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Exponentially small splitting: A direct approach

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Abstract

In this paper, we go beyond what was proposed in theory by Melnikov ([15]) to introduce a practical method to calculate the high order splitting distances of stable and unstable manifold in time-periodic equations. Not only we derive integral formula for splitting distances of all orders, but also we develop an analytic theory to evaluate the acquired multiple integrals. We reveal that the dominance of the exponentially small Poincaré/Melnikov function for equations of high frequency perturbation is caused by a certain symmetry embedded in the kernel functions of high order Melnikov integrals. This symmetry is beheld by many non-Hamiltonian equations.

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1. Introduction

The modern theory of chaos and dynamical systems was originated from the work of Henri Poincaré ([16]), particularly from his discovery of homoclinic tangles. Poincaré first observed that a transversal homoclinic intersection of the stable and unstable manifold of a saddle fixed point in a time-periodic equation induces into existence complicated dynamical structures, which he named as *homoclinic tangles*. He then tackled the problem of how to prove the existence of homoclinic tangles in a given time-periodic equation ([17], Chapter XXI, Volume II). He worked on an equation of a periodically perturbed pendulum, expanding the splitting distance $D(t_0, \varepsilon)$ into a formal power series in ε as

$$D(t_0, \varepsilon) = D_0(t_0) + \varepsilon D_1(t_0) + \cdots + \varepsilon^n D_n(t_0) + \cdots.$$

Poincaré evaluated $D_0(t_0)$ explicitly for his example to conclude that for all sufficiently small $\varepsilon \neq 0$, $D(t_0, \varepsilon) = 0$ admits a non-degenerate solution representing a point of transversal homoclinic intersection.

Poincaré's discovery of homoclinic tangle in time-periodic equations induced many important follow ups in history, including the work of Cartwright and Littlewood ([4]) on time-periodic second order equations, Levinson ([12]) on the Van der Pol equation, Alekseev and Sitnikov ([1], [19]) on the three-body problem. These studies, in turn, led to the Birkhoff/Smale theorem ([2] [14]), asserting the existence of a horseshoe in all homoclinic tangles. Melnikov ([15]) then generalized the computational method Poincaré employed on $D_0(t_0)$ to apply Birkhoff/Smale

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theorem to other time-periodic equations. Since then, the Poincaré/Melnikov method has been used in numerous occasions to prove the existence of homoclinic tangles in time-periodic equations ([9]).

Poincaré/Melnikov method is based on the derivation of an explicit integral formula for $D_0(t_0)$. After obtaining the integral for $D_0(t_0)$, Melnikov asked if we can also compute $D_1(t_0)$, $D_2(t_0)$ and so on, and he gave an affirmative answer to this question in theory in [15] by proposing an inductive scheme to calculate $D_n(t_0)$ for all n in ascending orders of ε. However, an affirmative answer in theory is not quite the same as a practical method. In the last fifty-five years, the inductive scheme proposed by Melnikov on D_n, $n \geq 1$, was not used to study time-periodic equations. As a matter of fact, we are not aware of any previous result on D_1, let alone on D_n for all n, for time-periodic equations in the literature ([5]).

The purpose of this paper is to go beyond what was proposed by Melnikov to develop a practical method to calculate $D_n(t_0)$ for $n \geq 1$. We first introduce a new recursive scheme to derive integral formula for $D_n(t_0)$ for all $n \geq 1$. We then apply this new method for $D_n(t_0)$ to a Duffing equation subjected to a high frequency non-Hamiltonian perturbation: we develop an analytic theory to evaluate $D_n(t_0)$ for all n to prove the existence of homoclinic tangles for all $0 < \varepsilon < \omega^{-k_0}$ where ω is the forcing frequency.

We note that Poincaré/Melnikov method can not be used to prove the existence of homoclinic tangles as stated just above. The mathematical challenge we face here is on the dominance of $D_0(t_0)$ over the rest of the power series expansions of the splitting distance $D(t_0, \varepsilon)$ in equations of high frequency perturbation. For equations of high frequency perturbation, $D_0(t_0)$ is exponentially small in magnitude. It can not be used as *a priori* to control the rest of the power series expansion of the splitting distance $D(t_0, \varepsilon)$. In order to verify that $D_0(t_0)$ remains in control, we need to prove that $\varepsilon^n D_n(t_0)$ are also exponentially small in magnitude for all $n \geq 1$. This problem was first raised by Holmes some forty five years ago ([18], [9]).

In this paper, we prove the dominance of $D_0(t_0)$ by evaluating $D_n(t_0)$ for all $n \geq 1$. To derive explicit integrals formula for D_n, and to develop an analytic theory to evaluate these integrals are what this paper is all about. We reveal that exponentially small splitting is a phenomenon induced by a certain symmetry embedded in the kernel functions of high order Melnikov integrals.

For Hamiltonian equations, there has been a theory on the dominance of $D_0(t_0)$ developed by a number of authors in a string of long and technically involved papers ([6], [7], [8], [11], [13], [10], [3], etc.). The equation we study in this paper, however, is not Hamiltonian, and the method we use is entirely different from the ones adopted in these studies on Hamiltonian equations.

1.1. Statement of results

In this paper, we study the time-periodic second order equation

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon \cos \omega t \cdot y^2. \tag{1.1}$$

We prove that there exist a constant $k_0 > 0$ and an $\omega_0 > 0$, so that for all $\omega > \omega_0$ and all $\varepsilon \in (0, \omega^{-k_0})$, the saddle fixed point $(x, y) = (0, 0)$ of equation (1.1) has a homoclinic solution over which the stable and unstable manifold intersect transversally to induce homoclinic tangles.

Let

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}. \tag{1.2}$$

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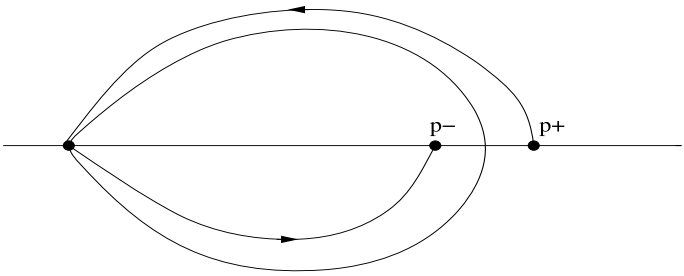

Fig. 1. Splitting Distance D.

It is easy to verify that $(a(t), b(t))$ is a homoclinic solution of the saddle fixed point $(0, 0)$ of the unperturbed equation

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3. \tag{1.3}$$

Let $\ell = \{(a(t), b(t)),\ t \in (-\infty, +\infty)\} \cup (0, 0)$ be the unperturbed homoclinic loop and D_ℓ be a small neighborhood of ℓ in phase space. Let t_0 be the initial time and $(x^s(t, \varepsilon, \omega), y^s(t, \varepsilon, \omega)), t \geq t_0$ be the stable solution of $(0, 0)$ satisfying $y^s(t_0, \varepsilon, \omega) = 0$ in $\mathcal{D}_\ell$. In parallel, let $(x^u(t, \varepsilon, \omega), y^u(t, \varepsilon, \omega)), t \leq t_0$ be the unstable solution satisfying $y^u(t_0, \varepsilon, \omega) = 0$ in $\mathcal{D}_\ell$. The quantity

$$D(t_0, \varepsilon, \omega) = \varepsilon^{-1}\left(x^s(t_0, \varepsilon, \omega) - x^u(t_0, \varepsilon, \omega)\right)$$

is the **splitting distance** that measures the relative positions of the stable and unstable manifold. See Fig. 1.

We start with

Proposition 1.1. *There exists a constant $\varepsilon_0 > 0$, independent of ω, so that for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and all $\omega \in (-\infty, +\infty)$, the splitting distance $D(t_0, \omega, \varepsilon)$ of equation (1.1) can be expanded as a convergent power series of ε, which we write as*

$$D(t_0, \omega, \varepsilon) = \varepsilon D_0(t_0, \omega) + \varepsilon^2 D_1(t_0, \omega) + \cdots + \varepsilon^{n+1} D_n(t_0, \omega) + \cdots. \tag{1.4}$$

Proposition 1.1 is proved at the end of Part 1 of this paper.

The classical Poincaré/Melnnikov method is based on an explicit integral formula on D_0. To apply this integral formula to equation (1.1), we obtain

$$D_0(t_0, \omega) = \frac{\sqrt{2}\pi}{30}\omega^5 e^{-\omega\pi/2}\left[1 + O(\omega^{-1})\right]\sin\omega t_0. \tag{1.5}$$

Note that $D_0(t_0, \omega)$ is exponentially small in ω. It fails to dominate $\varepsilon^n D_n(t_0, \omega)$ *a priori* when ω is large and $\varepsilon \approx \omega^{-k_0}$. In order to argue that $D_0(t_0, \omega)$ remains dominant, we move beyond $D_0(t_0, \omega)$ to evaluate $D_n(t_0, \omega)$ to acquire an upper bound that is also exponentially small for all $n \geq 1$. The main result of this paper is as follows.

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Main Theorem. *Let $D(t_0, \omega, \varepsilon)$ be the splitting distance of the equation (1.1) and $D_n(t_0, \omega)$ be as in (1.4). We have*

(a) *for all $n \geq 0$,*

$$D_n(t_0, \omega) = \sum_{k=1}^{4(n+1)} A_{k,n}(\omega) \sin k\omega t_0;$$

(b) *there exist positive constants κ_0 and ω_0, so that for all $\omega > \omega_0$,*

$$|A_{k,n}(\omega)| < \omega^{\kappa_0(n+1)} e^{-\omega\pi/2}$$

for all $A_{k,n}$ in (a).

Combining $D_0(t_0, \omega)$ in (1.5) with the Main Theorem (b), we conclude that $D_0(t_0, \omega)$ remains the dominating term over the rest of the power series expansion of $D(t_0, \omega, \varepsilon)$ assuming $|\varepsilon| < \omega^{-k_0}$ with $k_0 = 3\kappa_0$. The existence of the homoclinic tangles as claimed in the opening paragraph of this subsection then follows from a simply application of the implicit function theorem.

1.2. High order Melnikov integrals

The theory we develop in this paper has two main components. The first component of this theory is to introduce a method to derive integral formulas for D_n for all $n \geq 0$. We conclude that D_n for an arbitrarily $n > 0$ is a finite collection of well structured multiple integrals, which we name as *high order Melnikov integrals*. The multiplicity of the integrals for D_n are $\geq n$ but $\leq 4(n+1)$. The second component of this theory is for us to manipulate high order Melnikov integrals to prove Main Theorem (b).

To define a high order Melnikov integral N_p^s, we start with a *structure tree*. A structure tree is originated from a *root node*, from which some direct descendants are branched out. Each of these descendant nodes then serves as a root node for a subtree, from which the second generation descendants are branched out, and so on. Eventually the branching out stops, and we obtain a structure tree. See Fig. 2 for an example of a structure tree.

Assume a given structure tree has p nodes in total. We index the tree nodes from the bottom level to the top level, and at a fixed level, from the right to the left, as $N_1, \cdots, N_p$. The root node of the entire tree is then indexed as N_p. There are two types of nodes in a structure tree: the M-type and the W-type. To each node N_j we also assign (i) an integral variable t_j; (ii) a kernal function $f_j(t_j, t_0)$; (iii) an interval of integration I_j for t_j.

Details on item (ii): The kernel functions $f_j(t_j, t_0)$ is in the form of

$$f_j(t_j, t_0) = (\cos\omega(t_j + t_0))^{n_0(j)} d_j(t_j)$$

where $n_0(j)$ is either 0 or 1. We would need more technical preparations to write $d_j(t_j)$ explicitly for equation (1.1), but the symmetry that induces exponentially small splitting is embedded in $d_j(t_j)$: it is associated to the fact that, for equation (1.1), *all $d_j(t_j)$ are odd function of t_j*.

Details on item (iii): We have $I_p = (0, +\infty)$ for N_p. For $j < p$, let j' be such that N_j is directly branched out of $N_{j'}$. We have $I_j = (t_{j'}, +\infty)$ if N_j is an M-node, but $I_j = (0, t_{j'})$ if N_j is a W-node.

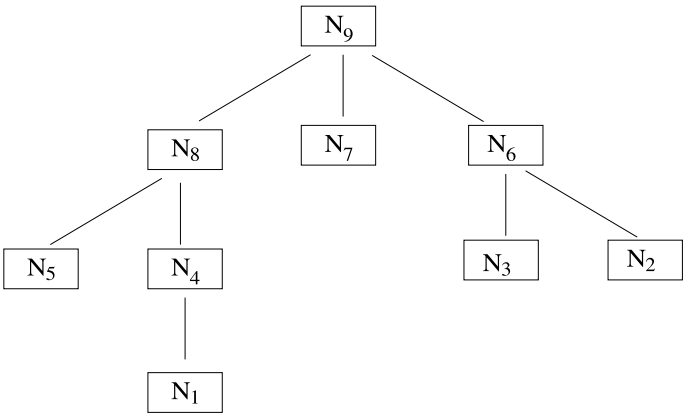

Fig. 2. An Example of a Structure Tree.

We define a *high order Melnikov integral* N_p^s by letting

$$N_p^s = \int_{I_p} f_p(t_p, t_0) \left(\cdots \left(\int_{I_j} f_j(t_j, t_0) \left(\cdots \left(\int_{I_1} f_1(t_1, t_0) dt_1 \right) \cdots \right) dt_j \right) \cdots \right) dt_p.$$

The dual N_p^u for a given N_p^s is obtained by changing I_j for all M-nodes from $(t_{j'}, +\infty)$ to $(t_{j'}, -\infty)$.

This paper is divided into three parts. In Part 1, we prove that, for all $n \geq 0$, there is a collection Λ_n of high order Melnikov integrals N_p^s so that

$$D_n(t_0, \omega) = \sum_{N_p^s \in \Lambda_n} d_{N_p}(t_0, \omega)$$

where

$$d_{N_p}(t_0, \omega) = N_p^s(t_0, \omega) - N_p^u(t_0, \omega).$$

To control the summation for $D_n(t_0, \omega)$, we also prove that the number of high order Melnikov integrals in Λ_n is $\leq K^n$, and $p \leq 4(n+1)$ for all $N_p^s \in \Lambda_n$.

Part 2 of this paper is then devoted to extract an exponentially small factor out of one $d_{N_p}(t_0, \omega)$ assuming *all* $d_j(t_j)$ *are odd functions*. We prove the Main Theorem in Sect. 6.2. Three rather involved combinatoric proofs, for Propositions 3.1, 6.1 and 6.2 respectively, are placed in Part 3.

With this new theory we also conclude that exponentially small splitting as a dynamical phenomenon is not exclusively tied to Hamiltonian equations. It is rather induced by a symmetry embedded in the kernel functions of high order Melnikov integrals. This symmetry beholds, for instances, by equations in the form of

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \varepsilon \cos \omega t \cdot P(x, y)$$

where $P(x, y)$ are such that

$$P(x, y) = P(x, -y).$$

In this paper, we elect to work with a specific case by letting $P(x, y) = y^2$ to better illustrate the new ideas and the new method of calculating $D_n(t_0)$ for all n.

Notation. Throughout this paper, the letter K is reserved for a *generic constant* that is independent of $t, t_0, \omega, \varepsilon$ and k, n. The value K represents can vary from place to place.

Part 1. High order Melnikov integrals

2. Recursion on primary stable solutions

In this section, we introduce a direct recursion to derive integral formula for $D_n(t_0, \omega)$ for all n. In Sect. 2.1, we introduce new variables to solve the equation of first variations of the unperturbed equation around $(a(t), b(t))$. In between the two new variables m and w introduced in Sect. 2.1, m is the classical Melnikov variable, but w is new.

A correspondence of the new variables is then applied to the perturbed equation (1.1) in Sect. 2.2, and the resulted differential equations are converted to integral equations for stable solutions in Sect. 2.3. The final change of variables introduced in Sect. 2.3 enables us to write the dynamic part of all kernel functions of high order Melnikov integrals as *odd* functions in t. This symmetry is critical for the success of the analysis introduced in Part 2. *It is the ultimate reason why exponentially small splitting occurs for certain periodically perturbed equations*. In Sect. 2.4, we introduce a new recursion to derive D_n for all n by using the integral equations acquired in Sect. 2.3. We present an explicit integral formula for $D_1(t_0)$ at the end of Sect. 2.4.

2.1. On the equation of first variations

Denote

$$a = a(t), \quad a' = a'(t), \quad b = b(t), \quad b' = b'(t).$$

We have

$$a' = b, \qquad b' = a - a^3, \qquad b^2 = a^2 - a^4/2$$

where the last equality reflects the fact that (a, b) is a zero energy solution of the unperturbed equation (1.3).

We start by solving the equation of first variations

$$\frac{d\xi}{dt} = \eta; \qquad \frac{d\eta}{dt} = (1 - 3a^2)\xi. \tag{2.1}$$

This equation is for the unperturbed equation (1.1) around $\ell(t) = (a, b)$. In what follows we let

$$h(t) = 3a^2(t)\int_0^t a^{-2}(\tau)d\tau = \frac{3(e^{2t} - e^{-2t} + 4t)}{2(e^t + e^{-t})^2}; \tag{2.2}$$

and

$$H(t) = \frac{1}{a(t)}\left[b(t)h(t) + a(t)\right]; \qquad \tilde{H}(t) = \frac{1}{a(t)}\left[b'(t)h(t) + 2b(t)\right]. \tag{2.3}$$

Lemma 2.1. *The function $a(t)$, $H(t)$ are even functions, but $b(t)$, $h(t)$ and $\tilde{H}(t)$ are odd functions in t. In addition, $h(t)$, $H(t)$ and $\tilde{H}(t)$ are all uniformly bounded for all $t \in (-\infty, +\infty)$.*

Proof. All are easily verifiable facts. □

We also denote

$$h = h(t),\ H = H(t),\ \tilde{H} = \tilde{H}(t).$$

Lemma 2.2. *Let m, w be such that*

$$\xi = \frac{1}{a}(bw - aHm); \qquad \eta = \frac{1}{a}\left(b'w - a\tilde{H}m\right). \tag{2.4}$$

The equation of first variations (2.1) is transformed in new variables (m, w) to

$$\frac{dm}{dt} = -\frac{b}{a}m; \qquad \frac{dw}{dt} = \frac{b}{a}w. \tag{2.5}$$

Proof. Let z_1, z_2 be such that

$$z_1 = \frac{1}{a}\left(b'\xi - b\eta\right); \qquad z_2 = \frac{1}{a}(2b\xi - a\eta). \tag{2.6}$$

We have by inversion that

$$\xi = \frac{1}{a}(bz_2 - az_1); \qquad \eta = \frac{1}{a}\left(b'z_2 - 2bz_1\right). \tag{2.7}$$

We note that the new variable z_1 is the projection of (ξ, η) onto the direction that is perpendicular to the unperturbed homoclinic solution. This is according to the original design of Melnikov. For z_2 we on purposely avoided to use the projection to the tangential direction because it would induce a dividing factor $b^2 + (b')^2$ when the change of coordinates is inverted. Though $b^2 + (b')^2 = 0$ has no real solution for t, it admits solutions on complex t-plane. Our design of z_2 is such that no new singularity in the complex t plane are added. Note that to invert (2.6) to obtain (2.7) we used

$$2b^2 - ab' = 2a^2 - a^4 - a(a - a^3) = a^2.$$

By this equality we avoided introducing new singularity in the complex t-plane in (2.7).

