau} \right).
\end{aligned}
$$

3.3.1. *General Reduction.* Let $y(z) : \mathbb{R} \to \mathbb{R}$ be such that

$$
y(z) = (\sqrt{z^2 + 4^3} + z)^{1/3} - (\sqrt{z^2 + 4^3} - z)^{1/3}. \tag{60}
$$

Proposition 3.3. *We have*

$$
\begin{aligned}
I_{n,m,3} =&(-1)^m\frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}\int_{-\infty}^{+\infty}\frac{e^{-\frac{imz}{24\varepsilon^3}}}{(y(z)-2i)^{2n-m+2}(y(z)+2i)^{2n+m+2}}dz;\\
I_{n,m,5} =&(-1)^m\frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}\int_{-\infty}^{+\infty}\frac{e^{-\frac{imz}{24\varepsilon^3}}}{(y(z)-2i)^{2n-m+3}(y(z)+2i)^{2n+m+3}}dz;\\
J_{n,m,1} =&(-1)^{m+1}(2\sqrt{2})^{4n+2}\varepsilon^{4n}\int_{-\infty}^{+\infty}\frac{y(z)e^{-\frac{mi}{24\varepsilon^3}z}}{(y-2i)^{2n-m+2}(y+2i)^{2n+m+2}}dz\\
J_{n,m,3} =&(-1)^{m+1}(2\sqrt{2})^{4n+2}\varepsilon^{4n+2}\int_{-\infty}^{+\infty}\frac{y(z)e^{-\frac{mi}{24\varepsilon^3}z}}{(y-2i)^{2n-m+3}(y+2i)^{2n+m+3}}dz.
\end{aligned}
\tag{61}
$$

Proof. First we work on

$$I_{n,m,5}=\varepsilon^{4n}\int_{-\infty}^{+\infty}a^{4n+5}e^{im\psi}d\tau.$$

Step 1. From τ to x: We let $x=e^{\tau}$ to obtain

$$I_{n,m,5} = \varepsilon^{4n}\int_0^{+\infty}\frac{(2\sqrt{2})^{4n+5}e^{2mi\tan^{-1}\frac{x-x^{-1}}{2}}e^{-\frac{mi}{48\varepsilon^3}\left((x^3-x^{-3})+9(x-x^{-1})\right)}}{x(x+x^{-1})^{4n+5}}dx.$$

Step 2. From x to y: Next we let

$$y=x-x^{-1}.$$

We have

$$dy=(1+x^{-2})dx=(x+x^{-1})x^{-1}dx$$

We also have

$$y^2=x^2+x^{-2}-2=(x+x^{-1})^2-4;\qquad (x+x^{-1})^2=y^2+4$$

and

$$x^3-x^{-3}=(x-x^{-1})(x^2+x^{-2}+1)=y(y^2+3).$$

This is to imply

$$
\begin{aligned}
I_{n,m,5} &= \varepsilon^{4n}(2\sqrt{2})^{4n+5}\int_{-\infty}^{+\infty}\frac{e^{2mi\tan^{-1}\frac{y}{2}}e^{-\frac{mi}{48\varepsilon^3}y(y^2+12)}}{(y^2+4)^{2n+2}}dy\\
&= \varepsilon^{4n}(2\sqrt{2})^{4n+5}\int_{-\infty}^{+\infty}\frac{\left(e^{i\tan^{-1}\frac{y}{2}}\right)^{2m}e^{-\frac{mi}{48\varepsilon^3}y(y^2+12)}}{(y^2+4)^{2n+2}}dy\\
&= \varepsilon^{4n}(2\sqrt{2})^{4n+5}\int_{-\infty}^{+\infty}\frac{\left(\cos\tan^{-1}\frac{y}{2}+i\sin\tan^{-1}\frac{y}{2}\right)^{2m}e^{-\frac{mi}{48\varepsilon^3}y(y^2+12)}}{(y^2+4)^{2n+2}}dy
\end{aligned}
$$