For z_1 we have

$$\frac{dz_1}{dt} = \frac{1}{a}\left(b''\xi + b'\xi' - a''\eta - a'\eta'\right) - \frac{a'}{a}z_1.$$

Recall that from $b' = a''$ and $\xi' = \eta$, we have $b'\xi' = a''\eta$. Also from $b'' = (1 - 3a^2)a'$ and $\eta' = (1 - 3a^2)\xi$, we have $b''\xi = a'\eta'$. We conclude that

$$\frac{dz_1}{dt} = -\frac{b}{a}z_1.$$

For z_2 we have

$$\begin{aligned}\frac{dz_2}{dt} &= \frac{1}{a}\left[2b'\xi + 2b\xi' - a'\eta - a\eta'\right] - \frac{a'}{a}z_2\\ &= \frac{1}{a}\left[2b'\xi + 2b\eta - a'\eta - a(1-3a^2)\xi\right] - \frac{a'}{a}z_2\\ &= \frac{1}{a^2}\left[(a+a^3)(a'z_2 - az_1) + b(b'z_2 - 2bz_1)\right] - \frac{a'}{a}z_2.\end{aligned}$$

It then follows that

$$\frac{dz_2}{dt} = -3z_1 + \frac{a'}{a}z_2.$$

Let $w, h(t)$ be such that

$$w = h(t)z_1 + z_2.$$

We have

$$\frac{dw}{dt} = \left(h' - \frac{2a'}{a}h - 3\right)z_1 + \frac{a'}{a}w.$$

We use

$$h = 3a^2(t)\int_0^t a^{-2}(\tau)d\tau$$

to obtain

$$h' - \frac{2a'}{a}h - 3 = 0.$$

With this choice of $h(t)$, we have

$$\frac{dw}{dt} = \frac{b}{a}w.$$

We use (2.7) and $z_2 = w - hz_1$, and re-write z_1 as m to obtain

$$\xi = \frac{1}{a}(bw - aHm); \qquad \eta = \frac{1}{a}(b'w - a\tilde{H}m)$$

where

$$H = \frac{1}{a}(bh + a); \qquad \tilde{H} = \frac{1}{a}(b'h + 2b).$$

This is (2.4) with (2.3). We also have

$$\frac{dm}{dt} = -\frac{b}{a}m; \qquad \frac{dw}{dt} = \frac{b}{a}w.$$

This is (2.5). □

2.2. *Differential equation for stable solutions*

Let t_0 be a given initial time, and $(\hat{x}(t), \hat{y}(t))$ be the stable solution of the perturbed equation satisfying $(\hat{x}(t_0), \hat{y}(t_0)) = (x_0, y_0)$. Let $(x(t), y(t)) = (\hat{x}(t + t_0), \hat{y}(t + t_0))$. Then $(x(t), y(t))$ is well-defined on $t \in [0, +\infty)$ satisfying

$$\frac{dx}{dt} = y; \qquad \frac{dy}{dt} = x - x^3 + \varepsilon \cos\omega(t + t_0)y^2$$

and $(x(0), y(0)) = (x_0, y_0)$. Let

$$X = x - a(t); \qquad Y = y - b(t). \tag{2.8}$$

We have

$$\frac{dX}{dt} = Y; \qquad \frac{dY}{dt} = (1 - 3a^2)X + Q(t, X) + \varepsilon \cos\omega(t + t_0)(Y + b)^2 \tag{2.9}$$

where

$$Q(t, X) = -3aX^2 - X^3.$$

Lemma 2.3. *Let M, W be such that*

$$X = \frac{1}{a}(bW - aHM); \qquad Y = \frac{1}{a}\left(b'W - a\tilde{H}M\right) \tag{2.10}$$

where h and $H, \tilde{H}$ are as in (2.2) and (2.3). Equation (2.9) is transformed in new variables (M, W) to

$$\begin{aligned} \frac{dM}{dt} &= -\frac{b}{a}M - \frac{b}{a}Q(t, X) - \varepsilon\frac{b}{a}\cos\omega(t + t_0)(Y + b)^2; \\ \frac{dW}{dt} &= \frac{b}{a}W - HQ(t, X) - \varepsilon H\cos\omega(t + t_0)(Y + b)^2 \end{aligned} \tag{2.11}$$

where

$$Q(t, X) = -3aX^2 - X^3. \tag{2.12}$$

Proof. Let

$$Z_1 = \frac{1}{a}\left(b'X - bY\right); \qquad Z_2 = \frac{1}{a}\left(2bX - aY\right). \tag{2.13}$$

We have

$$X = \frac{1}{a}(bZ_2 - aZ_1); \qquad Y = \frac{1}{a}(b'Z_2 - 2bZ_1). \tag{2.14}$$

For Z_1, we have

$$\begin{aligned}
\frac{dZ_1}{dt} &= \frac{1}{a}\left(b''X + b'X' - b'Y - bY'\right) - \frac{b}{a}Z_1 \\
&= \frac{1}{a}\left\{(1-3a^2)a'X - b\left[(1-3a^2)X + Q(t,X)\right]\right\} \\
&\quad -\varepsilon\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2 - \frac{b}{a}Z_1 \\
&= -\frac{b}{a}Z_1 - \frac{b}{a}Q(t,X) - \varepsilon\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2.
\end{aligned}$$

For Z_2, we have

$$\begin{aligned}
\frac{dZ_2}{dt} &= \frac{1}{a}[2b'X + 2bX' - a'Y - aY'] - \frac{b}{a}Z_2 \\
&= \frac{1}{a}\left\{2b'X + bY - a\left[(1-3a^2)X + Q(t,X)\right]\right\} \\
&\quad -\varepsilon\cos\omega(t+t_0)(Y+b)^2 - \frac{b}{a}Z_2 \\
&= \frac{1}{a^2}\left\{\left[2b' - a(1-3a^2)\right](a'Z_2 - aZ_1) + b(b'Z_2 - 2bZ_1)\right\} \\
&\quad -Q(t,X) - \varepsilon\cos\omega(t+t_0)(Y+b)^2 - \frac{b}{a}Z_2.
\end{aligned}$$

To continue, we obtain

$$\frac{dZ_2}{dt} = -3Z_1 + \frac{b}{a}Z_2 - Q(t,X) - \varepsilon\cos\omega(t+t_0)(Y+b)^2.$$

In summary,

$$\begin{aligned}
\frac{dZ_1}{dt} &= -\frac{b}{a}Z_1 - \frac{b}{a}Q(t,X) - \varepsilon\cdot\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2; \\
\frac{dZ_2}{dt} &= -3Z_1 + \frac{b}{a}Z_2 - Q(t,X) - \varepsilon\cos\omega(t+t_0)(Y+b)^2.
\end{aligned}$$

We again let

$$W = h(t)Z_1 + Z_2$$

to obtain

$$\frac{dW}{dt} = h'Z_1 + h\left[-\frac{b}{a}Z_1 - \frac{b}{a}Q(t,X) - \varepsilon\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2\right]$$
$$+\left[-3Z_1 + \frac{b}{a}(W - hZ_1) - Q(t,X) - \varepsilon\cos\omega(t+t_0)(Y+b)^2\right].$$

Recall that $h = h(t)$ is such that

$$h' - \frac{2b}{a}h - 3 = 0,$$

and

$$H = h\frac{b}{a} + 1.$$

We have

$$\frac{dW}{dt} = \frac{b}{a}W - HQ(t,X) - \varepsilon H\cos\omega(t+t_0)(Y+b)^2.$$

Altogether, we have

$$\frac{dZ_1}{dt} = -\frac{b}{a}Z_1 - \frac{b}{a}Q(t,X) - \varepsilon\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2;$$
$$\frac{dW}{dt} = \frac{b}{a}W - HQ(t,X) - \varepsilon H\cos\omega(t+t_0)(Y+b)^2$$

where

$$X = \frac{1}{a}\left[b(W - hZ_1) - aZ_1\right] = \frac{1}{a}\left(bW - aHZ_1\right),$$
$$Y = \frac{1}{a}\left[b'(W - hZ_1) - 2bZ_1\right] = \frac{1}{a}\left(b'W - a\tilde{H}Z_1\right).$$

Finally, we write Z_1 as M to obtain

$$\frac{dM}{dt} = -\frac{b}{a}M - \frac{b}{a}Q(t,X) - \varepsilon\frac{b}{a}\cos\omega(t+t_0)(Y+b)^2;$$
$$\frac{dW}{dt} = \frac{b}{a}W - HQ(t,X) - \varepsilon H\cos\omega(t+t_0)(Y+b)^2$$

where

$$Q(t, X) = -3aX^2 - X^3;$$

and

$$X = \frac{1}{a}(bW - aHM), \qquad Y = \frac{1}{a}\left[b'W - a\tilde{H}M\right].$$

This finishes the proof of Lemma 2.3. □

2.3. Integral equation for stable solutions

We introduce one more change of variables by letting

$$\mathbb{M} = \frac{a}{\varepsilon}M; \qquad \mathbb{W} = \frac{\sqrt{(2-a^2)}}{\varepsilon b}W. \tag{2.15}$$

In between the two new variables, $\mathbb{M}$ is straight forward to motivate: $1/\varepsilon$ is a rescale and a is multiplied to remove the linear term in the equation for M. In fact $\mathbb{M}$ is the original Melnikov variable. For $\mathbb{W}$, we note that

$$\frac{\sqrt{(2-a^2)}}{\varepsilon b} = \frac{\sqrt{a^2(2-a^2)}}{\varepsilon ba} = \frac{\sqrt{2}|b|}{\varepsilon ba}.$$

We have, for $t > 0$,

$$\mathbb{W} = \frac{\sqrt{2}}{\varepsilon a}W,$$

but for $t < 0$,

$$\mathbb{W} = -\frac{\sqrt{2}}{\varepsilon a}W.$$

This is to say, in computing stable and unstable solutions, we are using a different sign for $\mathbb{W}$. The purpose of $\mathbb{W}$ is to remove the linear term in the equation for W. The reason for us to use different signs for $t > 0$ and $t < 0$ will become clear as we move forward.

We now derive the equation for $\mathbb{M}$ and $\mathbb{W}$. Let

$$\mathbb{X} = \frac{1}{\varepsilon}X, \qquad \mathbb{Y} = \frac{1}{\varepsilon}Y.$$

We have

$$\mathbb{X} = \frac{1}{a}\left(\frac{b^2}{\sqrt{(2-a^2)}}\mathbb{W} - H\mathbb{M}\right), \qquad \mathbb{Y} = \frac{1}{a}\left[\frac{bb'}{\sqrt{(2-a^2)}}\mathbb{W} - \tilde{H}\mathbb{M}\right].$$

First for $\mathbb{M}$, we have

$$\frac{d\mathbb{M}}{dt} = \varepsilon b\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - b\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2.$$

We also have, for $t > 0$,

$$\begin{aligned}\frac{d\mathbb{W}}{dt} &= \frac{\sqrt{2}}{\varepsilon a}\frac{dW}{dt} - \frac{\sqrt{2}b}{\varepsilon a^2}W \\ &= \frac{\sqrt{2}}{a}\left(\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2\right).\end{aligned}$$

We also have for $t < 0$,

$$\begin{aligned}\frac{d\mathbb{W}}{dt} &= -\frac{\sqrt{2}}{\varepsilon a}\frac{dW}{dt} + \frac{\sqrt{2}b}{\varepsilon a^2}W \\ &= -\frac{\sqrt{2}}{a}\left(\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2\right).\end{aligned}$$

This implies that for all t, we have

$$\begin{aligned}\frac{d\mathbb{W}}{dt} &= \frac{\sqrt{2}|b|}{ab}\left(\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2\right) \\ &= \frac{\sqrt{(2-a^2)}}{b}\left(\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2\right)\end{aligned}$$

In summary, we have

$$\begin{aligned}\frac{d\mathbb{M}}{dt} &= \varepsilon b\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - b\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2; \\ \frac{d\mathbb{W}}{dt} &= \frac{2b}{a^2\sqrt{(2-a^2)}}\left[\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(t+t_0)(\varepsilon\mathbb{Y}+b)^2\right].\end{aligned} \tag{2.16}$$

Lemma 2.4. *The functions* $(\mathbb{M}(t), \mathbb{W}(t))$ *for primary stable solutions satisfy the integral equation*

$$\begin{aligned}\mathbb{M}(t) &= -\varepsilon\int_t^{+\infty} b(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3)d\tau + \int_t^{+\infty} b\cos\omega(\tau+t_0)(\varepsilon\mathbb{Y}+b)^2 d\tau; \\ \mathbb{W}(t) &= \int_0^t \frac{2b}{a^2\sqrt{(2-a^2)}}\left[\varepsilon H(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3) - H\cos\omega(\tau+t_0)(\varepsilon\mathbb{Y}+b)^2\right]d\tau\end{aligned} \tag{2.17}$$

where

$$\mathbb{X} = \frac{1}{a}\left[\frac{b^2}{\sqrt{(2-a^2)}}\mathbb{W} - H\mathbb{M}\right], \qquad \mathbb{Y} = \frac{1}{a}\left[\frac{bb'}{\sqrt{(2-a^2)}}\mathbb{W} - \tilde{H}\mathbb{M}\right]. \tag{2.18}$$

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Proof. We use (2.16) to obtain

$$\begin{aligned}\mathbb{M}(t) &= \mathbb{M}(0) + \varepsilon \int_0^t b(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3)d\tau - \int_0^t b\cos\omega(\tau+t_0)(\varepsilon\mathbb{Y}+b)^2 d\tau;\\ \mathbb{W}(t) &= \mathbb{W}(0) + \int_0^t \frac{2b}{a^2\sqrt{(2-a^2)}}\left[\varepsilon H(3a\mathbb{X}^2+\varepsilon\mathbb{X}^3) - H\cos\omega(\tau+t_0)(\varepsilon\mathbb{Y}+b)^2\right]d\tau.\end{aligned} \tag{2.19}$$

Recall that we have

$$Y = \frac{b'}{a}W - \tilde{H}M.$$

So for a primary stable solution,

$$W(0) = \frac{a(0)}{b'(0)}\left(Y(0) + \tilde{H}(0)M(0)\right) = 0.$$

Note that to obtain the last equality we use $Y(0) = 0$ and $\tilde{H}(0) = 0$. We also recall

$$\mathbb{W}(t) = \frac{\sqrt{(2-a^2)}}{\varepsilon b}W.$$

Consequently,

$$\mathbb{W}(0) = \lim_{t\to 0}\frac{\sqrt{2-a^2(t)}}{\varepsilon b(t)}W(t) = 0.$$

The integral equation for $\mathbb{W}$ in (2.17) follows directly from (2.19) for $\mathbb{W}$.

We now compute $\mathbb{M}(0)$. Observe that

$$\mathbb{M}(t) = \varepsilon^{-1}aM(t) = \varepsilon^{-1}a\left(\frac{b'}{a}X - \frac{a'}{a}Y\right).$$

We have for stable solution

$$\lim_{t\to+\infty}(X(t), Y(t)) = (0,0),$$

which implies

$$\mathbb{M}(+\infty) = 0.$$

It then follows that

$$\mathbb{M}(0) = -\varepsilon\int_0^{+\infty} b\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right)d\tau + \int_0^{+\infty} b\cos\omega(\tau+t_0)(\varepsilon\mathbb{Y}+b)^2 d\tau. \tag{2.20}$$

This formula for $\mathbb{M}(0)$ is then put back into (2.19) to obtain (2.17) for $\mathbb{M}$. □

2.4. Recursive derivation of splitting distance

With (2.17) in Lemma 2.4, we are now ready to present the first main result of this paper. Let us write $\mathbb{M}(t) = \mathbb{M}(t, t_0, \omega, \varepsilon)$, $\mathbb{W}(t) = \mathbb{W}(t, t_0, \omega, \varepsilon)$ formally as power series of ε. This is to say,

$$\mathbb{M}(t) = \sum_{n=0}^{+\infty} \varepsilon^n \mathbb{M}_n(t, t_0, \omega); \qquad \mathbb{W}(t) = \sum_{n=0}^{+\infty} \varepsilon^n \mathbb{W}_n(t, t_0, \omega).$$

We can use (2.17) to determine the functions $\mathbb{M}_n = \mathbb{M}_n(t, t_0, \omega)$, $\mathbb{W}_n = \mathbb{W}_n(t, t_0, \omega)$ recursively for all n: First we have, from (2.17),

$$\begin{aligned} \mathbb{M}_0(t, t_0, \omega) &= \int_t^{+\infty} \cos\omega(\tau + t_0) \cdot b^3 d\tau; \\ \mathbb{W}_0(t, t_0, \omega) &= -\int_0^t \cos\omega(\tau + t_0) \cdot \frac{2b^3 H}{a^2\sqrt{(2-a^2)}} d\tau. \end{aligned} \tag{2.21}$$

We then have

$$\begin{aligned} \mathbb{M}_1(t) &= -3\int_t^{+\infty} ba\mathbb{X}_0^2 d\tau + 2\int_t^{+\infty} \cos\omega(\tau + t_0) b^2 \mathbb{Y}_0 d\tau; \\ \mathbb{W}_1(t) &= \int_0^t \frac{2bH}{a^2\sqrt{(2-a^2)}} \left[3a\mathbb{X}_0^2 - 2\cos\omega(\tau + t_0) b\mathbb{Y}_0\right] d\tau \end{aligned} \tag{2.22}$$

where

$$\mathbb{X}_0 = \frac{1}{a}\left[\frac{b^2}{\sqrt{(2-a^2)}}\mathbb{W}_0 - H\mathbb{M}_0\right], \qquad \mathbb{Y}_0 = \frac{1}{a}\left[\frac{bb'}{\sqrt{(2-a^2)}}\mathbb{W}_0 - \tilde{H}\mathbb{M}_0\right]. \tag{2.23}$$

In general, assume that we have obtained $\mathbb{M}_k = \mathbb{M}_k(t, t_0, \omega)$, $\mathbb{W}_k = \mathbb{W}_k(t, t_0, \omega)$ for all $k \le n$. We can solve for $\mathbb{M}_{n+1} = \mathbb{M}_{n+1}(t, t_0, \omega)$, $\mathbb{W}_{n+1} = \mathbb{W}_{n+1}(t, t_0, \omega)$ in terms of $\mathbb{M}_k = \mathbb{M}_k(t, t_0, \omega)$, $\mathbb{W}_k = \mathbb{W}_k(t, t_0, \omega)$, $k \le n$ by using (2.17). This is because on the left hand side the only term of order ε^{n+1} are $\mathbb{M}_{n+1}$ and $\mathbb{W}_{n+1}$, but on the right hand side the terms of order ε^{n+1} only include $\mathbb{M}_k(t, t_0, \omega)$, $\mathbb{W}_k(t, t_0, \omega)$ with $k \le n$. All terms involving $\mathbb{M}_k$, $\mathbb{W}_k$, $k \ge n+1$ on the right hand side would be at least of order ε^{n+2}.

So far, we have derived the integral formula for primary stable solutions. We can compute primary unstable solutions the same way. To distinguish the two, let us denote a stable solution as $(\mathbb{M}^s(t, t_0, \omega, \varepsilon), \mathbb{W}^s(t, t_0, \omega, \varepsilon))$ and an unstable solution as $(\mathbb{M}^u(t, t_0, \omega, \varepsilon), \mathbb{W}^u(t, t_0, \omega, \varepsilon))$. We write

$$\mathbb{M}^s(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{M}_n^s(t, t_0, \omega); \qquad \mathbb{W}^s(t, t_0, \omega, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbb{W}_n^s(t, t_0, \omega);$$

and in dual,

$$\mathbb{M}^u(t,t_0,\omega,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\mathbb{M}_n^u(t,t_0,\omega);\qquad \mathbb{W}^u(t,t_0,\omega,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n\mathbb{W}_n^u(t,t_0,\omega).$$

The splitting distance, by definition, is

$$D(t_0,\omega,\varepsilon)=\mathbb{M}^s(0,t_0,\omega,\varepsilon)-\mathbb{M}^u(0,t_0,\omega,\varepsilon)=\sum_{n=0}^{\infty}\varepsilon^n D_n(t_0,\omega) \tag{2.24}$$

where

$$D_n(t_0,\omega)=\mathbb{M}_n^s(0,t_0,\omega)-\mathbb{M}_n^u(0,t_0,\omega). \tag{2.25}$$

Directly from (2.21), we obtain

$$D_0(t_0,\omega)=\int_{-\infty}^{+\infty}\cos\omega(\tau+t_0)\cdot b^3(\tau)d\tau. \tag{2.26}$$

We also have the following corollary for $D_1(t_0,\omega)$.

Corollary 2.1. *We have*

$$D_1(t_0,\omega)=\mathbb{M}_1^s-\mathbb{M}_1^u$$

where

$$\mathbb{M}_1^s=-3\int_0^{+\infty}ba\left[\mathbb{X}_0^s(\tau)\right]^2d\tau+2\int_0^{+\infty}\cos\omega(\tau+t_0)b^2\mathbb{Y}_0^s(\tau)d\tau;$$

and we substitute (2.21) into (2.23) to obtain

$$\mathbb{X}_0^s(t)=-\frac{1}{a}\left[\frac{b^2}{\sqrt{(2-a^2)}}\int_0^t\cos\omega(\tau+t_0)\cdot\frac{2b^3H}{a^2\sqrt{(2-a^2)}}d\tau+H\int_t^{+\infty}\cos\omega(\tau+t_0)\cdot b^3d\tau\right],$$

$$\mathbb{Y}_0^s(t)=-\frac{1}{a}\left[\frac{bb'}{\sqrt{(2-a^2)}}\int_0^t\cos\omega(\tau+t_0)\cdot\frac{2b^3H}{a^2\sqrt{(2-a^2)}}d\tau+\tilde{H}\int_t^{+\infty}\cos\omega(\tau+t_0)\cdot b^3d\tau\right].$$

In addition, $\mathbb{M}_1^u$, $\mathbb{X}_0^u$ and $\mathbb{Y}_0^u$ are obtained by changing $+\infty$ to $-\infty$ in $\mathbb{M}_1^s$, $\mathbb{X}_0^s$ and $\mathbb{Y}_0^s$ respectively.

3. High order Melnikov integrals

To the high order Melnikov integrals introduced earlier in Sect. 1.2, we add the following refinements in Sects 3.1 and 3.2: (1) the kernel functions $d_j(t_j)$ are made explicit; (2) additional restrictions are imposed on the structure tree; and (3) we define $N_p^s(t, t_0, \omega)$ instead of $N_p^s(t_0, \omega)$ ($N_p^s(t_0, \omega) = N_p^s(0, t_0, \omega)$).

In Sect. 3.1, we define the set of kernel functions in explicit terms. Symmetry acclaimed in Lemma 3.3 is why we have exponentially small splitting as a dynamics phenomenon in time-periodic equations. We redefine high order Melnikov integrals in Sect 3.2 and prove, at the end of Sect. 3.2, that all high order Melnikov integrals are absolutely convergent. The main result of this section is Proposition 3.1 in Sect. 3.3, the proof of which we postpone to Section 7 in Part 3.

3.1. Kernel functions

Let

$$A = A(t) = \frac{1}{2}\sqrt{(2 - a^2(t))}. \tag{3.1}$$

We note that

(i) $A(t)$ is an even function in t,

(ii) $|A(t)| < 1$ for all t, and

(iii) $|b(t)A^{-1}(t)|$ and $|\tilde{H}(t)A^{-1}|$ are uniformly bounded by a constant K as $t \to 0$.

We rewrite (2.17) as

$$\begin{aligned} \mathbb{M}(t) &= -\varepsilon \int_t^{+\infty} b(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3)d\tau + \int_t^{+\infty} b\cos\omega(\tau + t_0)(\varepsilon\mathbb{Y} + b)^2 d\tau; \\ \mathbb{W}(t) &= \int_0^t \frac{b}{a^2 A}\left[\varepsilon H\left(3a\mathbb{X}^2 + \varepsilon\mathbb{X}^3\right) - H\cos\omega(\tau + t_0)(\varepsilon\mathbb{Y} + b)^2\right]d\tau \end{aligned} \tag{3.2}$$

where

$$\mathbb{X} = \frac{1}{a}\left(a^2 A\mathbb{W} - H\mathbb{M}\right), \qquad \mathbb{Y} = \frac{1}{a}\left[\frac{bb'}{2A}\mathbb{W} - \tilde{H}\mathbb{M}\right]. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} \mathbb{M}(t) = &\int_t^{+\infty} \cos\omega(\tau + t_0)\cdot b^3 d\tau - 3\varepsilon \int_t^{+\infty} ba^3 A^2 \mathbb{W}^2 d\tau \\ &- 3\varepsilon \int_t^{+\infty} ba^{-1} H^2 \mathbb{M}^2 d\tau + 6\varepsilon \int_t^{+\infty} baAH\mathbb{W}\mathbb{M} d\tau \end{aligned}$$

$$
\begin{aligned}
&-\varepsilon^2\int_t^{+\infty} ba^3A^3\mathbb{W}^3d\tau+\varepsilon^2\int_t^{+\infty} ba^{-3}H^3\mathbb{M}^3d\tau+3\varepsilon^2\int_t^{+\infty} baA^2H\mathbb{W}^2\mathbb{M}d\tau\\
&-3\varepsilon^2\int_t^{+\infty} ba^{-1}AH^2\mathbb{W}\mathbb{M}^2d\tau+2\varepsilon\int_t^{+\infty}\cos\omega(\tau+t_0)ba^2A\mathbb{W}d\tau\\
&-2\varepsilon\int_t^{+\infty}\cos\omega(\tau+t_0)ba^4A\mathbb{W}d\tau-2\varepsilon\int_t^{+\infty}\cos\omega(\tau+t_0)b^2a^{-1}\tilde{H}\mathbb{M}d\tau\\
&+\frac{1}{2}\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)ba^2\mathbb{W}^2d\tau-\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)ba^4\mathbb{W}^2d\tau\\
&+\frac{1}{2}\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)ba^6\mathbb{W}^2d\tau+\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)b\tilde{H}^2a^{-2}\mathbb{M}^2d\tau\\
&-2\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)\tilde{H}aA\mathbb{W}\mathbb{M}d\tau+2\varepsilon^2\int_t^{+\infty}\cos\omega(\tau+t_0)a^3\tilde{H}A\mathbb{W}\mathbb{M}d\tau. \quad (3.4)
\end{aligned}
$$

Note that to derive this integral formula we also used $b'=a-a^3$ and $b^2=a^2-a^4/2$.

Lemma 3.1. *The integral function of every integral on the left hand side of (3.4) for $\mathbb{M}(t)$ is in the form*

$$f(t)=\cos^{n_0}\omega(t+t_0)\cdot b^{m_1}\tilde{H}^{m_2}a^{n_1}A^{n_2}H^{n_3}\mathbb{M}^{n_4}\mathbb{W}^{n_5}$$

where

(i) n_0 *is either* 0 *or* 1*;*
(ii) $n_1\geq -3$, $m_1,m_2,n_2,n_3,n_4,n_5\geq 0$, *and* $m_1+m_2+|n_1|+n_2+n_3\leq 7$*;*
(iii) m_1+m_2 *is either* 1 *or* 3*;*
(iv) $0\leq n_4+n_5\leq 3$*;*
(v) $m_1+n_1+3n_4\geq 3$.

Proof. All acclaimed items can be verified by going through the integrals on the right hand side of (3.4) one by one. Every item listed here is going to play an important role in our analysis, so let us now explain them one by one. Item (i) implies that we either have one copy of $\cos\omega(t+t_0)$ or none in a kernel function. Those with one copy are from the function of perturbation and those without are from high order terms of the unperturbed equation. Item (ii) represents obvious upper and lower bounds for the powers of all functions involved. The power n_1 is allowed to be negative because a^{-1} is in $\mathbb{X}$ and $\mathbb{Y}$. It is ≥ -3 because the highest order of the non-linear terms of equation (2.16) is three. Item (iii) is critically important. It implies that dropping the trigonometric function $\cos\omega(t+t_0)$, and $M^{n_4}W^{n_5}$ from $f(t)$, leaves the function

$$b^{m_1}\tilde{H}^{m_2}a^{n_1}A^{n_2}H^{n_3}$$

as an odd function in t. Item (iv) follows again from the fact that the highest order of the nonlinear terms in equation (2.16) is three. Item (v) facilitates the absolute convergence of all high order Melnikov integrals. See, in particular, the proof of (3.18) in Lemma 3.5. □

We now turn to the function $\mathbb{W}(t)$. We have, from (3.2) and (3.3), that

$$\begin{aligned}
\mathbb{W}(t) = &-2\int_0^t \cos\omega(\tau+t_0)\cdot bAHd\tau + 3\varepsilon\int_0^t baAH\mathbb{W}^2d\tau \\
&+3\varepsilon\int_0^t ba^{-3}A^{-1}H^3\mathbb{M}^2d\tau - 6\varepsilon\int_0^t ba^{-1}H^2\mathbb{W}\mathbb{M}d\tau \\
&+\varepsilon^2\int_0^t baA^2H\mathbb{W}^3d\tau - \varepsilon^2\int_0^t ba^{-5}A^{-1}H^4\mathbb{M}^3d\tau + 3\varepsilon^2\int_0^t ba^{-3}H^3\mathbb{W}\mathbb{M}^2d\tau \\
&-3\varepsilon^2\int_0^t ba^{-1}AH^2\mathbb{W}^2\mathbb{M}d\tau - 2\varepsilon\int_0^t \cos\omega(\tau+t_0)bH\mathbb{W}d\tau \\
&+2\varepsilon\int_0^t \cos\omega(\tau+t_0)ba^2H\mathbb{W}d\tau + 4\varepsilon\int_0^t \cos\omega(\tau+t_0)\tilde{H}a^{-1}AH\mathbb{M}d\tau \\
&-\varepsilon^2/2\int_0^t \cos\omega(\tau+t_0)bA^{-1}H\mathbb{W}^2d\tau - \varepsilon^2/2\int_0^t \cos\omega(\tau+t_0)ba^4A^{-1}H\mathbb{W}^2d\tau \\
&+\varepsilon^2\int_0^t \cos\omega(\tau+t_0)ba^2A^{-1}H\mathbb{W}^2d\tau - \varepsilon^2\int_0^t \cos\omega(\tau+t_0)b\tilde{H}^2a^{-4}A^{-1}H\mathbb{M}^2d\tau \\
&+2\varepsilon^2\int_0^t \cos\omega(\tau+t_0)\tilde{H}a^{-1}H\mathbb{W}\mathbb{M}d\tau - 2\varepsilon^2\int_0^t \cos\omega(\tau+t_0)\tilde{H}aH\mathbb{W}\mathbb{M}d\tau.
\end{aligned} \tag{3.5}$$

We again used $b' = a - a^3$ and $b^2 = a^2 - a^4/2$.

Lemma 3.2. *The integral function of every integral on the left hand side of (3.5) for $\mathbb{W}(t)$ is in the form*

$$f(t) = \cos^{n_0}\omega(t+t_0)\cdot b^{m_1}\tilde{H}^{m_2}a^{n_1}A^{n_2}H^{n_3}\mathbb{M}^{n_4}\mathbb{W}^{n_5}$$

where

(i) n_0 *is either* 0 *or* 1*;*
(ii) $n_1 \geq -5$, $n_2 \geq -1$, $m_1, m_2, n_3, n_4, n_5 \geq 0$, *and* $m_1 + m_2 + |n_1| + |n_2| + n_3 \leq 11$*;*
(iii) $m_1 + m_2$ *is either* 1 *or* 3*;*
(iv) $0 \leq n_4 + n_5 \leq 3$*;*
(v) $m_1 + n_1 + 3n_4 \geq 1$.

Proof. All acclaimed items can be verified by going through the integrals on the right-hand side of (3.5) one by one. The differences between Lemma 3.1 and this lemma is caused by the fact that the integral function of $\mathbb{W}$ in (3.2) is divided by $a^2 A$. Item (v) is again in anticipation of the proof of (3.19) in Lemma 3.5. □

Let $\mathcal{K}_M$ be the collection of all integral functions on the right hand side of (3.4) for $\mathbb{M}$. For $f \in \mathcal{K}_M$ we have

$$f = \cos^{n_0} \omega(t + t_0) \cdot b^{m_1} \tilde{H}^{m_2} a^{n_1} A^{n_2} H^{n_3} \mathbb{M}^{n_4} \mathbb{W}^{n_5}.$$

We drop $\mathbb{M}^{n_4} \mathbb{W}^{n_5}$ from f to obtain a function

$$\hat{f} := \cos^{n_0} \omega(t + t_0) \cdot b^{m_1} \tilde{H}^{m_2} a^{n_1} A^{n_2} H^{n_3}. \tag{3.6}$$

Let

$$\hat{\mathcal{K}}_M = \{\hat{f} : \ f \in \mathcal{K}_M\}.$$

The set $\mathcal{K}_W$ and $\hat{\mathcal{K}}_W$ are defined in parallel by using (3.5) for $\mathbb{W}$.

Definition 3.1. (a) We call a function $\hat{f} \in \hat{\mathcal{K}}_M \cup \hat{\mathcal{K}}_W$ a **kernel function**; and
(b) we write a kernel function as $\hat{f}(t) = g(t) \cdot d(t)$ where

$$g(t) = \cos^{n_0} \omega(t + t_0), \qquad d(t) = b^{m_1} \tilde{H}^{m_2} a^{n_1} A^{n_2} H^{n_3}. \tag{3.7}$$

We call $g(t)$ the **trigonometric part**, and $d(t)$ the **dynamic part** of this kernel function.

Lemma 3.3. (Symmetry) *We have* $d(t) = -d(-t)$. *This is to say that the dynamic part* $d(t)$ *of all kernel functions are odd functions in* t.

Proof. This follows directly from Lemmas 3.1(iii) and 3.2(iii), and the fact that $a(t)$, $A(t)$ and $H(t)$ are even, but $b(t)$ and $\tilde{H}(t)$ are odd. □

Remarks on Lemma 3.3. (1) The symmetry as stated in Lemma 3.3 is critically important for the analysis of Part 2 to go through.
(2) Recall that, for all real $t > 0$, we have

$$\mathbb{W} = \frac{\sqrt{2 - a^2}}{\varepsilon b} W = \frac{\sqrt{2}}{\varepsilon a} W,$$

but for all real $t < 0$, we have

$$\mathbb{W} = \frac{\sqrt{2-a^2}}{\varepsilon b} W = -\frac{\sqrt{2}}{\varepsilon a} W.$$

This is because $b = a\sqrt{1-a^2/2}$ for $t > 0$, but $b = -a\sqrt{1-a^2/2}$ for $t < 0$.

(3) We can remove the linear terms of the new equation for $\mathbb{W}$ by either letting $\mathbb{W} = (\varepsilon a)^{-1} W$ or $\mathbb{W} = -(\varepsilon a)^{-1} W$. By using

$$\mathbb{W} = \frac{\sqrt{2-a^2}}{\varepsilon b} W,$$

we adopt a mixture of the two choices: to solve for primary stable solutions, we adopt the positive sign, but to solve for primary unstable solutions, we adopt the negative sign.

(4) The reason we use different signs to solve for stable and unstable solution is to obtain Lemma 3.3. If we use, say the positive sign for all t, then Lemma 3.3 would be false for the resulted integrals for $\mathbb{W}$. Recall that

$$X = \frac{1}{a}(bW - aHM), \qquad Y = \frac{1}{a}\left(b'W - a\tilde{H}M\right).$$

By letting $\mathbb{M} = aM$, $\mathbb{W} = a^{-1}W$, we would obtain

$$X = b\mathbb{W} - a^{-1}H\mathbb{M}, \qquad Y = b'\mathbb{W} - a^{-1}\tilde{H}\mathbb{M} \tag{3.8}$$

Observe that, with $\mathbb{W}$ so elected, the function in front of $\mathbb{W}$ and that in front of $\mathbb{M}$ for both X and Y are opposite in terms of even/odd symmetry. On the other hand, by letting $\mathbb{W} = \frac{2A}{b}W$, we obtain

$$X = \frac{1}{a}\left(b^2(2A)^{-1}\mathbb{W} - H\mathbb{M}\right), \qquad Y = \frac{1}{a}\left(b'b(2A)^{-1}\mathbb{W} - \tilde{H}\mathbb{M}\right). \tag{3.9}$$

Now the coefficients in front of $\mathbb{W}$ and $\mathbb{M}$ are both even functions in t for X and both odd for Y.

3.2. *Definition of high order Melnikov integrals*

With the detailed layout of the kernel functions in Sect. 3.1, we are now ready to define high order Melnikov integrals in precise terms. To define these integrals we start with a *structure tree* of Sect. 1.2.

Assume a given structure tree has p nodes in total. We index the tree nodes from the bottom level to the top level, and at a fixed level, from the right to the left, as $N_1, \cdots, N_p$. The root node of the entire tree is then indexed as N_p. For $j \le p$, we define three index sets, which we denote as $C(j)$, $T(j)$ and $P(j)$ respectively: $C(j)$ is the index set of the direct descendants branched out of N_j; $T(j)$ is the index set of all nodes located in the sub-tree rooted at N_j (including j); and $P(j)$ is the index set of all nodes that are in the *ancestry line* of N_j (excluding j). We note that j' ($\neq j$) is in $P(j)$ if and only if $j \in T(j')$.

All nodes N_j are of two types: the M-type and the W-type. To each node N_j, we assign an integral variable which we denote as t_j. We also associate N_j with a function f_j. We let $f_j \in \mathcal{K}_M$ if N_j is M-type and $f_j \in \mathcal{K}_W$ if N_j is W-type. Two things are defined by the function f_j: the first is the kernel function, which is $\hat{f}_j(t_j)$. The second is that N_j has n_4 many direct descendant of

M-type and n_5 many direct descendant of W-type. It follows from $n_4 + n_5 \leq 3$ (Lemma 3.1(iv) and Lemma 3.2(iv)) that every tree node in a given structure tree can only have up to 3 direct descendants. We also assign an interval of integration I_j to every N_j. Assume N_j is branched out directly from $N_{j'}$. We let $I_j = (t_{j'}, +\infty)$ if N_j is an M-node but $I_j = (0, t_{j'})$ if it is a W-node. For N_p, we let $t_{j'} = t$.

What make a structure tree finite is $f(t) = \cos\omega(t+t_0)b^3$ in $\mathcal{K}_M$ for $\mathbb{M}_0$ and $f(t) = \cos\omega(t+t_0)bAH$ in $\mathcal{K}_W$ for $\mathbb{W}_0$. All ending nodes in a structure tree must be one of these two.

We define a **high order Melnikov integral** $N_p^s(t, t_0, \omega)$ as a multiple integral of multiplicity p as follows: (a) the integral function for N_p^s is $\prod_{j=1}^p \hat{f}_j(t_j)$; (b) the order of variables of this integration is set to be $dt = dt_1 dt_2 \cdots dt_p$; and (c) the interval of integration for t_j is I_j.

We can also adopt the following alternative (but equivalent) definition of high order Melnikov integrals in inductive form, which is more convenient to use in certain technical proofs. For $p = 1$, N_1^s is either $\mathbb{M}_0(t)$ or $\mathbb{W}_0(t)$ in (2.21). Assume $p > 1$. If $\hat{f}_p \in \hat{\mathcal{K}}_M$, we let

$$N_p^s(t, t_0, \omega) = \int_t^{+\infty} \hat{f}_p(t_p) \prod_{j \in C(p)} N_j^s(t_p, t_0, \omega) dt_p. \tag{3.10}$$

Alternatively, if $\hat{f}_p \in \hat{\mathcal{K}}_W$, we let

$$N_p^s(t, t_0, \omega) = \int_0^{t} \hat{f}_p(t_p) \prod_{j \in C(p)} N_j^s(t_p, t_0, \omega) dt_p. \tag{3.11}$$

Note that, for M-type, the integration is from t to $+\infty$, but for W-type, it is from 0 to t.

Lemma 3.4. *There exists a constant $K_0 > 0$ so that for all $t > 0$, we have*

$$\begin{aligned} &|H(t)|,\ |\tilde{H}(t)|,\ |A(t)| < K_0; \quad |b(t)/a(t)| < K_0; \\ &|b(t)/A(t)| < K_0 e^{-t}; \quad K_0^{-1} e^{-t} < |a(t)| < K_0 e^{-t}. \end{aligned} \tag{3.12}$$

Proof. This is a good place to have a brief review of all functions involved. We have

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}, \tag{3.13}$$

and $a(t), b(t)$ are such that

$$b = a'; \qquad b' = a - a^3; \qquad b^2 = a^2 - \frac{1}{2}a^4. \tag{3.14}$$

Also, by definition,

$$A(t) = \frac{1}{2}\sqrt{(2 - a^2)}; \tag{3.15}$$

$$h(t) = 3a^2(t)\int_0^t a^{-2}(\tau)d\tau = \frac{3(e^{2t} - e^{-2t} + 4t)}{2(e^t + e^{-t})^2}; \tag{3.16}$$

and

$$H(t) = \frac{1}{a(t)}\left[b(t)h(t) + a(t)\right]; \qquad \tilde{H}(t) = \frac{1}{a(t)}\left[b'(t)h(t) + 2b(t)\right]. \tag{3.17}$$

All functions involved in this lemma are explicitly defined, and all statements are straight forward to verify. □

Let $r_j = m_1(j) + n_1(j)$ for N_j. We also have $d_j(t_j) \sim e^{-r_j t_j}$ as $t_j \to +\infty$. We caution that r_j is not necessarily positive, and in the case of $r_j < 0$, $d_j(t_j)$ diverges to ∞ exponentially fast as $t_j \to +\infty$. This poses a potential threat to convergence.

Our next lemma assures that convergence is *not* a problem for high order Melnikov integrals.

Lemma 3.5. *All high order Melnikov integrals are absolutely convergent. In fact, there exists a constant $\mathbb{K} > 0$ such that, for all $p \geq 1$ and $t \geq 0$,*

$$|N_p^s(t, t_0, \omega)| \leq \mathbb{K}^p a^3(t) \tag{3.18}$$

if N_p is M-type; and

$$|N_p^s(t, t_0, \omega)| \leq \mathbb{K}^p \tag{3.19}$$

if N_p is W-type.

Proof. First, the acclaimed estimation of this lemma holds for both $\mathbb{M}_0(t)$ and $\mathbb{W}_0(t)$ because we have $d(t) \sim b^3$ for $\mathbb{M}_0$ and $d(t) \sim b$ for $\mathbb{W}_0$ as $t \to +\infty$. For $p > 1$, we assume (3.18) and (3.19) hold for all $N_k^s(t, t_0, \omega)$, $k < p$. Assume N_p is M-type. We have

$$\begin{aligned} |N_p^s(t, t_0, \omega)| &= \left| \int_t^{+\infty} b^{m_1} |\tilde{H}|^{m_2} a^{n_1} A^{n_2} |H|^{n_3} \prod_{j \in C(p)} |N_j(\tau, t_0, \omega)| d\tau \right| \\ &\leq K_0^7 \mathbb{K}^{p-1} \left| \int_t^{+\infty} a^{m_1+n_1} a^{3n_4} d\tau \right|. \end{aligned}$$

Here we used (3.12) and $m_1 + m_2 + |n_1| + n_2 + n_3 \leq 7$. We also used (3.18), (3.19) for all N_j, $j \in C(p)$, and the fact that

$$\sum_{j \in C(p)} p_j = p - 1$$

where p_j is the multiplicity of the integral N_j^s defined by the subtree rooted at N_j. It then follows that

$$
\begin{aligned}
|N_p^s(t,t_0,\omega)| &\le K_0^{7+(m_1+n_1+3n_4)}\mathbb{K}^{p-1}\left|\int_t^{+\infty} e^{-(m_1+n_1+3n_4)\tau}d\tau\right| \\
&\le K_0^{7+12}\mathbb{K}^{p-1}\left|\int_t^{+\infty} e^{-3\tau}d\tau\right| \\
&\le K_0^{22}\mathbb{K}^{p-1}a^3(t).
\end{aligned}
$$

Note that to obtain the second inequality we use Lemma 3.1(v), which is $m_1+n_1+3n_4 \ge 3$. We finally let $\mathbb{K} > K_0^{22}$ to obtain

$$|N_p(t,t_0,\omega)| \le \mathbb{K}^p a^3(t).$$

The proof for (3.19) is similar. The result differs because in the place of Lemma 3.1(v), we use Lemma 3.2(v), which is $m_1+n_1+3n_4 \ge 1$. □

We end this subsection by recording a technical fact that will be used later.