To continue, we have

$$
\begin{aligned}
I_{n,m,5} &= \varepsilon^{4n}(2\sqrt{2})^{4n+5}\int_{-\infty}^{+\infty}\frac{(2+iy)^{2m}e^{\frac{-im}{48\varepsilon^3}y(y^2+12)}}{(y^2+4)^{2(n+1)+m)}}dy\\
&= \varepsilon^{4n}(-1)^m(2\sqrt{2})^{4n+5}\int_{-\infty}^{+\infty}\frac{e^{\frac{-im}{48\varepsilon^3}y(y^2+12)}}{(y-2i)^{2(n+1)-m}(y+2i)^{2(n+1)+m}}dy.
\end{aligned}
$$

Step 3. From y to z: Finally, we let z be such that

$$2z = y^3 + 12y. \tag{62}$$

We have,

$$dz = 3(y^2+4)dy,$$

and in reverse we let

$$y(z) = (\sqrt{z^2+4^3}+z)^{1/3} - (\sqrt{z^2+4^3}-z)^{1/3}. \tag{63}$$

We note that as a change of real variable to real variable, the maps $y(z):\mathbb{R}\to\mathbb{R}$ is 1-1 and onto. It then follows that we can write $I_{n,m,5}$ as

$$I_{n,m,5} \;=\; (-1)^m \frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}\int_{-\infty}^{+\infty} f(z)e^{-\frac{imz}{24\varepsilon^3}}dz$$

where

$$f(z) = \frac{1}{(y(z)-2i)^{2n-m+3}(y(z)+2i)^{2n+m+3}}. \tag{64}$$

The other three are obtained by using the same sequence of change of variables. □

3.3.2. *Upper Bound Estimation.* We treat z as a complex variable and $y(z)$ as a complex function. Recall that $y(z)$ is such that

$$2z = y^3 + 12y.$$

Sublemma 3.1. *We have (i) the only complex solution for $y(z)+2i=0$ is $z=-8i$ and the only complex solution for $y(z)-2i=0$ is $z=8i$; (ii) the function $Y(z) := (y(z)+2i)^2$ is analytic at $z=-8i$, and $h'(-8i)=\frac{1}{6}i$.*

Proof. For (i), we write $y(z)+2i=0$ explicitly as

$$(\sqrt{z^2+4^3}+z)^{1/3} - (\sqrt{z^2+4^3}-z)^{1/3} + 2i = 0$$

Let $X = (\sqrt{z^2+4^3}+z)^{1/3}$, we have

$$X - \frac{4}{X} + 2i = 0, \quad X^2 + 2iX - 4 = 0, \quad X = -i \pm \sqrt{3}.$$

This is to have

$$\sqrt{z^2+4^3}+z = (-i\pm\sqrt{3})^3 = +i - 9i \mp 3\sqrt{3} \pm 3\sqrt{3} = -8i.$$

It then follows that $z=8i$. Similarly, $z=-8i$ is the only solution of the equation $y(z)+2i=0$. Equation $y(z)-2i=0$ is solved the same way.

For item (ii) we start from

$$\frac{dy(z)}{dz} = \frac{1}{3(y-2i)(y+2i)}$$

to obtain

$$\frac{d(y(z)+2i)^2}{dz} = \frac{2}{3(y(z)-2i)}.$$

We have, at $z=-8i$,

$$\left.\frac{d(y(z)+2i)^2}{dz}\right|_{z=-8i} = \frac{1}{6}i.$$

□

Sublemma 3.1 implies that that the integral functions for $I_{n,m,5}$, $J_{n,m,3}$ take $z=-8i$ as poles of order $n+k+1$ if $m=2k-1$. For $m=2k$, $z=-8i$ is no longer a pole, but a branch point for the integral functions due to the appearance of a copy of $(z+8i)^{-1/2}$. This dependency on m is switched for $I_{n,m,3}$, $J_{n,m,1}$: $z=-8i$ is a pole if m is even, but for odd m, $z=-8i$ is no longer a pole, but a branch point. In the case of poles, the corresponding integrals can be evaluated in precise terms by use of the residue theorem. In the case of branch singularity, it would be less direct if we insist on precise evaluation. Our task here, fortunately, is easier because, instead of precise evaluation, it suffices for us to identify a dominating term, which we need to evaluate in precise terms, and an upper bound estimation for the rest.