Lemma 3.6. *Let $R_j = \sum_{j'\in T(j)}(m_1(j')+n_1(j'))$. We have $R_j \ge 3$ if N_j is an M-node and $R_j \ge 1$ if N_j is a W-node.*

Proof. Assume N_j is an M-node. We have

$$R_j = m_1(j)+n_1(j)+\sum_{j'\in C(j)} R_{j'} \ge m_1(j)+n_1(j)+3n_4(j) \ge 3$$

where the first equality is by definition; the next inequality is obtained by inductively assuming $R_{j'} \ge 3$ if $N_{j'}$ is an M-node and $R_{j'} \ge 1$ if $N_{j'}$ is a W-node; and the last inequality is Lemma 3.1(v). If N_j is a W-node, we have

$$R_j = m_1(j)+n_1(j)+\sum_{j'\in C(j)} R_{j'} \ge m_1(j)+n_1(j)+3n_4(j) \ge 1$$

where the last inequality is Lemma 3.2(v). □

3.3. Splitting distance as collections of high order Melnikov integrals

Our purpose is to reduce the exponentially small estimate on $A_{k,n}$ to that of a single difference of a high order Melnikov integral and its dual. This is made possible because $\mathbb{M}_n(t)$ and $\mathbb{W}_n(t)$ for stable solutions are sums of collections of high order Melnikov integrals. To make a legitimate reduction we also need to bound the multiplicity, and the number of these well-structured multiple integrals in the respective collections in terms of n.

Let $n \ge 0$ be an arbitrary integer.

Proposition 3.1. *There exist two collections of Melnikov integrals for $\mathbb{M}_n(t)$ and $\mathbb{W}_n(t)$, which we denote as $\Lambda_{M,n}$ and $\Lambda_{W,n}$ respectively, such that*

$$\mathbb{M}_n(t) = \sum_{N_p \in \Lambda_{M,n}} c_{N_p} N_p(t), \qquad \mathbb{W}_n(t) = \sum_{N_p \in \Lambda_{W,n}} c_{N_p} N_p(t) \tag{3.20}$$

where c_{N_p} are constants. In addition, there exist constants $K_1 = 4$, K_2 and K_3 so that

(1) *(Order of Integrals) For all $N_p(t) \in \Lambda_{M,n} \cup \Lambda_{W,n}$, $n \le p \le K_1(n+1)$.*
(2) *(Number of Integrals) The total number of Melnikov integral in $\Lambda_{M,n} \cup \Lambda_{W,n}$ is $< K_2^{n+1}$.*
(3) *(Coefficients) We have $|c_{N_p}| < K_3^n$ for all $N_p(t) \in \Lambda_{M,n} \cup \Lambda_{W,n}$.*

The fact that $\mathbb{M}_n(t)$, $\mathbb{W}_n(t)$ are summations of collections of high order Melnikov integral follows rather directly from the recursion introduced in Sect. 2.4 based on the integral equation (2.17) of Lemma 2.4. The proof for items (1)-(3) are less straight forward and they afford rather involved combinatoric proofs.

The proof of this proposition is placed in Part 3 as Section 7.

Proof of Proposition 1.1. Proposition 1.1 follows directly from Proposition 3.1 combined with Lemma 3.5. □

Part 2. Exponentially small splitting

In the rest of this paper, the letter i is reserved for $\sqrt{-1}$.

4. Preliminaries on high order Melnikov integrals

This section is a preliminary study on high order Melnikov integrals. We introduce a new convention of presenting a multiple integral by using a structure tree in Sect. 4.1. We then use this new convention to define extended Melnikov integrals, pure integrals (in Sect. 4.1), and complex Melnikov integrals (in Sect. 4.2). We prove Main Theorem (a) in Sect. 4.2. In Sect. 4.3, we use a simple example to demonstrate how to extract an exponentially small factor out of the difference of a real integral and its dual by moving to the complex plane. Equality (4.7) and the example of Sect. 4.3 together serve as a launching pad for the theory to be constructed in the rest of Part 2.

4.1. Extended and pure Melnikov integrals

Our first challenge is on how to present a multiple integral that would also allow us to

(a) explain in detail on how we are going to manipulate this integral;
(b) present the end product in precise terms after a proposed manipulation is adopted;
(c) rigorously derive the end product from the integral that is manipulated.

The traditional convention of using multiple integral signs with an implicitly or explicitly defined integral domain and integral function would still be helpful on occasion, but it is far from sufficient in elucidating the details needed to prove many of the subtle assertions in Part 2, in particular, when such details are tied to that of a defining structure tree.

Note that a multiple integral is well-defined as far as we have the following:

(i) a set of integral variables;
(ii) a well-defined integral function;
(iii) a well-defined interval of integration for each of the integral variables;
(iv) an assigned order of integral variables in which integrations are carried out.

We will use a structure tree $\mathcal{T}$ to represent a multiple integral. Tree nodes of $\mathcal{T}$ are indexed from the bottom level to the top level, and from the right to the left at the same level, as $N_1, \cdots, N_p$. Stored in the memory of a tree node N_j is a variable of integration, which we denote as t_j; an interval of integration, which we denote as I_j; and a kernel function, which we denote as f_j. To define a multiple integral by using $\mathcal{T}$, we let the integral variables be $(t_1, \cdots, t_p)$, the integral function be the product of all kernel functions, the interval of integration for t_j be I_j. Finally, we let the integration be carried out in the order of $dt_1 dt_2 \cdots dt_p$.

Let $\mathcal{T}$ be the structure tree for a high order Melnikov integral. We denote this integral as $\mathcal{T} = \mathcal{T}(t)$. We also denote the subtree rooted at N_j as $\mathcal{T}_j$ and call it a W-subtree if N_j is a W-node. The integral defined by $\mathcal{T}_j$ is denoted as $\mathcal{T}_j = \mathcal{T}_j(t_{j'})$ where j' is such that N_j is a direct child of $N_{j'}$.

The next hurdle we encounter is the mixed nature of the integral bounds. For a high order Melnikov integral, the integration is from 0 to t for a tree node of W-type, but from t to $+\infty$ for a tree node of M-type. We need to apply

$$\int_0^t = \int_0^{+\infty} - \int_t^{+\infty} \tag{4.1}$$

to all W-node to make all upper integral bounds uniformly $+\infty$. A formal treatment is as follows.

Definition 4.1. Let $\mathcal{T}$ be the structure tree for a high order Melnikov integral $\mathcal{T}(0)$. We obtain an extended Melnikov integral $\tilde{\mathcal{T}}(0)$ from $\mathcal{T}(0)$ by adopting the following changes to $\mathcal{T}$. We change the label of all W-node in $\mathcal{T}$ to either W_1 or W_2; and we change the interval of integration to $[0, +\infty)$ for all newly assigned W_1-node, but to $[t_{j'}, +\infty)$ for all newly assigned W_2-node.

Lemma 4.1. *Let $\mathcal{E}(\mathcal{T}(0))$ be the collection of all extended Melnikov integrals $\tilde{\mathcal{T}}(0)$ induced by using $\mathcal{T}(0)$. Then,*

(a) *all extended Melnikov integrals $\tilde{\mathcal{T}}(0)$ are absolutely convergent;*
(b) *the number of extended integrals in $\mathcal{E}(\mathcal{T}(0))$ is equal to $2^{\hat{p}}$, where $\hat{p}$ is the total number of W-node in $\mathcal{T}(0)$; and*
(c) *we have*

$$\mathcal{T}(0) = \sum_{\tilde{\mathcal{T}}(0) \in \mathcal{E}(\mathcal{T}(0))} (-1)^{w(\tilde{\mathcal{T}})} \tilde{\mathcal{T}}(0)$$

where $w(\tilde{\mathcal{T}})$ is the total number of W_2-node in $\tilde{\mathcal{T}}(0)$.

Proof. To prove item (a), we repeat the proof of Lemma 3.5, treating W_1 and W_2 node as W-node. Item (b) holds because changing every W to either W_1 or W_2 is a process of binary splitting. Item (c) is straight forward from (4.1). □

Definition 4.2. Let $\tilde{\mathcal{T}}(0)$ be an extended integral induced by using $\mathcal{T}(0)$. We define **pure blocks** of $\tilde{\mathcal{T}}(0)$ as follows:

(a) the remainder tree obtained by dropping all W_1-subtree from $\tilde{\mathcal{T}}(0)$ is a pure block;
(b) a W_1-subtree is a pure block if it contains no smaller subtree of W_1-type; and
(c) if a W_1-subtree contains other W_1-subtree inside, then the remainder tree obtained by deleting all inside W_1-subtree is a pure block.

We also call a pure block a **pure integral**.

Lemma 4.2. *An extended integral $\tilde{\mathcal{T}}(0)$ with m W_1-node has $m+1$ pure blocks, which we denote as $\mathcal{B}_1(0), \cdots, \mathcal{B}_{m+1}(0)$; and we have*

$$\tilde{\mathcal{T}}(0) = \mathcal{B}_1(0)\mathcal{B}_2(0)\cdots\mathcal{B}_{m+1}(0).$$

Proof. This is because the interval of integration for a W_1-node is $[0, +\infty)$, causing the subtree rooted at it to factor out. □

4.2. Complex Melnikov integrals

Let $\mathcal{T}(0)$ be a Melnikov integral of order p. Recall that the kernel function for N_j is in the form of

$$f_j(t_j) = \cos^{n_0} \omega(t_j + t_0) d_j(t_j)$$

where $n_0(f_j)$ is either 0 or 1 and

$$d_j(t_j) = b^{m_1}(t_j)\tilde{H}^{m_2}(t_j)a^{n_1}(t_j)A^{n_2}(t_j)H^{n_3}(t_j).$$

Definition 4.3. Let

$$\mathbf{q} = (q_1, \cdots, q_p)$$

where q_j is either $n_0(f_j)$ or $-n_0(f_j)$. For a given $\mathbf{q}$, we define a **complex Melnikov integral** for $\mathcal{T}(0)$, which we denote as $\mathcal{T}_{\mathbf{q}}(0)$, by changing $\cos^{n_0(f_j)} \omega(t_j + t_0)$ in $f_j(t_j)$ to

$$e^{i\omega q_j t_j}.$$

Let

$$\mathcal{S} = \left\{\mathbf{q} = (q_1, \cdots, q_p);\ \ q_j \in \{n_0(f_j), -n_0(f_j)\}\right\}.$$

The set $\mathcal{S}$ is the collection of all possible $\mathbf{q}$ vectors.

Lemma 4.3. *We have*

$$\mathcal{T}(0) = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q}\in\mathcal{S}} e^{i\omega Q_{\mathbf{q}} t_0} \mathcal{T}_{\mathbf{q}}(0)$$

where

$$\hat{p} = n_0(f_1) + \cdots + n_0(f_p), \qquad Q_{\mathbf{q}} = q_1 + \cdots + q_p.$$

Proof. We substitute all $\cos\omega(t_j + t_0)$ in $f_j(t_j)$ by using

$$\frac{1}{2}\left(e^{i\omega(t_j+t_0)} + e^{-i\omega(t_j+t_0)}\right). \qquad \square$$

Definition 4.4. Let $\mathcal{T}_{\mathbf{q}}(0)$ be a complex Melnikov integral for a given $\mathcal{T}(0)$ where $\mathbf{q} = (q_1, \cdots, q_p)$ and $\mathcal{T}_j$ be the subtree of $\mathcal{T}_{\mathbf{q}}(0)$ rooted at N_j.

(i) We call p_j, the total number of tree nodes in the subtree rooted at N_j, the **order** of the integral $\mathcal{T}_j$. In particular, the order of $\mathcal{T}_{\mathbf{q}}(0)$ is p.
(ii) We call

$$Q_j = \sum_{j'\in T(j)} q_{j'}$$

the **total index** of the subtree $\mathcal{T}_j$. In particular, the total index of $\mathcal{T}_{\mathbf{q}}(0)$ is Q_p, which we sometimes also denote as $Q_{\mathbf{q}}$.
(iii) We use σ_j to denote the **sign** of Q_j. This is to say that if $Q_j > 0$ then $\sigma_j = 1$, if $Q_j < 0$ then $\sigma_j = -1$, and if $Q_j = 0$ then $\sigma_j = 0$.
(iv) We call $\mathcal{T}_j$ a **zero subtree** if $Q_j = 0$.

We now use superscript s and u to distinguish an integral for stable solutions from its dual. Let $\mathcal{T}^s_{\mathbf{q}}(0)$ be a complex Melnikov integral for $\mathcal{T}^s(0)$ and $\mathcal{T}^u_{\mathbf{q}}(0)$ be the dual of $\mathcal{T}^s_{\mathbf{q}}(0)$. We write $\mathcal{T}^s_{\mathbf{q}}(0)$ as a multiple integral defined on a region $\mathcal{R} \subset (0, +\infty)^p$ as

$$\mathcal{T}^s_{\mathbf{q}}(0) = \int_{\mathcal{R}} e^{i\omega\mathbf{q}\cdot\mathbf{t}} F(\mathbf{t}) d\mathbf{t} \tag{4.2}$$

where $\mathbf{t} = (t_1, \cdots, t_p)$ and

$$F(\mathbf{t}) = \prod_{j=1}^{p} d_j(t_j).$$

By definition,

$$\mathcal{T}^u_{\mathbf{q}}(0) = \int_{-\mathcal{R}} e^{i\omega\mathbf{q}\cdot\mathbf{t}} F(\mathbf{t}) d\mathbf{t}. \tag{4.3}$$

We let

$$\mathbb{D}_{\mathbf{q}}(0) := e^{i\omega Q_{\mathbf{q}} t_0}\left(\mathcal{T}^s_{\mathbf{q}}(0) - \mathcal{T}^u_{\mathbf{q}}(0)\right).$$

Proposition 4.1. *By definition,*

$$\mathbb{D}_{\mathbf{q}}(0) + \mathbb{D}_{-\mathbf{q}}(0) = -4\sin(\omega Q_{\mathbf{q}} t_0)\int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}. \tag{4.4}$$

Proof. We have, by using Lemma 3.3, that

$$\begin{aligned}\int_{\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} &= \int_{-\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t},\\ \int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} &= -\int_{-\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}.\end{aligned} \tag{4.5}$$

It then follows that

$$\begin{aligned}\mathbb{D}_{\mathbf{q}}(0) + \mathbb{D}_{-\mathbf{q}}(0) &= e^{i\omega Q_{\mathbf{q}} t_0}\left(\mathcal{T}^s_{\mathbf{q}}(0) - \mathcal{T}^u_{\mathbf{q}}(0)\right) + e^{-i\omega Q_{\mathbf{q}} t_0}\left(\mathcal{T}^s_{-\mathbf{q}}(0) - \mathcal{T}^u_{-\mathbf{q}}(0)\right)\\
&= \left(\cos(\omega Q_{\mathbf{q}} t_0) + i\sin(\omega Q_{\mathbf{q}} t_0)\right)\left(\int_{\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} + i\int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}\right)\\
&\quad - \left(\cos(\omega Q_{\mathbf{q}} t_0) + i\sin(\omega Q_{\mathbf{q}} t_0)\right)\left(\int_{-\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} + i\int_{-\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}\right)\\
&\quad + \left(\cos(\omega Q_{\mathbf{q}} t_0) - i\sin(\omega Q_{\mathbf{q}} t_0)\right)\left(\int_{\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} - i\int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}\right)\\
&\quad - \left(\cos(\omega Q_{\mathbf{q}} t_0) - i\sin(\omega Q_{\mathbf{q}} t_0)\right)\left(\int_{-\mathcal{R}} \cos(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t} - i\int_{-\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}\right)\\
&= -4\sin(\omega Q_{\mathbf{q}} t_0)\int_{\mathcal{R}} \sin(\omega\mathbf{q}\cdot\mathbf{t})F(\mathbf{t})d\mathbf{t}.\end{aligned}$$

We emphasize that cancellations caused by (4.5) are induced from the symmetry of Lemma 3.3. □

We now use

$$\sin(\omega\mathbf{q}\cdot\mathbf{t}) = \frac{1}{2i}\left(e^{i\omega\mathbf{q}\cdot\mathbf{t}} - e^{-i\omega\mathbf{q}\cdot\mathbf{t}}\right)$$

to rewrite (4.4) as

$$\mathbb{D}_{\mathbf{q}}(0)+\mathbb{D}_{-\mathbf{q}}(0)=2i\sin(\omega Q_{\mathbf{q}}t_0)\left(\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)\right), \tag{4.6}$$

and we have from Lemma 4.3 and (4.6),

$$d_{\mathcal{T}}:=\mathcal{T}^s(0)-\mathcal{T}^u(0)=\frac{1}{2^{\hat{p}}}\sum_{\mathbf{q}\in\mathcal{S}^+}2i\sin(\omega Q_{\mathbf{q}}t_0)\left(\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)\right) \tag{4.7}$$

where $\mathcal{S}^+$ is the set of all $\mathbf{q}\in\mathcal{S}$ so that $Q_{\mathbf{q}}>0$.

4.3. A simple example

Let

$$\mathcal{T}^s_1(0)=\int_0^{+\infty}e^{i\omega t}b(t)A^{-1}(t)a(t)dt,\quad \mathcal{T}^s_{-1}(0)=\int_0^{+\infty}e^{-i\omega t}b(t)A^{-1}(t)a(t)dt.$$

We prove in this subsection that:

Proposition 4.2. *There exists a constant K such that*

$$\left|\mathcal{T}^s_1(0)-\mathcal{T}^s_{-1}(0)\right|<(K\omega)^2e^{-\omega\pi/2}.$$

Here, our purpose is to introduce to the reader a critical insight on *why* we can extract an exponentially small factor $e^{-\omega\pi/2}$ out of $\mathcal{T}^s_{\mathbf{q}}(0)-\mathcal{T}^s_{-\mathbf{q}}(0)$ (See (4.7)).

We start by treating $b(z)A^{-1}(z)$ as a complex function where $z=t+is$. Let

$$D=\{z=t+is:\ \ t\in(0,+\infty),\ s\in(-\pi/2,\pi/2)\}.$$

As an analytic extension of $b(t)A^{-1}(t), t>0$, $b(z)A^{-1}(z)$ is well-defined on D. This function can also be extended analytically to the closure of D minus three points: the first one is $z=0$, which is a branch point, and the other two are $z=\pm i\pi/2$, poles of order one. To stay an ω^{-1} distance away from the poles, we let

$$D_\omega=\{z=t+is:\ \ t\in(0,+\infty),\ s\in(-\pi/2+\omega^{-1},\pi/2-\omega^{-1})\}.$$

Lemma 4.4. *The function $B(z)=b(z)A^{-1}(z)$ is analytic on D_ω, continuously defined on the closure of D_ω. In addition, there exists a constant $K_0>0$ so that $|B(z)|,|a(z)|\le K_0\omega e^{-t}$ on $\bar{D}_\omega$ where $z=t+is$.*

Proof. Easy facts ready to be directly verified. □

Let

$$\ell_0=\{z=t+is,\quad t\in(0,+\infty),\quad s=0\}.$$

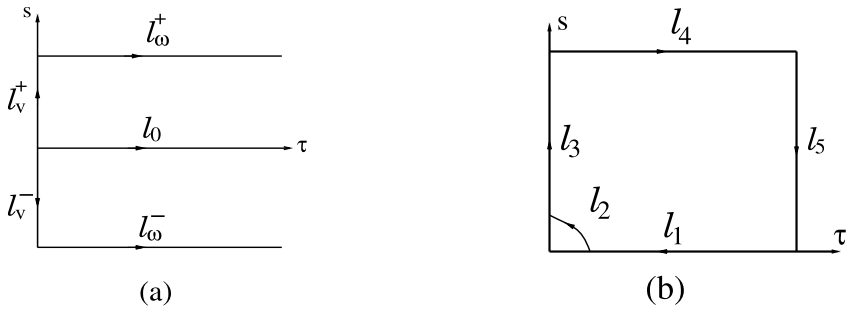

Fig. 3. Moving to Complex Plane.

We have

$$\mathcal{T}_1^s(0) = \int_0^{+\infty} e^{i\omega t} b(t) A^{-1}(t) a(t) dt = \int_{\ell_0} e^{i\omega z} b(z) A^{-1}(z) a(z) dz.$$

To convert this integral to complex integrals, we let

$$\ell_\omega^+ = \left\{ z = t + is, \quad t \in (0, +\infty), \quad s = \pi/2 - \omega^{-1} \right\};$$
$$\ell_v^+ \quad = \left\{ z = t + is, \quad t = 0, \quad s \in (0, \pi/2 - \omega^{-1}] \right\}.$$

See Fig. 3(a).

Lemma 4.5. *We have*

$$\int_{\ell_0} e^{i\omega z} b(z) A^{-1}(z) a(z) dz = \int_{\ell_v^+} e^{i\omega z} b(z) A^{-1}(z) a(z) dz + \int_{\ell_\omega^+} e^{i\omega z} b(z) A^{-1}(z) a(z) dz.$$

Proof. We use the Cauchy integral theorem to prove this lemma. See Fig. 3(b). The integration over the boundaries of the region enclosed by $\ell_1, \ell_2, \ell_3, \ell_4$ and ℓ_5 is zero. We then push ℓ_5 to infinity, and shrink the small arc ℓ_2 to $z = 0$. The integral over ℓ_5 approaches zero because the magnitude of the integral function declines to zero exponentially fast as $\ell_5 \to +\infty$. The integral over ℓ_2 also approaches zero because the integral function is bounded on ℓ_2 but the length of ℓ_2 approaches zero. □

In parallel, we let

$$\ell_\omega^- = \left\{ z = t + is, \quad t \in (0, +\infty), \quad s = -\pi/2 + \omega^{-1} \right\};$$
$$\ell_v^- = \left\{ z = t + is, \quad t = 0, \quad s \in (0, -\pi/2 + \omega^{-1}] \right\}.$$

Lemma 4.6. *We have*

$$\int_{\ell_0} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz = \int_{\ell_v^-} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz + \int_{\ell_\omega^-} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz.$$

Proof. The proof is parallel to that of Lemma 4.5. □

Proof of Proposition 4.2. By Lemma 4.5,

$$\begin{aligned}\mathcal{T}_1^s(0) &= \int_{\ell_v^+} e^{i\omega z} b(z) A^{-1}(z) a(z) dz + \int_{\ell_\omega^+} e^{i\omega z} b(z) A^{-1}(z) a(z) dz \\ &= i \int_0^{\pi/2-\omega^{-1}} e^{-\omega s} b(is) A^{-1}(is) a(is) ds + e^{-\omega\pi/2+1} \int_0^{+\infty} e^{i\omega t} b(t_+) A^{-1}(t_+) a(t_+) dt\end{aligned}$$

where $t_+ = t + i(\pi/2 - \omega^{-1})$. We also have, by Lemma 4.6,

$$\begin{aligned}\mathcal{T}_{-1}^s(0) &= \int_{\ell_v^-} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz + \int_{\ell_\omega^-} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz \\ &= i \int_0^{-\pi/2+\omega^{-1}} e^{\omega s} b(is) A^{-1}(is) a(is) ds + e^{-\omega\pi/2+1} \int_0^{+\infty} e^{-i\omega t} b(t_-) A^{-1}(t_-) a(t_-) dt\end{aligned}$$

where $t_- = t - i(\pi/2 + \omega^{-1})$. We obtain, by changing s to $-s$, that

$$\int_0^{-\pi/2+\omega^{-1}} e^{\omega s} b(is) A^{-1}(is) a(is) ds = \int_0^{\pi/2-\omega^{-1}} e^{-\omega s} b(is) A^{-1}(is) a(is) ds.$$

Here, it is crucial that $b(t)$ is odd and that $A(t)$, $a(t)$ are even. This implies

$$\begin{aligned}\mathcal{T}_1^s(0) - \mathcal{T}_{-1}^s(0) &= e^{-\omega\pi/2+1} \int_0^{+\infty} e^{i\omega t} b(t_+) A^{-1}(t_+) a(t_+) dt \\ &\quad - e^{-\omega\pi/2+1} \int_0^{+\infty} e^{-i\omega t} b(t_-) A^{-1}(t_-) a(t_-) dt.\end{aligned}$$

We conclude, by using Lemma 4.4, that

$$\left|\mathcal{T}_1^s(0) - \mathcal{T}_{-1}^s(0)\right| < (K\omega)^2 e^{-\omega\pi/2}. \qquad \square$$

By changing the integral path from the real axis to one that is ω^{-1}-distance away from the singular set of the integral function in complex plane, we magnify the dynamic part of the kernel function by a factor $O(\omega^2)$, but gain from the trigonometric part of the kernel function an exponentially small factor $e^{-\omega\pi/2}$. We emphasize that for Proposition 4.2 to hold, it is critical that

$$\int_{\ell_v^+} e^{i\omega z} b(z) A^{-1}(z) a(z) dz$$

cancels

$$\int_{\ell_v^-} e^{-i\omega z} b(z) A^{-1}(z) a(z) dz.$$

This cancellation relies on the symmetry of the integral function.

5. Manipulating pure integrals

This section is a study on *complex pure integrals*. Let $\mathcal{T}_{\mathbf{q}}(0)$ be a complex pure integral for stable solutions where $\mathbf{q} = (q_1, \cdots q_p)$. For pure integrals, all upper integral bounds are uniformly $+\infty$.

In Sect. 5.1, we use a new set of variables τ_j to transform the integral domain of all pure integrals to $(0, +\infty)^p$. This would allow us to switch the order of the new variables of integration without worrying the integral bounds. We then introduce complex variables $z_j = \tau_j + is_j$, one j at a time, to use the Cauchy integral theorem to replace an integral defined on $(0, +\infty)$ of the real τ_j-axis by using two integrals defined on paths in complex z_j-plane in Sects. 5.2-5.4. This process is formally presented as an induction of p steps. We initiate this induction in Sect. 5.2. The inductive assumptions are presented in Sect. 5.3, and this induction is moved forward in Sect. 5.4. The main result of this section is the decomposition formula (5.13) for pure integrals obtained at the end of this induction.

Notation. We recall that $C(j)$ is the set of all k so that N_k is a direct descendant of N_j, $T(j)$ is the set of all k so that N_k is inside of the subtree rooted at N_j (including j), and $P(j)$ is the set of all k such that $j \in T(k)$ (excluding j). The set of N_k, $k \in P(j)$ is the *ancestry line* of N_j.

5.1. Unification of all integral bounds

We regard a complex pure integral $\mathcal{T}_{\mathbf{q}}(0)$ as a multiple integral defined on a region $\mathcal{R}$ in $(0, +\infty)^p$. Let $\tau = (\tau_1, \cdots, \tau_p)$ be such that

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'} \tag{5.1}$$

for all j. Recall that $P(j)$ is the index set for the ancestry line of N_j (excluding j). We change the integral variables from $(t_1, \cdots, t_p)$ to $(\tau_1, \cdots, \tau_p)$. Corresponding to this change of variables, we modify $\mathcal{T}_{\mathbf{q}}(0)$ as follows to obtain a new structure tree $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ in τ variables. For all N_j, $j \le p$,

(i) The integral variable for N_j is changed from t_j to τ_j.
(ii) The interval of integration is changed to $(0, +\infty)$ for τ_j.
(iii) The new kernel function is

$$\hat{f}_j(\tau) = \hat{g}_j(\tau_j)\hat{d}_j(\tau)$$

where

$$\hat{g}_j(\tau_j) = e^{i\omega Q_j \tau_j}$$

is the new trigonometric part, and

$$\hat{d}_j(\tau) = d_j\left(\tau_j + \sum_{j' \in P(j)} \tau_{j'}\right)$$

is the dynamic part of the new kernel function. Note that in the above,

$$Q_j = \sum_{j' \in T(j)} q_{j'}$$

is the total index of the subtree rooted in N_j, and

$$d_j(t) = b^{m_1}(t)\tilde{H}^{m_2}(t)a^{n_1}(t)A^{n_2}(t)H^{n_3}(t).$$

The integral defined by $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ is a multiple integral defined on $(0, +\infty)^p$ in the space of $\tau = (\tau_1, \cdots, \tau_p)$. We have

$$\hat{\mathcal{T}}_{\mathbf{q}}(0) = \int_{(0,+\infty)^p} \prod_{j=1}^{p} \hat{f}_j(\tau) d\tau \tag{5.2}$$

where $d\tau = d\tau_1 \cdots d\tau_p$.

Proposition 5.1. *We have* $\mathcal{T}_p(0) = \hat{\mathcal{T}}_p(0)$.

Proof. The determinant of the Jacobian of the coordinate change from t to τ defined by using

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'}$$

is 1. The integral interval for τ_j is $(0, +\infty)$ because, by definition,

$$\tau_j = t_j - t_{j'}$$

where j' is such that $j \in C(j')$. For the trigonometric part of the kernel function, we observe that

$$\sum_{j=1}^{p} q_j t_j = \sum_{j=1}^{p} q_j \left(\tau_j + \sum_{j' \in P(j)} \tau_{j'} \right) = \sum_{j=1}^{p} Q_j \tau_j \tag{5.3}$$

where

$$Q_j = \sum_{j' \in T(j)} q_{j'}.$$

In (5.3), the first equality is by definition and the second equality follows from the fact that, as far as variable τ_j is concerned, the contribution from $t_{j'}$ to $\sum_{j=1}^{p} q_j t_j$ is $q_{j'}\tau_j$ if $j' \in T(j)$, but none if $j' \notin T(j)$. It then follows that, for the trigonometric part of the kernel function, we have

$$\prod_{j=1}^{p} e^{i\omega q_j t_j} = \prod_{j=1}^{p} e^{i\omega q_j (\tau_j + \sum_{j' \in P(j)} \tau_{j'})} = \prod_{j=1}^{p} e^{i\omega Q_j \tau_j}. \qquad \square$$

The geometry of the domain of integration of $\hat{\mathcal{T}}_p(0)$ is completely trivialized. Consequently, we are free to rewrite $d\tau_j d\tau_{j'}$ as $d\tau_{j'} d\tau_j$ in $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ for any given pair j, j'. This simplicity is, however, not achieved without a price to pay: the argument of the dynamical part of the integral functions are now convoluted in new variables.

Assumption. For the rest of this section, we assume $Q_p \neq 0$.

5.2. *Moving to the complex plane*

In the rest of this section, we are going to introduce complex variables $z_j = \tau_j + is_j$, starting with $z_p = \tau_p + is_p$, one j at a time, to transform a pure integral that was originally defined on the positive half of the real axis $\tau_j \in [0, +\infty)$, to two integrals; one is defined on an interval of the purely imaginary z_j-axis, and the other is defined on a half line in the complex plane that is parallel to the real axis. By a careful design of integral paths in the complex plane, we move the argument of the integral functions towards the complex singularity of the function $a(z)$ (but keep it at least an ω^{-1}-distance away), in the hope of trading a worsening factor of polynomial power in ω from the dynamical part of the kernel function for an exponentially small factor $e^{-\omega\pi/2}$ from the trigonometric part of the kernel function. This method is motivated by the computations presented in Sect. 4.3.

In between the two integrals, the one that is defined on the purely imaginary axis is passed to the next step, and the other is put aside, which we will collect at the end. For the integral that is passed forward, the variables of integration are turned from τ_j to s_j one at a time. At the end, the integral passed forward in this induction is entirely in s-variables.

We start with a pure integral $\mathcal{T}_{\mathbf{q}}(0)$ in t variables to obtain the corresponding integral $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ in τ variables. What is outlined in the above for $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ is an inductive process of p steps. We use $k = 0, \cdots, p-1$ as the inductive index.

The Initial Step of Induction. To initiate the intended induction, we start by decomposing the integral $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ into two complex valued integrals, which we denote as $\mathbb{T}(0, v)$ and $\mathbb{T}(0, \omega)$.

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By definition, we have

$$\hat{\mathcal{T}}_{\mathbf{q}}(0) = \int\limits_{(0,+\infty)^p} F(\tau)d\tau$$

where

$$F(\tau) = \prod_{j=1}^{p} e^{i\omega Q_j \tau_j} d_j \left(\tau_j + \sum_{j' \in P(j)} \tau_{j'} \right).$$

Since we have a free hand in switching the order of integration in this case, we can re-write this integral as

$$\hat{\mathcal{T}}_{\mathbf{q}}(0) = \int\limits_{(0,+\infty)^{p-1}} \left(\int\limits_{0}^{+\infty} F(\tau)d\tau_p \right) d\hat{\tau}$$

where $d\hat{\tau} = d\tau_1 \cdots d\tau_{p-1}$. Separating the variable τ_p from the rest, we write $F(\tau) = F(\hat{\tau}, \tau_p)$. We regard $\hat{\tau} = (\tau_1, \cdots \tau_{p-1})$ as real parameters and τ_p as a complex variable z_p, which we denote as $z_p = \tau_p + is_p$. Let

$$\mathbb{D}_\ell = \int\limits_{\ell} F(\hat{\tau}, z_p)dz_p$$

where ℓ is a continuous path in the complex z_p plane.

Lemma 5.1. *Under the assumption that $s_p \in [-\pi/2, \pi/2]$ and $\tau_p > 1$, we have*

$$|F(\hat{\tau}, z_p)| \le K(\hat{\tau})e^{-R_p \tau_p} \tag{5.4}$$

where

$$R_p := \sum_{j=1}^{p} (m_1(j) + n_1(j)). \tag{5.5}$$

Proof. By definition,

$$F(\hat{\tau}, \tau_p + is_p) = \prod_{j=1}^{p} e^{i\omega Q_j \tau_j} d_j \left(\tau_j + \sum_{j' \in P(j)\setminus\{p\}} \tau_{j'} + \tau_p + is_p \right).$$

Recall that

$$d_j(t) = b^{m_1}(t)\tilde{H}^{m_2}(t)a^{n_1}(t)A^{n_2}H^{n_3}(t).$$

We also note that N_p is in the ancestry line of all N_j. Denote

$$t_j = \tau_j + \sum_{j' \in P(j)\setminus\{p\}} \tau_{j'} + \tau_p + i s_p.$$

Under the assumption that $s_p \in [-\pi/2, \pi/2]$ and $\tau_p > 1$, we have

$$|b(t_j)| < K_j(\hat{\tau}) e^{-\tau_p}, \quad K_j^{-1}(\hat{\tau}) e^{-\tau_p} < |a(t_j)| < K_j(\hat{\tau}) e^{-\tau_p},$$
$$|A(t_j)|, \ |H(t_j)|, \ |\tilde{H}(t_j)| < K_j(\hat{\tau});$$

and (5.4) follows directly from these estimates. □

Note that $R_p \geq 1$ by Lemma 3.6. This lemma then implies

$$\lim_{\tau_p \to +\infty} \int_{\ell_5} F(\hat{\tau}, z_p) dz_p = 0$$

where

$$\ell_5 = \left\{ z_p = \tau_p + i s_p, \quad s_p \in [-\pi/2 + \omega^{-1}, \ \pi/2 - \omega^{-1}] \right\}.$$

See Fig. 3(b).

Let

$$\ell_0 = \{z_p = \tau_p + i s_p : \ s_p = 0, \ \tau_p \in (0, +\infty)\}$$
$$\ell_v = \{z_p = \tau_p + i s_p : \ \tau_p = 0, \ s_p \in (0, \sigma_p(\pi/2 - \omega^{-1})]\}$$
$$\ell_\omega = \{z_p = \tau_p + i s_p : \ s_p = \sigma_p(\pi/2 - \omega^{-1}), \ \tau_p \in (0, +\infty)\}.$$

Recall that σ_p is the sign of Q_p: $\sigma_p = 1$ if $Q_p > 0$, $\sigma_p = -1$ if $Q_p < 0$. We move up towards $i\pi/2$ if $Q_p > 0$, but move down towards $-i\pi/2$ if $Q_p < 0$.

We have

Lemma 5.2.

$$\mathbb{D}_{\ell_0} = \mathbb{D}_{\ell_v} + \mathbb{D}_{\ell_\omega}.$$

Proof. This is again proved by applying the Cauchy integral theorem. See the proof of Lemma 4.4 in Sect. 4.3 and Fig. 3(b) for the integral paths used. Lemma 5.1 assures that both $\mathbb{D}_{\ell_v}$ and $\mathbb{D}_{\ell_\omega}$ are absolutely convergent. It also assures that the integration on ℓ_5 goes away as we move ℓ_5 towards $+\infty$. The only point of concern left is again the singularity of $b^{m_1}(z)\tilde{H}^{m_2}(z)A^{n_2}(z)$ at $z = 0$. By $n_2 \geq -1$ and $m_1 + m_2 \geq 1$ (Lemma 3.1 and 3.2), this point is a branch point that causes no harm because both $|b(z)A^{-1}(z)|$ and $|\tilde{H}(z)A^{-1}(z)|$ are bounded as $z \to 0$. □

We obtain, by using Lemma 5.2,

$$\begin{aligned}\hat{\mathcal{T}}_{\mathbf{q}}(0) &= \int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} \mathbb{D}_{\ell_0} d\hat{\tau} = \int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} \mathbb{D}_{\ell_v} d\hat{\tau} + \int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} \mathbb{D}_{\ell_\omega} d\hat{\tau} \\ &= i \int_0^{\sigma_p(\pi/2-\omega^{-1})} \left(\int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} F(\hat{\tau}, is_p) d\hat{\tau} \right) ds_p \\ &\quad + \int_0^{+\infty} \left(\int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} F(\hat{\tau}, \tau_p + i\sigma_p(\pi/2-\omega^{-1})) d\hat{\tau} \right) d\tau_p.\end{aligned}$$

By definition,

$$F(\hat{\tau}, \tau_p + i\sigma_p(\pi/2-\omega^{-1})) = e^{-\sigma_p Q_p \omega(\pi/2-\omega^{-1})} \prod_{j=1}^{p} e^{i\omega Q_j \tau_j} d_j(w_j)$$

where

$$w_j = \tau_j + \sum_{j'\in P(j)} \tau_{j'} + i\sigma_p(\pi/2-\omega^{-1}). \tag{5.6}$$

We have

$$\hat{\mathcal{T}}_{\mathbf{q}}(0) = i\mathbb{T}(0, v) + e^{-\sigma_p Q_p \omega(\pi/2-\omega^{-1})} \mathbb{T}(0, \omega)$$

where

$$\mathbb{T}(0, v) = \int_0^{\sigma_p(\pi/2-\omega^{-1})} \left(\int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} F(\hat{\tau}, is_p) d\hat{\tau} \right) ds_p,$$

$$\mathbb{T}(0, \omega) = \int_0^{+\infty} \left(\int_{\hat{\tau}\in\mathbb{R}_+^{p-1}} \prod_{j=1}^{p} e^{i\omega Q_j \tau_j} d_j(w_j) d\hat{\tau} \right) d\tau_p$$

and w_j is as in (5.6). This is the initial step of the intended induction.

The traditional convention of integral signs is used in the above to represent multiple integrals. This is as much as this convention can help us. To present the general step of this induction, we need to switch to the new convention of representing multiple integrals by using structure trees. To make a smooth transition, we give the following description on the structure trees for $\mathbb{T}(0, v)$ and $\mathbb{T}(0, \omega)$.

Description on $\mathbb{T}(0,v)$ *and* $\mathbb{T}(0,\omega)$*:* (a) We obtain $\mathbb{T}(0,v)$ by modifying $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ as follows: (i) in all kernel functions, change τ_p to is_p; and (ii) for the tree node N_p, change the variable of integration from τ_p to s_p; change the interval of integration from $[0,+\infty)$ to $[0,\sigma_p(\pi/2-\omega^{-1})]$.

(b) We obtain $\mathbb{T}(0,\omega)$ by replacing, in the dynamic part of all kernel functions of $\hat{\mathcal{T}}_{\mathbf{q}}(0)$, the variable τ_p by using $\tau_p+i\sigma_p(\pi/2-\omega^{-1})$.

5.3. *Structure tree for* $\mathbb{T}(k,v)$ *and* $\mathbb{T}(k,\omega)$

At k-th step, starting from $k=0$, we denote the integral that is going to be passed forward as $\mathbb{T}(k,v)$, and the integral that is going to be put aside as $\mathbb{T}(k,\omega)$. The detailed definition of $\mathbb{T}(k,v)$ and $\mathbb{T}(k,\omega)$ in this subsection will serve as inductive assumptions. We again modify the memories of all tree nodes in $\mathcal{T}_{\mathbf{q}}(0)$ to define $\mathbb{T}(k,v)$ and $\mathbb{T}(k,\omega)$.

The Structure Tree $\mathbb{T}(k,v)$. We divide all tree nodes into two groups. Group (a) is the top part of the tree. That is, all N_j such that $p-k\le j\le p$. Group (b) is the rest. At step k, integral variables for tree nodes in Group (a) have been converted from τ to s. For N_j, $p-k\le j\le p$, we set the variable of integration as s_j, the interval of integration as

$$I_j=\left[0,\sigma_j(\pi/2-\omega^{-1})-\sum_{j'\in P(j)}s_{j'}\right],$$

the trigonometric part of the kernel function as $e^{-\omega Q_j s_j}$, and the dynamic part of the kernel function as

$$d_j(iS_j)=b^{m_1}(iS_j)\tilde{H}^{m_2}(iS_j)a^{n_1}(iS_j)A^{n_2}(iS_j)H^{n_3}(iS_j)$$

where

$$S_j=s_j+\sum_{j'\in P(j)}s_{j'}. \tag{5.7}$$

Here, S_j is obtained by adding to s_j all variables of integration along the ancestry line of N_j.

The lower bound of I_j is easy to understand: moving from the real axis to the complex plane always starts from $s_j=0$. Recall that σ_j is the sign of Q_j: $\sigma_j=1$ if $Q_j>0$, $\sigma_j=-1$ if $Q_j<0$ and $\sigma_j=0$ if $Q_j=0$. The end of I_j,

$$\sigma_j(\pi/2-\omega^{-1})-\sum_{j'\in P(j)}s_{j'},$$

is a tricky design. It ensures that all S_j are in between $[-\pi/2+\omega^{-1},\pi/2-\omega^{-1}]$ so the complex singularities of $a(z)$, $b(z)$, $A(z)$ and $H(z)$ do not interfere with our intended use of the Cauchy integral theorem. See the upcoming Lemma 5.3.

At this step of the induction, the process of converting τ_j to s_j has not yet reached tree nodes in Group (b). So for N_j, $1\le j<p-k$, we set the variable of integration as τ_j, the interval of integration as $(0,+\infty)$, and the trigonometric part of the kernel function remains $e^{i\omega Q_j\tau_j}$. For the dynamic part of the kernel function, we recall that it was

$$\hat{d}_j(\tau) = b^{m_1}(t_j)\tilde{H}^{m_2}(t_j)a^{n_1}(t_j)A^{n_2}(t_j)H^{n_3}(t_j)$$

in $\hat{\mathcal{T}}_{\mathbf{q}}$ where

$$t_j = \tau_j + \sum_{j' \in P(j)} \tau_{j'}.$$

We now change it to

$$\hat{d}_j(\tau, s) = b^{m_1}(w_j)\tilde{H}^{m_2}(w_j)a^{n_1}(w_j)A^{n_2}(w_j)H^{n_3}(w_j)$$

where w_j is obtained by changing $\tau_{j'}$ in t_j to $is_{j'}$ if $N_{j'}$ is in Group (a). This is to say we have

$$w_j = \tau_j + \sum_{j' \in P(j),\ j' < p-k} \tau_{j'} + i \sum_{j' \in P(j),\ j' \geq p-k} s_{j'}. \tag{5.8}$$

This finishes the definition of $\mathbb{T}(k, v)$.