Lemma 3.1. *We have*

$$\begin{aligned}
|I_{n,m,3}| &\leq K^n\varepsilon^n\left(\frac{1}{\varepsilon^{\frac{3}{2}}}e^{-\frac{1}{3\varepsilon^3}}\right)^m; \quad |I_{n,m,5}| \leq K^n\varepsilon^{n-\frac{3}{2}}\left(\frac{1}{\varepsilon^{\frac{3}{2}}}e^{-\frac{1}{3\varepsilon^3}}\right)^m;\\
|J_{n,m,1}| &\leq K^n\varepsilon^n\left(\frac{1}{\varepsilon^{\frac{3}{2}}}e^{-\frac{1}{3\varepsilon^3}}\right)^m; \quad |J_{n,m,3}| \leq K^n\varepsilon^{n+\frac{1}{2}}\left(\frac{1}{\varepsilon^{\frac{3}{2}}}e^{-\frac{1}{3\varepsilon^3}}\right)^m.
\end{aligned}$$

Proof. In what follows, we let L be a piece-wise smooth curve in the complex z-plane and let

$$I^L_{n,m,3}=(-1)^m\frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}\int_L\frac{1}{(y(z)-2i)^{2n-m+2}(y(z)+2i)^{2n+m+2}}e^{-\frac{imz}{24\varepsilon^3}}dz.$$

Let

$$L_\varepsilon=\{z=t+i(-8+\varepsilon^3): \quad t\in(-\infty,+\infty)\}.$$

We first estimate $I_{n,m,3}$. By Cauchy integral theorem,

$$I_{n,m,3}=-I^{L_\varepsilon}_{n,m,3}.$$

We have

$$\begin{aligned}
|I_{n,m,3}| &= \left|\int_L\frac{\frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}e^{-\frac{im(t+i(-8+\varepsilon^3))}{24\varepsilon^3}}}{(y(t+i(-8+\varepsilon^3))-2i)^{2n-m+2}(y(t+i(-8+\varepsilon^3))+2i)^{2n+m+2}}dt\right|\\
&\leq e^{\frac{m(-8+\varepsilon^3)}{24\varepsilon^3}}\int_L\frac{\frac{2}{3}(2\sqrt{2})^{4n+5}\varepsilon^{4n}}{|y(t+i(-8+\varepsilon^3))-2i|^{2n-m+2}|y(t+i(-8+\varepsilon^3))+2i|^{2n+m+2}}dt.
\end{aligned}$$

It follows from Sublemma 3.1(ii)

$$\begin{aligned}
|y(t+i(-8+\varepsilon^3))-2i|^{2n-m+2} &\geq K^{-2n-m+2}(|t|+1)^{n-m/2+1}\\
|y(t+i(-8+\varepsilon^3))+2i|^{2n+m+2} &\geq K^{-2n+m+2}(|t|+\varepsilon^3)^{n+m/2+1}.
\end{aligned}$$

We have, in conclusion,

$$|I_{n,m,3}|\leq K^n\varepsilon^n\left(\frac{1}{\varepsilon^{\frac{3}{2}}}e^{-\frac{1}{3\varepsilon^3}}\right)^m.$$

All other integrals are estimated the same way. □

3.3.3. *Lower Bound Estimation.* In this paragraph, we first evaluate $I_{1,1,5}$. By definition,

$$I_{1,1,5} = -\frac{2}{3}(2\sqrt{2})^9\varepsilon^4\int_{-\infty}^{+\infty}\frac{e^{-\frac{iz}{24\varepsilon^3}}}{(y(z)-2i)^4(y(z)+2i)^6}dz \tag{65}$$