For the integral represented by $\mathbb{T}(k, v)$, we no longer have a free hand in altering the order of variables of integration. Consequently, it is necessary to fix a specific order, which we define as usual from bottom to top level, and from right to left at the same level of the structure tree. With this integral, however, we still have some freedom: we can divide the integral variables into two groups: the s-group and the τ-group. Let

$$S(k) = \{p, p-1, \cdots, p-k\}.$$

We can write $\mathbb{T}(k, v)$ as

$$\int_{R_s} \left(\int_{\mathbb{R}_+^{p-k-1}} F d\tau \right) dS$$

where

$$dS = ds_{p-k} \cdots ds_p, \qquad d\tau = \prod_{j' \notin S(k)} d\tau_{j'},$$

and

$$R_s = \left\{ s_j \in I_j = \left[0, \sigma_j(\pi/2 - \omega^{-1}) - \sum_{j' \in P(j)} s_{j'} \right],\ j \in S(k) \right\}.$$

We note that the order of $d\tau_{j'}$ in $d\tau$ is still allowed to be arbitrary.

Lemma 5.3. *Assume $s_j \in I_j$ for all $j \in S(k) := \{p, p-1, \cdots, p-k\}$. Let*

$$S_j = s_j + \sum_{j' \in P(j)} s_{j'}.$$

We have, for all $j \in S(k)$,

$$S_j \in \left[-\pi/2 + \omega^{-1},\ \ \pi/2 - \omega^{-1}\right]. \tag{5.9}$$

Proof. From

$$s_j \in \left[0,\ \ \sigma_j(\pi/2 - \omega^{-1}) - \sum_{j' \in P(j)} s_{j'}\right],$$

it follows that

$$S_j \in \left[\sum_{j' \in P(j)} s_{j'},\ \ \sigma_j(\pi/2 - \omega^{-1})\right].$$

Observe that $S_{j''} = \sum_{j' \in P(j)} s_{j'}$ assuming $j \in C(j'')$. We obtain (5.9) for S_j by inductively assuming it for $S_{j''}$. □

The Structure Tree $\mathbb{T}(k, \omega)$. To define $\mathbb{T}(k, \omega)$, we divide all tree nodes into three groups and denote them as Group (a), (b) and (c) respectively. Group (a) is again at the top, but it differs from Group (a) in $\mathbb{T}(k, v)$ by one node: N_{p-k} is excluded. The rest of the tree nodes are divided into Group (b), containing the remaining tree nodes that reside outside of the subtree rooted at N_{p-k}, and Group (c), containing all that reside inside the subtree rooted at N_{p-k}.

A tree node N_j in Group (a) for $\mathbb{T}(k, \omega)$ is identical to the corresponding N_j in $\mathbb{T}(k, v)$ as far as it is not in the ancestry line of N_{p-k}. For N_j that is on the ancestry line of N_{p-k}, all again remain the same as in $\mathbb{T}(k, v)$ except the trigonometric part of the kernel function, which we change from $e^{-\omega Q_j s_j}$ to

$$e^{-\omega(Q_j - Q_{p-k})s_j}$$

for $\mathbb{T}(k, \omega)$.

Tree nodes in Group (b) are identical to their correspondences in $\mathbb{T}(k, v)$. Finally, for N_j in Group (c), that is, a tree node in the subtree rooted at N_{p-k} (including N_{p-k}), all are the same as in Group (b) except the dynamic part of the kernel function, which remains in the form of

$$d_j(w_j) = b^{m_1}(w_j)\tilde{H}^{m_2}(w_j)a^{n_1}(w_j)A^{n_2}(w_j)H^{n_3}(w_j).$$

However, in this case,

$$w_j = \tau_j + \sum_{j' \in P(j) \cap T(p-k)} \tau_{j'} + i\sigma_{p-k}(\pi/2 - \omega^{-1}). \tag{5.10}$$

Comparing this to w_j in Group (b), all s-variables are dropped but we add a constant shift $i\sigma_{p-k}(\pi/2-\omega^{-1})$.

5.4. Decomposition of pure integrals

We complete this induction by moving from step k to step $k+1$.

Proposition 5.2. *We have, for $k \geq 0$,*

$$\mathbb{T}(k,v) = i\mathbb{T}(k+1,v) + e^{-\sigma_{p-k-1}Q_{p-k-1}\omega(\pi/2-\omega^{-1})}\mathbb{T}(k+1,\omega). \tag{5.11}$$

Proof. We start with

$$\mathbb{T}(k,v) = \int_{R_s}\left(\int_{(0,+\infty)^{p-k-1}} F d\tau\right) dS$$

where

$$dS = ds_{p-k}\cdots ds_p, \quad d\tau = d\tau_{p-k-1}\cdots d\tau_1.$$

We have

$$\mathbb{T}(k,v) = \int_{R_s}\int_{(0,+\infty)^{p-k-2}}\left(\int_0^{+\infty} F d\tau_{p-k-1}\right) d\hat{\tau} dS$$

where $\hat{\tau} = (\tau_1,\cdots,\tau_{p-k-2})$. The integral function F is a function of $S,\hat{\tau},\tau_{p-k-1}$, which we write as

$$F = F(S,\hat{\tau},\tau_{p-k-1}).$$

Regarding $S,\hat{\tau}$ as real parameters and letting z_{p-k-1} be a complex variable, we define

$$\mathbb{D}_\ell := \int_\ell F(S,\hat{\tau},z_{p-k-1})dz_{p-k-1}$$

where ℓ is a continuous path in the complex z_{p-k-1}-plane, $z_{p-k-1} = \tau_{p-k-1} + is_{p-k-1}$. Let

$$\ell_0 = \{\tau_{p-k-1} \in (0,+\infty), \quad s_{p-k-1} = 0\}$$

$$\ell_v = \left\{\tau_{p-k-1} = 0,\ s_{p-k-1} \in \left(0, \sigma_{p-k-1}(\pi/2-\omega^{-1}) - \sum_{j'\in P(p-k-1)} s_{j'}\right]\right\}$$

$$\ell_\omega = \left\{\tau_{p-k-1} \in (0,+\infty),\ \ s_{p-k-1} = \sigma_{p-k-1}(\pi/2-\omega^{-1}) - \sum_{j'\in P(p-k-1)} s_{j'}\right\}.$$

We have

$$\mathbb{D}_{\ell_0} = \mathbb{D}_{\ell_v} + \mathbb{D}_{\ell_\omega}. \tag{5.12}$$

Note that the Cauchy integral theorem applies here because Lemma 5.3 assures that on the complex region bounded by ℓ_0, ℓ_v, and ℓ_ω, the dynamic part of the kernel functions are free of singularities. To obtain (5.12), we also need to assure that

$$\lim_{\tau_{p-k-1}\to+\infty} \mathbb{D}_{\ell_5} = 0$$

where ℓ_5 is as in Fig. 3(b). This again follows from

$$|F(S, \hat{\tau}, z_{p-k-1})| < K(\hat{\tau}, \omega)e^{-R_{p-k-1}\tau_{p-k-1}},$$

the proof of which is similar to that of Lemma 5.1, and $R_{p-k-1} \geq 1$ by Lemma 3.6. Also note that by the same reason as presented in Lemma 4.4, the branch point of the function $b^{m_1}(z)\tilde{H}^{m_2}(z)A^{n_2}(z),\ n_2 \geq -1$ at $z = 0$ causes no problem.

It then follows that

$$\mathbb{T}(k, v) = \int_{R_s} \int_{(0,+\infty)^{p-k-2}} \left(\mathbb{D}_{\ell_0}\right) d\hat{\tau} dS = \mathcal{T}_1 + \mathcal{T}_2$$

where

$$\mathcal{T}_1 = \int_{R_s} \int_{(0,+\infty)^{p-k-2}} \left(\mathbb{D}_{\ell_v}\right) d\hat{\tau} dS \quad \mathcal{T}_2 = \int_{R_s} \int_{(0,+\infty)^{p-k-2}} \left(\mathbb{D}_{\ell_\omega}\right) d\hat{\tau} dS.$$

On ℓ_v, we have $dz_{p-k-1} = ids_{p-k-1}$. To obtain $\mathbb{D}_{\ell_v}$, we replace τ_{p-k-1} by using is_{p-k-1} in $\mathbb{D}_\ell$. Let

$$\mathcal{T}_1 = i\mathbb{T}(k+1, v)$$

where i comes out of $d\tau_{p-k-1} = ids_{p-k-1}$. We obtain $\mathbb{T}(k+1, v)$ by revising $\mathbb{T}(k, v)$ so that the node N_{p-k-1} is switched from Group (b) to Group (a).

We move on to revise the structure tree $\mathbb{T}(k, v)$ to define $\mathcal{T}_2$. We start with N_{p-k-1}. We have on ℓ_ω

$$z_{p-k-1} = \tau_{p-k-1} + i\left(\sigma_{p-k-1}(\pi/2 - \omega^{-1}) - \sum_{j'\in P(p-k-1)} s_{j'}\right)$$

where τ_{p-k-1} remains the integral variable, and the interval of integration remains $(0, +\infty)$. We also note that

$$\begin{aligned} e^{iQ_{p-k-1}\omega z_{p-k-1}} &= e^{iQ_{p-k-1}\omega\{\tau_{p-k-1}+i[\sigma_{p-k-1}(\pi/2-\omega^{-1})-\sum_{j'\in P(p-k-1)} s_{j'})]\}} \\ &= e^{-\sigma_{p-k-1}Q_{p-k-1}\omega(\pi/2-\omega^{-1})} \cdot e^{Q_{p-k-1}\omega\sum_{j'\in P(p-k-1)} s_{j'}} \cdot e^{iQ_{p-k-1}\omega\tau_{p-k-1}}. \end{aligned}$$

The first factor is a constant, which we take out. The second factor, we push upward along the ancestry path of N_{p-k-1}, multiplying $e^{Q_{p-k-1}\omega s_{j'}}$ to the trigonometric part of the kernel function of $N_{j'}$ for all $j' \in P(p-k-1)$. This is to say that we duly revise all $N_{j'}$, $j' \in S(k) \cap P(p-k-1)$ by multiplying $e^{Q_{p-k-1}\omega s_{j'}}$ to the trigonometric part of the kernel function of $N_{j'}$. Finally, the third factor we keep as the trigonometric kernel of N_{p-k-1}. For the dynamic part of the kernel function, observe that the new argument of this function is

$$z_{p-k-1} + i \sum_{j' \in P(p-k-1)} s_{j'} = \tau_{p-k-1} + i\sigma_{p-k-1}(\pi/2 - \omega^{-1}).$$

The argument of the dynamical part of the kernel function for the entire subtree rooted at N_{p-k-1} is duly revised: for $j \in T(p-k-1)$, the new argument is

$$\tau_j + \sum_{j' \in P(j) \setminus (S(k) \cup \{p-k-1\})} \tau_{j'} + z_{p-k-1} + i \sum_{j' \in P(j) \cap S(k)} s_{j'},$$

which is, by using $z_{p-k-1} = \tau_{p-k-1} + i[\sigma_{p-k-1}(\pi/2 - \omega^{-1}) - \sum_{j' \in P(p-k-1)} s_{j'})]$, to become

$$\tau_j + \sum_{j' \in P(j) \setminus S(k)} \tau_{j'} + i\sigma_{p-k-1}(\pi/2 - \omega^{-1}).$$

With these revisions, we turn $\mathbb{T}(k, v)$ to $\mathbb{T}(k+1, \omega)$. $\square$

By recursively using (5.11), we obtain

Proposition 5.3. *Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral. Then*

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) + \sum_{k=0}^{p-1} i^k e^{-\sigma_{p-k} Q_{p-k}\omega(\pi/2-\omega^{-1})} \mathbb{T}(k, \omega). \tag{5.13}$$

6. Exponentially small splitting of separatrix

Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral of order p. In Section 5, we first introduced τ variables to convert the integral domain for $\mathcal{T}_{\mathbf{q}}(0)$ to $[0, +\infty)^p$. We then moved to complex domain to convert τ variables to s variables one by one by using the Cauchy integral theorem to obtain (5.13). In Section 5, we have worked exclusively on $\mathcal{T}_{\mathbf{q}}(0)$ with a fixed $\mathbf{q} \in \mathcal{S}$. This is now no longer the case. We start by making the dependency of $\mathbb{T}(k, v)$ and $\mathbb{T}(k, \omega)$ on $\mathbf{q}$ explicit and re-write (5.13) as

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}_{\mathbf{q}}(p-1, v) + \sum_{k=0}^{p-1} i^k e^{-\sigma_{p-k} Q_{p-k}\omega(\pi/2-\omega^{-1})} \mathbb{T}_{\mathbf{q}}(k, \omega). \tag{6.1}$$

There are three kind of terms on the right hand side of (6.1). The first kind is all that are induced by subtrees of a non-zero total index. For these terms, we have

Proposition 6.1. *There exists a constant $K > 1$ so that we have, under the assumption that $Q_p \neq 0$ and $Q_{p-k} \neq 0$,*

$$|e^{-\sigma_{p-k} Q_{p-k} \omega(\pi/2-\omega^{-1})} \mathbb{T}_{\mathbf{q}}(k, \omega)| \leq (3\pi)^k (K\omega)^{27p} e^{-\omega\pi/2}.$$

The proof of this proposition is placed in Part 3 as Section 8. In the rest of this paper we call the terms of this kind *exponentially small terms*.

The second kind is the first term $i^p \mathbb{T}_{\mathbf{q}}(p-1, v)$. This term is an integral completely in s-variables. The third kind is all that are induced by zero-subtrees. Integrals of the second and the third kind are *not* exponentially small in size.

6.1. Cancellations based on symmetry

The first term on the right hand is completely in s variables. We assert that, as far as $\hat{\mathcal{T}}_{\mathbf{q}}(0) - \hat{\mathcal{T}}_{-\mathbf{q}}(0)$ is concerned, this term is canceled by its dual $\mathbb{T}_{-\mathbf{q}}(p-1, v)$ in $\hat{\mathcal{T}}_{-\mathbf{q}}(0)$ because of the symmetry of Lemma 3.3.

Lemma 6.1. *We have $\mathbb{T}_{\mathbf{q}}(p-1, v) = \mathbb{T}_{-\mathbf{q}}(p-1, v)$ assuming $Q_p \neq 0$.*

Proof. Note that both $\mathbb{T}_{\mathbf{q}}(p-1, v)$ and $\mathbb{T}_{-\mathbf{q}}(p-1, v)$ are completely in s variables. To prove this lemma, we let $\hat{s}_j = -s_j$ to convert $\mathbb{T}_{\mathbf{q}}(p-1, v)$ to $\mathbb{T}_{-\mathbf{q}}(p-1, v)$. Let Q_j for $\mathbf{q}$ be denoted as $Q_j(\mathbf{q})$, and σ_j for $\mathbf{q}$ be denoted as $\sigma_j(\mathbf{q})$. We have

$$Q_j(\mathbf{q}) = -Q_j(-\mathbf{q}); \quad \sigma_j(\mathbf{q}) = -\sigma_j(-\mathbf{q}).$$

By adopting $\hat{s}_j = -s_j$ as the new integral variable for N_j, we are obliged to change the corresponding interval of integration to

$$-\left[0,\ \sigma_j(\mathbf{q})(\pi/2-\omega^{-1}) - \sum_{j' \in P(j)} s_j\right] = \left[0,\ \sigma_j(-\mathbf{q})(\pi/2-\omega^{-1}) - \sum_{j' \in P(j)} \hat{s}_j\right].$$

We also rewrite the trigonometric part of the kernel function in $\hat{s}$ as

$$e^{-\omega Q_j(\mathbf{q}) s_j} = e^{-\omega Q_j(-\mathbf{q}) \hat{s}_j};$$

and the dynamic part in $\hat{s}$, by using Lemma 3.3, as

$$d_j(iS_j) = -d_j(i\hat{S}).$$

In addition we observe that the negative sign in the last equality is nullified because $ds_j = -d\hat{s}_j$. □

The terms induced by zero-subtrees on the right hand side of (6.1), unfortunately, are much harder to deal with. Assuming k_0 is such that $Q_{p-k_0} = 0$, this term is

$$i^{k_0} \mathbb{T}_{\mathbf{q}}(k_0, \omega).$$

The corresponding term in $\mathcal{T}_{-\mathbf{q}}(0)$ is

$$i^{k_0}\mathbb{T}_{-\mathbf{q}}(k_0,\omega).$$

Our problem is that (a) these two terms are not exponentially small in size, and (b) unlike $\mathbb{T}_{\mathbf{q}}(p-1,v)$ and $\mathbb{T}_{-\mathbf{q}}(p-1,v)$, the non-exponentially small part of these two terms *do not* cancel each other in $\mathcal{T}_{\mathbf{q}}(0)-\mathcal{T}_{-\mathbf{q}}(0)$.

We, however, do not have to prove that $\mathcal{T}_{\mathbf{q}}(0)-\mathcal{T}_{-\mathbf{q}}(0)$ is exponentially small in size for a fixed $\mathbf{q}$. Recall that we have

$$T^s(0)-T^u(0)=\frac{1}{2^{\hat{p}}}\sum_{\mathbf{q}\in\mathcal{S}^+}2i\sin\omega Q_{\mathbf{q}}t_0\left(\mathcal{T}^s_{\mathbf{q}}-\mathcal{T}^s_{-\mathbf{q}}\right). \tag{6.2}$$

This equality is (4.7) in Sect. 4.2. To make the terms induced by zero-subtrees (at least the non-exponentially small part of these terms) go away in $T^s(0)-T^u(0)$, we need to enlist the help of the terms

$$\sin\omega Q_{\mathbf{q}}t_0\left(\mathcal{T}^s_{\mathbf{q}}-\mathcal{T}^s_{-\mathbf{q}}\right)$$

for different $\mathbf{q}\in\mathcal{S}^+$.

Let us now make clear what other $\mathbf{q}\in\mathcal{S}^+$ are needed to cancel the non-exponentially small part of $\mathcal{T}_{\mathbf{q}}(0)-\mathcal{T}_{-\mathbf{q}}(0)$. Denote $\mathbf{q}=(q_p,\cdots,q_1)$. For a given $\mathbf{q}\in\mathcal{S}^+$, we define *basic* $\mathbf{q}$*-block* as follows. First, we count $\mathbf{q}_R$, the part of $\mathbf{q}$ that defines the remainder tree obtained by deleting all zero subtrees in $\mathcal{T}_{\mathbf{q}}(0)$, as one basic $\mathbf{q}$-block. We also count the part of $\mathbf{q}$ that defines a zero subtree $\mathcal{T}_k$ as a basic $\mathbf{q}$-block if $\mathcal{T}_k$ contains no other zero subtree. Finally, if a zero subtree $\mathcal{T}_k$ contains other zero subtree, we take the part of $\mathbf{q}$ that defines the remainder tree of $\mathcal{T}_k$ obtained by deleting all zero subtrees in $\mathcal{T}_k$ as a basic $\mathbf{q}$-block.

Lemma 6.2. *Under the assumption that $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees, we have in total, counting $\mathbf{q}_R$, $n+1$ basic $\mathbf{q}$-block which we denote as*

$$\mathbf{q}_R,\ \mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_n.$$

These are, by definition, mutually disjoint subsets of $\mathbf{q}$ and we have

$$\mathbf{q}=\mathbf{q}_R\cup\mathbf{q}_1\cup\cdots\cup\mathbf{q}_n.$$

Proof. Let q_j be the component of $\mathbf{q}$ for N_j. Starting from N_j, we go backward along the ancestry line of N_j. If there is no root node of zero subtree on this ancestry line, then q_j is a component of $\mathbf{q}_R$. Otherwise, q_j is in the basic $\mathbf{q}$-block defined by the first root node of a zero subtree encountered on the way. □

For a given $\mathbf{q}\in\mathcal{S}^+_{\mathbf{q}}$, let

$$\mathbb{S}_{\mathbf{q}}=\{\hat{\mathbf{q}}=\mathbf{q}_R\cup\pm\mathbf{q}_1\cup\cdots\cup\pm\mathbf{q}_n\} \tag{6.3}$$

where $\pm$ is either $+$ or $-$ if the basic $\mathbf{q}$-block it marks is not a zero vector. We set $\pm=0$ if the basic $\mathbf{q}$-block it marks is a zero vector.

Lemma 6.3. *We have (i) for all* $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$, $Q_{\hat{\mathbf{q}}} = Q_{\mathbf{q}}$; *and (ii) if* $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$, *then* $\mathbb{S}_{\hat{\mathbf{q}}} = \mathbb{S}_{\mathbf{q}}$.

Proof. Item (i) holds because the total index of all basic blocks are zero except that of $\mathbf{q}_R$, the total index of which is the total index of the entire $\mathbf{q}$. Item (ii) follows directly from (6.3). □

Let

$$D_{\mathbb{S}_{\mathbf{q}}} = \sum_{\hat{q} \in \mathbb{S}_{\mathbf{q}}} (T_{\hat{\mathbf{q}}}(0) - T_{-\hat{\mathbf{q}}}(0)).$$

We have

Proposition 6.2. *There exists a constant* $K > 0$ *so that*

$$\left| \mathbb{D}_{\mathbb{S}_{\mathbf{q}}} \right| \leq (K\omega)^{27p} e^{-\omega\pi/2}.$$

The proof of this proposition is rather involved. It is placed in Part 3 as Section 9.

6.2. *Proof of the Main Theorem*

In this subsection, we pull the various things we have done so far into one place to prove the Main Theorem.

Let $\ell = (a(t), b(t))$ be the homoclinic solution of the unperturbed equation (1.1) satisfying $b(0) = 0$ and D_ℓ be a small neighborhood of ℓ. For a $t_0 \in [0, 2\pi\omega^{-1})$, there exists a primary stable solution $(x^s(t, \omega, \varepsilon), y^s(t, \omega, \varepsilon))$ of the perturbed equation (1.1) in D_ℓ satisfying $y^s(t_0, \omega, \varepsilon) = 0$. We started by introducing a sequence of new variables in Section 2 to solve for $x^s(t_0, \omega, \varepsilon)$. Our conclusion is that, expanding $x^s(t_0, \omega, \varepsilon)$ into a formal power series in ε as

$$x^s(t_0, \omega, \varepsilon) = \varepsilon x_0(t_0, \omega) + \cdots + \varepsilon^{n+1} x_n^s(t_0, \omega) + \cdots,$$

we can write, for all $n \geq 0$, $x_n^s(t_0, \omega)$ as a linear combination of a collection $\Lambda_{M,n}$ of certain well-structured multiple integrals, which we named *high order Melnikov integrals*. We have

$$x_n^s(t_0, \omega) = \sum_{\mathcal{T} \in \Lambda_{M,n}} c_{\mathcal{T}} \mathcal{T}.$$

This is (3.20) in Proposition 3.1.

A high order Melnikov integral is defined by the help of a structure tree. The details on how to define a high order Melnikov integral are presented in Sect. 3.2. We use $\mathcal{T}$ to represent both a structure tree and the multiple integral it helps define and assume that $\mathcal{T}$ has p tree nodes, which we index from the bottom level to the top level, and at the same level, from the right to the left of the tree, as $N_1, \cdots, N_p$. The following are stored in N_j (Sect. 3.2).

(i) A node type, which is either an M or a W.
(ii) An integral variable, which we denote as t_j.

(iii) An interval of integration I_j. We have $I_j = (t_{j'}, +\infty)$ for an M-node and $I_j = (0, t_{j'})$ for a W-node where j' is such that N_j is directly branched out of $N_{j'}$. For $j = p$ we let $I_p = (0, +\infty)$.

(iv) A kernel function in the form of

$$\hat{f}(t_j) = g_j(t_j)d_j(t_j)$$

where

$$g_j(t_j) = \cos^{n_0} \omega(t_j + t_0)$$

is the *trigonometric part* of the kernel function and

$$d_j(t_j) = b^{m_1}(t_j)\tilde{H}^{m_2}(t_j)a^{n_1}(t_j)A^{n_2}(t_j)H^{n_3}(t_j)$$

is the *dynamic part* of the kernel function. Note that n_0 in $g_j(t_j)$ is either 0 or 1.

We use $\mathcal{T}$ to represent both the structure tree and the multiple integral it defines. The functions $a(t), b(t), H(t), \tilde{H}(t), A(t)$ are all explicitly defined, and $a(t)$, $H(t)$ and $A(t)$ are even functions, while $b(t)$ and $\tilde{H}(t)$ are odd. See Lemma 2.1. For the dynamic part of the kernel function we have

$$d_j(t_j) = -d_j(-t_j). \tag{6.4}$$

This is Lemma 3.3.

In parallel, we also have a primary unstable solution $(x^u(t, \varepsilon, \omega), y^u(t, \varepsilon, \omega))$ in D_ℓ satisfying $y^u(t_0, \varepsilon, \omega) = 0$. We use superscripts s and u to distinguish the corresponding quantities for primary stable and for primary unstable solutions. Every $\mathcal{T} \in \Lambda_{M,n}$ for the primary stable solution, which we also write as $\mathcal{T}^s \in \Lambda^s_{M,n}$, has a dual for the primary unstable solution, which we denote as $\mathcal{T}^u$. To convert $\mathcal{T}^s$ to $\mathcal{T}^u$, we change $I_j = (t_{j'}, +\infty)$ for all M-nodes in $\mathcal{T}^s$ to $(t_{j'}, -\infty)$. We expand $x^u(t_0, \omega, \varepsilon)$ into a formal power series in ε as

$$x^u(t_0, \omega, \varepsilon) = \varepsilon x^u_0(t_0, \omega) + \cdots + \varepsilon^{n+1} x^u_n(t_0, \omega) + \cdots,$$

to obtain

$$x^u_n(t_0, \omega) = \sum_{\mathcal{T} \in \Lambda_{M,n}} c_{\mathcal{T}} \mathcal{T}^u.$$

Finally, let

$$D(t_0, \omega, \varepsilon) = x^s(t_0, \omega, \varepsilon) - x^u(t_0.\omega, \varepsilon) \tag{6.5}$$

be the *splitting distance*. We write

$$D(t_0, \omega, \varepsilon) = \varepsilon D_0(t_0, \omega) + \varepsilon^2 D_1(t_0, \omega) + \cdots + \varepsilon^{n+1} D_n(t_0, \omega) + \cdots \tag{6.6}$$

to obtain

$$D_n(t_0,\omega)=\sum_{\mathcal{T}\in\Lambda_{M,n}} c_{\mathcal{T}}\left(\mathcal{T}^s-\mathcal{T}^u\right). \tag{6.7}$$

The first step in manipulating a given integral $\mathcal{T}$ is to use

$$\int_0^t=\int_0^\infty-\int_t^\infty$$

to split $\mathcal{T}$ into a collection $\mathcal{E}(\mathcal{T})$ of *extended integrals*, each of which we denote as $\tilde{\mathcal{T}}$. To obtain an extended integral $\tilde{\mathcal{T}}$ from $\mathcal{T}$, we reassign all W node in $\mathcal{T}$ as either a W_1 node or a W_2 node. We then change the interval of integration from $[0,t_{j'}]$ to $[0,\infty)$ for a W_1 node but to $[t_{j'},\infty)$ for a W_2 node. We have

$$\mathcal{T}=\sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})}(-1)^{w(\tilde{\mathcal{T}})}\tilde{\mathcal{T}} \tag{6.8}$$

where $w(\tilde{\mathcal{T}})$ is the number of W_2 nodes in $\tilde{\mathcal{T}}$. This is Lemma 4.1(c). We combine (6.7) and (6.8) to obtain

$$D_n(t_0,\omega)=\sum_{\mathcal{T}\in\Lambda_{M,n}} c_{\mathcal{T}}\sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})}(-1)^{w(\tilde{\mathcal{T}})}\left(\tilde{\mathcal{T}}^s-\tilde{\mathcal{T}}^u\right). \tag{6.9}$$

Assume an extended integral $\tilde{\mathcal{T}}$ has m W_1 nodes, which we denote as

$$N_{j_1},\cdots,N_{j_m}.$$

We also denote the subtree rooted at N_{j_k} as $\mathcal{T}_{j_k}$. The **pure blocks** of $\tilde{\mathcal{T}}$ are defined as follows: (i) the remainder tree obtained by deleting all W_1-subtrees from $\tilde{\mathcal{T}}$ is a pure block; (ii) a W_1 subtree $\mathcal{T}_{j_k}$ is a pure block if it contains no smaller subtree of W_1-type; and (iii) if $\mathcal{T}_{j_k}$ contains a W_1 subtree inside, then the remainder tree obtained by deleting all W_1 subtree it contains is a pure block (Definition 4.2). An extended integral $\tilde{\mathcal{T}}$ with m W_1 nodes would have $m+1$ pure blocks, which we denote as $\mathcal{B}_1,\cdots,\mathcal{B}_{m+1}$. We have, in addition, (Lemma 4.2)

$$\tilde{\mathcal{T}}=\mathcal{B}_1\cdots\mathcal{B}_{m+1}.$$

We continue to write

$$D_n(t_0,\omega)=\sum_{\mathcal{T}\in\Lambda_{M,n}} c_{\mathcal{T}}\sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})}(-1)^{w(\tilde{\mathcal{T}})}\left(\mathcal{B}_1^s\cdots\mathcal{B}_{m+1}^s-\mathcal{B}_1^u\cdots\mathcal{B}_{m+1}^u\right).$$

For $0\le k\le m+1$, let

$$\mathbb{B}_k:=\left[\mathcal{B}_1^s\cdots\mathcal{B}_k^s\right]\cdot\left[\mathcal{B}_{k+1}^u\cdot\mathcal{B}_{m+1}^u\right].$$

We have

$$\begin{aligned}\tilde{\mathcal{T}}^s - \tilde{\mathcal{T}}^u &= \mathcal{B}_1^s \cdots \mathcal{B}_{m+1}^s - \mathcal{B}_1^u \cdots \mathcal{B}_{m+1}^u \\ &= \mathbb{B}_{m+1} - \mathbb{B}_0 \\ &= \sum_{k=0}^{m} (\mathbb{B}_{k+1} - \mathbb{B}_k) \\ &= \sum_{k=0}^{m} \left[\mathcal{B}_1^s \cdots \mathcal{B}_k^s\right]\left[\mathcal{B}_{k+2}^u \cdot \mathcal{B}_{m+1}^u\right]\left(\mathcal{B}_{k+1}^s - \mathcal{B}_{k+1}^u\right).\end{aligned}$$

This is to imply

$$D_n(t_0, \omega) = \sum_{\mathcal{T} \in \Lambda_{M,n}} c_{\mathcal{T}} \sum_{\tilde{\mathcal{T}} \in \mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \sum_{k=0}^{m} \left[\mathcal{B}_1^s \cdots \mathcal{B}_k^s\right] \cdot \left[\mathcal{B}_{k+2}^u \cdot \mathcal{B}_{m+1}^u\right]\left(\mathcal{B}_{k+1}^s - \mathcal{B}_{k+1}^u\right). \quad (6.10)$$

From Lemma 3.5, we have

$$\left|\mathcal{B}_1^s \cdots \mathcal{B}_k^s \cdot \mathcal{B}_{k+2}^u \cdot \mathcal{B}_{m+1}^u\right| < K^p. \quad (6.11)$$

See also Lemma 4.1(a). To prove Main Theorem (b), we now focus on extracting an exponentially small factor $e^{-\omega\pi/2}$ out of $\mathcal{B}^s - \mathcal{B}^u$ for a pure block $\mathcal{B}$. We also call a pure block a *pure integral*.

Let $\mathcal{B}$ be a pure integral of order p for stable solutions. We re-index all tree nodes of $\mathcal{B}$ from bottom level to top level, and from right to left, as $N_1, \cdots N_p$. Recall that the trigonometric part of the kernel function for N_j is

$$g_j(t_j) = \cos^{n_0(j)} \omega(t_j + t_0)$$

where $n_0(j)$ is either 0 or 1. If $n_0(j) = 0$ for all j, then $g_j(t_j) = 1$ for all j. In this case, $\mathcal{B}^s - \mathcal{B}^u = 0$ by the symmetry prescribed in (6.4). This is a degenerate case in which $\mathcal{B}^s$ cancels $\mathcal{B}^u$.

We assume, from this point on, that there exists at least one $n_0(j) \neq 0$. Let

$$\mathcal{S} = \{\mathbf{q} = (q_p, \cdots, q_1), \quad q_j \in \{-n_0(j), n_0(j)\}\}.$$

For $\mathbf{q} \in \mathcal{S}$, we define $\mathcal{B}_{\mathbf{q}}$, a *high order Melnikov integral in complex form*, by changing, in all $N_j \in \mathcal{B}$, the trigonometric part of the kernel function from $\cos^{n_0} \omega(t_j + t_0)$ to

$$e^{i\omega q_j t_j}.$$

We have

$$\mathcal{B}^s - \mathcal{B}^u = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q} \in \mathcal{S}} e^{i\omega Q_{\mathbf{q}} t_0} \left(\mathcal{B}_{\mathbf{q}}^s - \mathcal{B}_{\mathbf{q}}^u\right)$$

where

$$\hat{p} = \sum_{j=1}^{p} n_0(j), \qquad Q_{\mathbf{q}} = \sum_{j=1}^{p} q_j.$$

We name $Q_{\mathbf{q}}$ *the total index* of the integral $\mathcal{B}_{\mathbf{q}}$. This is Lemma 4.3.

Symmetry (6.4) is then used to prove Proposition 4.1, the conclusion of which is as follows. Let $\mathcal{S}^+$ be the subset of $\mathcal{S}$ so that $Q_{\mathbf{q}} > 0$. We have,

$$\mathcal{B}^s - \mathcal{B}^u = \frac{1}{2^{\hat{p}}} \sum_{\mathbf{q}\in\mathcal{S}^+} 2i \sin \omega Q_{\mathbf{q}} t_0 \left(\mathcal{B}^s_{\mathbf{q}} - \mathcal{B}^s_{-\mathbf{q}}\right). \tag{6.12}$$

This is (4.7). Observe that only $\mathcal{B}_{\mathbf{q}}$ satisfying

$$Q_{\mathbf{q}} \neq 0 \tag{6.13}$$

is on the right hand side of (6.12).

For a pure integral $\mathcal{B}_{\mathbf{q}}$ satisfying (6.13), we use $\mathcal{T}_j$ to denote the subtree rooted at N_j in $\mathcal{B}_{\mathbf{q}}$, and denote the sum of all components of $\mathbf{q}$ that defined $\mathcal{T}_j$ as Q_j. We call Q_j the total index of the subtree $\mathcal{T}_j$. A subtree is a **zero subtree** if its total index is zero. (Definition 4.4 in Sect. 4.1).

Assume that $\mathcal{B}_{\mathbf{q}}$ has m zero subtrees, which we denote as

$$\mathcal{T}_{j_1}, \ \cdots, \ \mathcal{T}_{j_m}.$$

We defined *basic blocks* for $\mathcal{B}_{\mathbf{q}}$ as follows: (1) the remainder tree obtained by dropping all zero subtrees from $\mathcal{B}_{\mathbf{q}}$ is a basic block; (2) a zero subtree that contains no zero subtree inside is a basic block; and (3) a remainder tree of a zero subtree obtained by dropping all zero subtrees it contains is a basic block. Every basic block is associated with a $\mathbf{q}$-vector, which we name a basic $\mathbf{q}$-block. In total, we have $m+1$ basic $\mathbf{q}$-block. The first is associated to N_p, which we denote as $\mathbf{q}_R$, and the rest are associated to the root node of the zero subtrees, which we denote as $\mathbf{q}_k$ for the one associated to N_{j_k}. We have (Lemma 6.2)

$$\mathbf{q} = \mathbf{q}_R \cup \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_m.$$

For a given $\mathbf{q} \in \mathcal{S}^+$, let

$$\mathbb{S}_{\mathbf{q}} = \{\hat{\mathbf{q}} \in \mathcal{S}^+, \ \ \hat{\mathbf{q}} = \mathbf{q}_R \cup \pm\mathbf{q}_1 \cup \cdots \cup \pm\mathbf{q}_m\}$$

where $\pm$ is either $+$ or $-$. By definition, we have

$$Q_{\hat{\mathbf{q}}} = Q_{\mathbf{q}}$$

for all $\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}$. We also observe that $\mathcal{S}^+$ is a union of mutually disjoint $\mathbb{S}_{\mathbf{q}}$. Regarding every $\mathbb{S}_{\mathbf{q}}$ as one element, we obtain a quotient set, which we denote as $\mathcal{S}^+/\sim$. This allows us to re-write (6.12) as

$$\mathcal{B}^s - \mathcal{B}^u = \frac{1}{2^{\hat{p}}} \sum_{\mathcal{S}^+/\sim} 2i \sin \omega Q_{\mathbf{q}} t_0 \cdot D_{\mathbb{S}_{\mathbf{q}}} \tag{6.14}$$

where

$$D_{\mathbb{S}_{\mathbf{q}}} = \sum_{\hat{q}\in\mathbb{S}_{\mathbf{q}}} (\mathcal{B}^s_{\hat{\mathbf{q}}} - \mathcal{B}^s_{-\hat{\mathbf{q}}}). \tag{6.15}$$

Putting (6.10), (6.14), (6.15) together, we have

$$\begin{aligned} D_n(t_0,\omega) = &\sum_{\mathcal{T}\in\Lambda_{M,n}} c_{\mathcal{T}} \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} (-1)^{w(\tilde{\mathcal{T}})} \sum_{k=0}^{m} \left[\mathcal{B}^s_1\cdots\mathcal{B}^s_k\right] \\ &\cdot \left[\mathcal{B}^u_{k+2}\cdot\mathcal{B}^u_{m+1}\right] \left(\frac{1}{2^{\hat{p}}} \sum_{\mathcal{S}^+/\sim} 2i\sin\omega Q_{\mathbf{q}} t_0 \cdot D_{\mathbb{S}_{\mathbf{q}}}\right)_{(k+1)} \end{aligned} \tag{6.16}$$

where the subscript $(k+1)$ is to indicate that the bracketed quantity is from $\mathcal{B}_{k+1}$. We also need the conclusion of Proposition 6.2. That is,

$$\left|D_{\mathbb{S}_{\mathbf{q}}}\right| < (K\omega)^{27p} e^{-\omega\pi/2}. \tag{6.17}$$

The proof of this estimate is placed in Section 9.

Proof of Main Theorem (a). To prove Main Theorem (a), we combine Proposition 4.1 with $|Q_{\mathbf{q}}| \le q$ and $q \le 4(n+1)$ (Proposition 3.1(1)). □

Proof of Main Theorem (b). We have, from (6.16),

$$\begin{aligned} |A_{k,n}| &\le K^p \sum_{\mathcal{T}\in\Lambda_{M,n}} \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} \sum_{k=0}^{m} \left(\sum_{\mathcal{S}^+/\sim} \left|D_{\mathbb{S}_{\mathbf{q}}}\right|\right)_{(k+1)} \\ &\le K^p \sum_{\mathcal{T}\in\Lambda_{M,n}} \sum_{\tilde{\mathcal{T}}\in\mathcal{E}(\mathcal{T})} \sum_{k=0}^{m} \sum_{\mathcal{S}^+/\sim} (K\omega)^{27p} e^{-\omega\pi/2} \end{aligned}$$

where for the first inequality, we used $|c_{\mathcal{T}}| < K^p$ (Proposition 3.1(3)) and $\left|\mathcal{B}^s_1\cdots\mathcal{B}^s_k \cdot \mathcal{B}^u_{k+2}\cdot\mathcal{B}^u_{m+1}\right| < K^p$ (see (6.11)). For the second inequality, we used (6.17). We now count the number of terms implied by the four summations. The count for the inner most summation is $\le 2^p$ (because the cardinality of the set $\mathcal{S}$ is $\le 2^p$) and so is that for the summation over the set of extended integrals. The count for the summation from $k=0$ to m is bounded by a simple factor p, and the count for the summation over $\Lambda_{M,n}$ is bounded by K^p (Proposition 3.1(2)). In conclusion, we have

$$|A_{k,n}| < K^p (K\omega)^{27p} e^{-\omega\pi/2} < (\mathbb{K}\omega)^{27p} e^{-\omega\pi/2} \le \omega^{\kappa_0(n+1)} e^{-\omega\pi/2}$$

for all $\omega > \omega_0$ assuming $\omega_0 > \mathbb{K}$, $\kappa_0 = 8\times 27$. The estimate $p < 4(n+1)$ (Proposition 3.1(1)) is again used for the last inequality. This proves Main Theorem (b). □

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Part 3. Combinatoric proofs

7. Proof of Proposition 3.1

In this section, we prove Proposition 3.1. First, we prove that $\mathbb{M}_n$ and $\mathbb{W}_n$ are linear combinations of high order Melnikov integrals. After that there are three more things we need to estimate. The first is the *order* (the multiplicity) of all multiple integrals in $\mathbb{M}_n$ and $\mathbb{W}_n$ for a given n; the second is the number of multiple integrals in $\mathbb{M}_n$ and $\mathbb{W}_n$; the third is the magnitude of the combinational constants in viewing $\mathbb{M}_n$ and $\mathbb{W}_n$ as linear combinations of high order Melnikov integrals. Here, we are compelled to deal with rather involved combinatoric counting.