We let $w = z + 8i$ to have

$$\begin{aligned} I_{1,1,5} &= -\frac{2\pi i(2\sqrt{2})^9}{3}\varepsilon^4 e^{-\frac{1}{3\varepsilon^3}}\ \frac{d^2}{dw^2}\left(\frac{w^3 e^{-\frac{iw}{24\varepsilon^3}}}{(y(w-8i)-2i)^4(y(w-8i)+2i)^6}\right)\Bigg|_{w=0} \\ &= -\frac{\pi 2^{11}\sqrt{2}}{3^6}\left(\frac{1}{\varepsilon^2}e^{-\frac{1}{3\varepsilon^3}}\right)\left(1+O(\varepsilon^3)\right). \end{aligned}$$

Compare to the upper bound estimation of Lemma 3.1, it follows that $I_{1,1,5}$ dominates, in magnitude, all $I_{n,m,3}, I_{n,m,5}, J_{n,m,1}, J_{n,m,3}$. Here we have two different cases: The first is the case of $m \geq 2$. In this case, increasing m by one would induce a copy of exponentially small factor $\sim e^{\frac{1}{24\varepsilon^3}}$ to be multiplied to the upper bound estimation. The second is the case of $m = 1$. In this case, the difference in magnitude of these integrals are not caused by the exponentially small factors, but by the order of the poles at $z = -8i$.

Proof of Proposition 3.1: The Fourier expansion of $\mathbb{D}_0(\theta_0)$ is a sine series so that we have $\mathbb{D}_0(0) = 0$. We also have

$$|\partial_{\theta_0}\mathbb{D}_0(0)| \geq K^{-1}|I_{1,1,5}| \geq K^{-1}\left(\frac{1}{\varepsilon^2}e^{-\frac{1}{3\varepsilon^3}}\right)\left(1+O(\varepsilon^{\frac{1}{2}})\right).$$

What is claimed in Proposition 3.1 follows directly from this estimation and the fact that $\mathbb{D}(\theta_0, \rho, \varepsilon)$ is real analytic at $\rho = 0$. □

Proof of Theorem 1: Directly follows from Proposition 3.1 and 2.1. □

References

[1] I. Baldomá, E. Fontich, M. Guardia and T. M. Seara *Exponentially small splitting of separatrices beyond Melnikov analysis: Rigorous results*, Journal of Differential Equations, 253 (12): 3304-3439 (2012)

[2] F. Chen and Q. Wang, *High Order Melnikov Method: Theory and Applications*, Journal of Differential Equations, 267 (2) (2019), 1095-1128.

[3] D. Florin and P. Holmes, *Celestial encounters: the origins of chaos and stability*, Princeton University Press, Princeton, 1999.

[4] M. Guardia, P. Martin and T. M. Seara *Oscillatory motions for the restricted planar circular three body problem*, Invent. Math. 203 (2016), 417-492.

[5] J. Llibre and C. Simo, *Oscillatory solutions in the planar restricted three-body problem*, Math. Ann. 248 (1980), 153-184.

[6] R. McGehee, *A stable manifold theorem for degenerate fixed points with applications to celestial mechanics*, J. Differential Equations 14 (1) (1973), 70-88

[7] R. McGehee, *Triple collision in collinear three-body problem.* Inventione Mathematicae, **27** (1974), 191-227.

[8] V.K. Melnikov, *On the stability of the center for time periodic perturbations*, Trans. Moscow Math. Soc., **12** (1963), 1-57.

[9] H. Poincaré, *Mémoire sur les courbes définies par une équation différetielle*, Journal de Mathematiques, **7** (1881), 375-422, and **8** (1882),251-296.

[10] H. Poincaré, *Sur le probléme des trios corps et les équations de la dynamique*, Acta Mathematica, **13** (1890), 1-270.

[11] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, 3 Vols. Gauthier-Villars: Paris (1899)

[12] A. Wintner *The Analytic Foundations of Celestial Mechanics*, Princeton University Press, (1941).

[13] Z. Xia; *Melnikov method and transversal homoclinic points in the restricted three-body problem*, J. Differ. Equ. 96(1), (1992) 170–184

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