We inductively assume (3.20) for $\mathbb{M}_k$, $\mathbb{W}_k$ for all $k \le n$ to prove (3.20) for $\mathbb{M}_{n+1}$, $\mathbb{W}_{n+1}$. For $\mathbb{M}_{n+1}$, we re-write (3.4) as

$$\begin{aligned}\mathbb{M}(t) = \mathbb{M}_0(t) + \varepsilon^{-1} \sum_{f\in\mathcal{K}_M, n_0(f)=0} c_f \int_t^{+\infty} \hat{f}(\tau)(\varepsilon\mathbb{M})^{n_4(f)}(\varepsilon\mathbb{W})^{n_5(f)} d\tau \\ + \sum_{f\in\mathcal{K}_M, n_0(f)=1} c_f \int_t^{+\infty} \hat{f}(\tau)(\varepsilon\mathbb{M})^{n_4(f)}(\varepsilon\mathbb{W})^{n_5(f)} d\tau\end{aligned} \tag{7.1}$$

where $|c_f| \le 6$. Here, the two sums are to distinguish integrals from the autonomous part and integrals from the non-autonomous part of the equation (2.16). For the first sum, ε^{-1} is the rescaling factor. For the second sum, this rescaling factor is canceled by the ε in front the non-autonomous forcing function of the equation (2.9). We also note that for all integrals in the first sum, we have

$$n_4 + n_5 \ge 2 \tag{7.2}$$

because autonomous linear terms are removed from equation (2.16), and for all integrals in the second sum we have

$$n_4 + n_5 \ge 1. \tag{7.3}$$

We compute M_{n+1} by first writing

$$\begin{aligned}\varepsilon\mathbb{M} &= \varepsilon\mathbb{M}_0 + \varepsilon^2\mathbb{M}_1 + \cdots + \varepsilon^{n+1}\mathbb{M}_n;\\ \varepsilon\mathbb{W} &= \varepsilon\mathbb{W}_0 + \varepsilon^2\mathbb{W}_1 + \cdots + \varepsilon^{n+1}\mathbb{W}_n\end{aligned}$$

to obtain

$$\begin{aligned}(\varepsilon\mathbb{M})^{n_4} &= \sum_I \varepsilon^{i_0+2i_1+\cdots+(n+1)i_n} C^{n_4,I} (\mathbb{M}_0)^{i_0}\cdots(\mathbb{M}_n)^{i_n};\\ (\varepsilon\mathbb{W})^{n_5} &= \sum_J \varepsilon^{j_0+2j_1+\cdots+(n+1)j_n} C^{n_5,J} (\mathbb{W}_0)^{j_0}\cdots(\mathbb{W}_n)^{j_n}\end{aligned} \tag{7.4}$$

where I is over all positive integers $i_0, \cdots, i_n$ satisfying

$$i_0 + i_1 + \cdots + i_n = n_4;$$

J is over all positive integers $j_0, \cdots, j_n$ satisfying

$$j_0 + j_1 + \cdots + j_n = n_5;$$

and

$$C^{n_4,I} = \frac{n_4!}{i_0! \cdots i_n!}, \qquad C^{n_5,J} = \frac{n_5!}{j_0! \cdots j_n!}.$$

Though this looks long, $C^{n_4,I}$ and $C^{n_5,J}$ are relatively small numbers because $n_4 + n_5 \leq 3$. In fact, all $C^{n_4,I}, C^{n_5,J}$ are ≤ 6.

We also denote

$$\kappa_I = i_0 + 2i_1 + \cdots + (n+1)i_n, \quad \kappa_J = j_0 + 2j_1 + \cdots + (n+1)j_n$$

to write

$$(\varepsilon\mathbb{M})^{n_4}(\varepsilon\mathbb{W})^{n_5} = \sum_{I,J} C^{n_4,I} C^{n_5,J} \varepsilon^{\kappa_I+\kappa_J} (\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} \tag{7.5}$$

where $\sum_{I,J}$ are over all indexes $i_0, \cdots, i_n$; $j_0, \cdots, j_n$ satisfying

$$n_4 = i_0 + \cdots + i_n; \qquad n_5 = j_0 + \cdots + j_n.$$

Substituting (7.5) into (7.1), we obtain

$$\mathbb{M}_{n+1} = \sum_{f \in \mathcal{K}_M} c_f \sum_{I,J} C^{n_4,I} C^{n_5,J} \int_0^{+\infty} \hat{f}(\tau)(\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} d\tau. \tag{7.6}$$

In the case of $n_0(f) = 0$, $\sum_{I,J}$ is over all $i_0, \cdots, i_n$, $j_0, \cdots, j_n$ satisfying

$$\begin{aligned} n_4 &= i_0 + \cdots + i_n; \\ n_5 &= j_0 + \cdots + j_n; \\ n + 2 = \kappa_I + \kappa_J &:= i_0 + j_0 + 2(i_1 + j_1) + \cdots + (n+1)(i_n + j_n) \end{aligned} \tag{7.7}$$

where $n_4 + n_5 \geq 2$. In the case $n_0(f) = 1$, $\sum_{I,J}$ is over all $i_0, \cdots, i_n$, $j_0, \cdots, j_n$ satisfying

$$\begin{aligned} n_4 &= i_0 + \cdots + i_n; \\ n_5 &= j_0 + \cdots + j_n; \\ n + 1 = \kappa_I + \kappa_J &:= i_0 + j_0 + 2(i_1 + j_1) + \cdots + (n+1)(i_n + j_n) \end{aligned} \tag{7.8}$$

where $n_4 + n_5 \geq 1$. We then further substitute all $\mathbb{M}_k$, $\mathbb{W}_k$, $k \leq n$ in (7.6) by using (3.20). It then follows that $\mathbb{M}_{n+1}$ is a sum of integrals, each of which is defined by using (3.10). The proof for $\mathbb{W}_{n+1}$ is similar by using (3.5).

We note that the index set for i_k, j_k satisfying (7.7) or (7.8) is not as big as one might initially perceive because $n_4 + n_5 \leq 3$ imposes severe restrictions on the number of non-zero i_k, j_k to be admitted. Another strict restriction is also imposed on i_k, j_k by the fact that, in κ_I and κ_J, $(k+1)$ is multiplied to i_k and j_k.

Proof of Proposition 3.1(1) (Order of Integrals). Let $p(n)$ be the highest order of Melnikov integrals in $\Lambda_{M,n} \cup \Lambda_{W,n}$. We prove

$$p(n) \leq 4(n+1) - 2$$

for all $n \geq 0$. This is obviously true for $n = 0$. Assume inductively that

$$p(m) \leq 4(m+1) - 2 \tag{7.9}$$

for all $m \leq n$. We have from (7.6),

$$\begin{aligned} p(n+1) &\leq 1 + (i_0 + j_0)p(0) + (i_1 + j_1)p(1) + \cdots + (i_n + j_n)p(n) \\ &\leq 1 + 4 \cdot 1(i_0 + j_0) + 4 \cdot 2(i_1 + j_1) + \cdots + 4 \cdot (n+1)(i_n + j_n) \\ &\quad -2(i_0 + \cdots + i_n + j_0 + \cdots + j_n). \end{aligned}$$

If f is such that $n_0(f) = 0$, then we have, by using (7.7),

$$p(n+1) \leq 4(n+2) + 1 - 2(n_4 + n_5) < 4(n+2) - 2$$

where $n_4 + n_5 \geq 2$ is used to obtain the second inequality. In the case of $n_0(f) = 1$, we use (7.8) to obtain

$$p(n+1) \leq 4(n+1) + 1 - 2(n_4 + n_5) < 4(n+2) - 2.$$

This proves Proposition 3.1(1). □

Proof of Proposition 3.1(2) (Number of Integrals). Let $\mathcal{N}_n$ be the cardinality of $\Lambda_{M,n} \cup \Lambda_{W,n}$. We prove

$$\mathcal{N}_n \leq (n+1)^{-2}\mathbb{K}^{2n+1}$$

for all $n \geq 0$. Assume inductively

$$\mathcal{N}_m \leq (m+1)^{-2}\mathbb{K}^{2m+1} \tag{7.10}$$

for all $m \leq n$. To compute $\mathcal{N}_{n+1}$, we note that there are only a fixed number of $f \in \mathcal{K}_M$ (18 to be exact). For a given $f \in \mathcal{K}_M$, we let $\mathcal{N}_{n+1}^f$ be the number of integrals the sum

$$\sum_{I,J} C^{n_4,I} C^{n_5,J} \int_0^{+\infty} \hat{f}(\tau)(\mathbb{M}_0)^{i_0} \cdots (\mathbb{M}_n)^{i_n} \cdot (\mathbb{W}_0)^{j_0} \cdots (\mathbb{W}_n)^{j_n} d\tau$$

contributes to $\mathbb{M}_{n+1}$. We have, from (7.6),

$$\mathcal{N}_{n+1} \leq \sum_{f \in \mathcal{K}_M \cup \mathcal{K}_W} \mathcal{N}_{n+1}^f. \tag{7.11}$$

We also have

$$\mathcal{N}_{n+1}^f \leq \sum [\mathcal{N}_0]^{w_0} \cdots [\mathcal{N}_n]^{w_n} \tag{7.12}$$

where the sum is over all $w_k = i_k + j_k$ satisfying

$$\begin{aligned} n_4 + n_5 &= w_0 + w_1 + \cdots + w_n; \\ n + 2 &= w_0 + 2w_1 + \cdots + (n+1)w_n \end{aligned} \tag{7.13}$$

if $n_0(f) = 0$; but

$$\begin{aligned} n_4 + n_5 &= w_0 + w_1 + \cdots + w_n; \\ n + 1 &= w_0 + 2w_1 + \cdots + (n+1)w_n \end{aligned} \tag{7.14}$$

if $n_0(f) = 1$.

If $n_4(f) + n_5(f) = 1$, then we must have $n_0(f) = 1$ and $w_n = 1$. This is to say that

$$\mathcal{N}_{n+1}^f = \mathcal{N}_n \leq \frac{(n+2)^2}{(n+1)^2}(n+2)^{-2}\mathbb{K}^{2n+1} < 4(n+2)^{-2}\mathbb{K}^{2n+1}. \tag{7.15}$$

If $n_4(f) + n_5(f) = 2$. Assuming $n_0(f) = 0$, we have

$$\begin{aligned} 2 &= w_0 + w_1 + \cdots + w_n; \\ n + 2 &= w_0 + 2w_1 + \cdots + (n+1)w_n. \end{aligned}$$

In this case, our choices for w_k are limited: we either have two non-zero w_k, which we denote as $k_1 \neq k_2$ so that $w_{k_1} = w_{k_2} = 1$, or we have one non-zero w_k. In the first case, we have $k_1 + k_2 = n$ because

$$n + 2 = (k_1 + 1)w_{k_1} + (k_2 + 1)w_{k_2};$$

and in the second case, we have $k = n/2$ because $2(k+1) = n + 2$. Counting both cases, we obtain

$$\begin{aligned}\mathcal{N}_{n+1}^f &\le \sum_{k=1}^{n/2} \mathcal{N}_k \mathcal{N}_{n-k} \\ &\le \mathbb{K}^{2k+1+2(n-k)+1} \sum_{k=1}^{n/2} \frac{1}{(k+1)^2(n-k+1)^2} \\ &\le 4n^{-2}\mathbb{K}^{2(n+1)} \sum_{k=1}^{n/2} \frac{1}{(k+1)^2}\end{aligned}$$

where (7.12) and $k_2 = n - k_1$ are used for the first inequality; the inductive assumption (7.10) for the second inequality; and $n - k + 1 \ge n/2$ (because $k \le n/2$) for the last inequality. In conclusion, we have

$$\mathcal{N}_{n+1}^f \le 16(n+2)^{-2} K_0 \mathbb{K}^{2(n+1)} \tag{7.16}$$

where $K_0 = \sum_{k=1}^{+\infty} k^{-2}$.

We move on to consider the case $n_4(f) + n_5(f) = 3$. Assuming $n_0(f) = 0$, we have

$$\begin{aligned}3 &= w_0 + w_1 + \cdots + w_n; \\ n+2 &= w_0 + 2w_1 + \cdots + (n+1)w_n.\end{aligned}$$

In this case, there are at most three non-zero w_k, which we denote as $w_{k_1}, w_{k_2}, w_{k_3}$. We consider first the case $w_{k_1} = w_{k_2} = w_{k_3} = 1$. We obtain

$$k_1 + k_2 + k_3 = n - 1,$$

from

$$(k_1+1)w_{k_1} + (k_2+1)w_{k_2} + (k_3+1)w_{k_3} = n + 2.$$

We also count the cases of two non-zero w_k and one non-zero w_k to obtain

$$\begin{aligned}\mathcal{N}_{n+1}^f &\le 6 \sum_{k_1 \le k_2 \le k_3 \le n-1} \mathcal{N}_{k_1}\mathcal{N}_{k_2}\mathcal{N}_{k_3} \\ &\le 6\mathbb{K}^{2k_1+2k_2+2k_3+3} \sum_{k_1 \le k_2 \le k_3 \le n-1} (k_1+1)^{-2}(k_2+1)^{-2}(k_3+1)^{-2} \\ &\le 6\mathbb{K}^{2n+1} \sum_{k_1 \le k_2 \le k_3 \le n-1} (k_1+1)^{-2}(k_2+1)^{-2}(k_3+1)^{-2}.\end{aligned}$$

We note that, by summing over $k_1 \le k_2 \le k_3$ (instead of $k_1 < k_2 < k_3$), the cases of two and one non-zero w_k are counted here. This implies

$$\mathcal{N}_{n+1}^f \le 108(n+2)^{-2} K_0^2 \mathbb{K}^{2n+1} \tag{7.17}$$

where $K_0 = \sum_{k=1}^{+\infty} 1/k^2$. Here we used $k_3+1 = n-k_1-k_2 \geq n/3$, which follows from $k_1+k_2 \leq 2n/3$. We have $k_1+k_2 \leq 2n/3$ because if $k_1+k_2 > 2n/3$, then $k_2 > n/3$, leading to $k_3 > n/3$ hence $k_1+k_2+k_3 > n$, contradicting $k_1+k_2+k_3 = n-1$.

The estimate for the case of $n_0(f) = 1$ is similar.

Finally, it follows from (7.11), (7.15), (7.16) and (7.17) that

$$\mathcal{N}_{n+1} \leq (n+2)^{-2}\hat{K}\mathbb{K}^{2(n+1)}$$

where

$$\hat{K} = \#\{f \in \mathcal{K}_M \cup \mathcal{K}_W\}(4+16K_0+108K_0^2).$$

We let $\mathbb{K}$ be such that $\mathbb{K} > \hat{K}$ to conclude

$$\mathcal{N}_{n+1} \leq (n+2)^{-2}\mathbb{K}^{2(n+1)+1}.$$

This proves Proposition 3.1(2). □

Proof of Proposition 3.1(3) (Coefficients). Let $c(n)$ be the max of the magnitude of coefficients in front of an $N \in \Lambda_{M,n} \cup \Lambda_{W,n}$. We prove that there exists $K_3 > 0$ so that

$$c(n) \leq 6K_3^n$$

for all $n \geq 0$. This inequality holds for $n = 0$ because $c(0) < 6$. See (2.21). Assume inductively

$$c(m) \leq 6K_3^m \tag{7.18}$$

for all $m \leq n$. From (7.6) and

$$|c_f|,\ C^{n_4,I},\ C^{N_5,J} \leq 6,$$

we have

$$\begin{aligned} c(n+1) &\leq 6^3 \cdot 6^{n_4+n_5}[c(0)]^{i_0+j_0}[c(1)]^{i_1+j_1}\cdots[c(n)]^{i_n+j_n} \\ &\leq 6^6 \cdot K_3^{(i_0+j_0)+2(i_i+j_i)+(n+1)(i_n+j_n)-(i_0+j_0+i_1+j_1+\cdots+i_n+j_n)}. \end{aligned}$$

We again have the two cases of $n_0(f) = 0$ and $n_0(f) = 1$. Recall that if $n_0(f) = 0$, then $n_4(f) + n_5(f) \geq 2$. We have

$$c(n+1) \leq 6^6 \cdot K_3^{n+2-(n_4+n_5)} \leq 6^6 K_3^n < K_3^{n+1}$$

assuming $K_3 > 6^6$. If $n_0(f) = 1$, we have

$$c(n+1) \leq 6^6 \cdot K_3^{n+1-(n_4+n_5)} < 6^6 K_3^n < K_3^{n+1}$$

assuming $K_3 > 6^6$. This proves Proposition 3.1(3). □

8. Proof of Proposition 6.1

Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral of order p, and $\mathbb{T}(k,\omega), k=0,1,\cdots,p-1$ be as in Proposition 5.3. In this section, we prove Proposition 6.1. We work exclusively on $\mathbb{T}(k,\omega)$ assuming $Q_p \neq 0$ and $Q_{p-k} \neq 0$.

To prove Proposition 6.1, we first revise the memory of all tree nodes in $\mathbb{T}(k,\omega)$ to construct a new integral, which we denote as $\|\mathbb{T}\|(k,\omega)$ and name the majorant of $\mathbb{T}(k,\omega)$. We denote the correspondence of N_j in $\|\mathbb{T}\|(k,\omega)$ as $\|N\|_j$. Recall that

$$\hat{S}(k) = \{p-k+1,\cdots,p\}.$$

For all j, the node type, integral variable, and interval of integration for $\|N\|_j$ remain the same as that of N_j.

On the trigonometric part of the kernel functions: For $j \in \hat{S}(k)$, the trigonometric part of the kernel function for $\|N\|_j$ remains the same as that of N_j. For $j \notin \hat{S}(k)$, we set the trigonometric part of the kernel function, which is $e^{-i\omega Q_j \tau_j}$ for N_j, to 1 for $\|N\|_j$.

On the dynamic part of the kernel functions: We denote the dynamic part of the kernel function for $||N\|_j$ as $\|d\|_j$. For $j \in \hat{S}(k)$, we let

$$\|d\|_j = (K\omega)^{27}, \tag{8.1}$$

and for $j \notin \hat{S}(k)$, we let

$$\|d\|_j = (K\omega)^{27} e^{-(m_1+n_1)(\tau_j + \sum_{j' \in P(j)\setminus \hat{S}(k)} \tau_{j'})}, \tag{8.2}$$

for certain sufficiently large constant K.

We have

Lemma 8.1. *Let d_j be the dynamic part of the kernel function for N_j and $\|d\|_j$ be that for $\|N\|_j$. Under the assumption that the constant K in (8.1) and (8.2) is sufficiently large. We have*

$$|d_j| \leq \|d\|_j,$$

for all $j \leq p$.

Proof. Recall that for $j \in \hat{S}(k)$, the dynamic part of the kernel function for N_j is

$$d_j(iS_j) = b^{m_1}(iS_j)\tilde{H}^{m_2}(iS_j)a^{n_1}(iS_j)A^{n_2}(iS_j)H^{n_3}(iS_j)$$

where

$$S_j = s_j + \sum_{j' \in P(j)} s_{j'}$$

is obtained by adding all s-variables tracing backward along the ancestry line. By Lemma 5.3, we have $S_j \in [-\pi/2+\omega^{-1},\ \pi/2-\omega^{-1}]$, which implies

$$\begin{aligned}&|a(iS_j)| < K\omega, \quad |b(iS_j)| < K\omega^2, \quad |A(iS_j)| < K\omega, \quad |H(iS_j)| < K\omega^3\\ &|\tilde{H}(iS_j)| < K\omega^4, \quad |a^{-1}(iS_j)| \le K, \quad |b(iS_j)A^{-1}(iS_j)| < K\omega.\end{aligned} \tag{8.3}$$

In view of this upper bound, we have

$$|d_j(iS_j)| \le (K\omega)^{27} := \|d\|_j.$$

For $j \notin \hat{S}(k)$, the dynamic part of the kernel function for N_j is

$$d_j = b^{m_1}(w_j)\tilde{H}^{m_2}(w_j)a^{n_1}(w_j)A^{n_2}(w_j)H^{n_3}(w_j)$$

where for $j \notin T(p-k)$,

$$w_j = \tau_j + \sum_{j' \in P(j),\ j' < p-k} \tau_{j'} + i \sum_{j' \in P(j),\ j' > p-k} s_{j'},$$

but for $j \in T(p-k)$,

$$w_j = \tau_j + \sum_{j' \in P(j),\ j' < p-k} \tau_{j'} + i\sigma_{p-k}(\pi/2 - \omega^{-1}).$$

In both cases, w_j has an imaginary part that is $\in [-\pi/2 + \omega^{-1},\ \pi/2 - \omega^{-1}]$. This is to imply

$$\begin{aligned}&K^{-1}e^{-(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})} < |a(w_j)| < K\omega e^{-(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})};\\ &|b(w_j)| < K\omega^2 e^{-(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})};\\ &|b(w_j)A^{-1}(w_j)| \le K\omega e^{-(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})};\\ &|A(w_j)| < K\omega; \quad |\tilde{H}(w_j)| < K\omega^4; \quad |H(w_j)| < K\omega^3.\end{aligned} \tag{8.4}$$

We again use the upper bounds offered by these inequalities to obtain

$$|d_j| \le (K\omega)^{27} e^{-(m_1+n_1)(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})} := \|d\|_j. \qquad \square$$

Lemma 8.2. *We have $|\mathbb{T}(k,\omega)| \le \|\mathbb{T}\|(k,\omega)$. This is to say that the value of $\mathbb{T}(k,\omega)$ is bounded by its majorant.*

Proof. This lemma follows directly from Lemma 8.1. $\square$

We move on to evaluate $\|\mathbb{T}\|(k,\omega)$. Let

$$P(p-k) = \{j_1, \cdots, j_m\}$$

be listed in descending order. We note that $j_1 = p$. Define

$$\mathbb{P} = \int_0^{\sigma_{j_1}(\pi/2-\omega^{-1})} e^{-\omega(Q_{j_1}-Q_{p-k})s_{j_1}} \left(\int_0^{\sigma_{j_2}(\pi/2-\omega^{-1})-s_{j_1}} e^{-\omega(Q_{j_2}-Q_{p-k})s_{j_2}} \right.$$
$$\left. \cdots \left(\int_0^{\sigma_{j_m}(\pi/2-\omega^{-1})-s_{j_1}-s_{j_2}-\cdots-s_{j_{m-1}}} e^{-\omega(Q_{j_m}-Q_{p-k})s_{j_m}} ds_{j_m} \right) \cdots \right) ds_{j_1}.$$

Lemma 8.3. *We have* $\|\mathbb{T}\|(k,\omega) \le (K\omega)^{27p}|\mathbb{P}|$.

Proof. By definition, the dynamic part of the kernel function for $\|N\|_j$, $j \notin \hat{S}(k)$ is

$$\|d\|_j = (K\omega)^{27} e^{-(m_1+n_1)(\tau_j + \sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'})}.$$

We now make the following adjustment: we leave the factor

$$e^{-(m_1+n_1)\tau_j}$$

in $\|N\|_j$, but kick the factor

$$e^{-(m_1+n_1)\sum_{j' \in P(j)\setminus\hat{S}(k)} \tau_{j'}}$$

upward, and distribute the factor

$$e^{-(m_1+n_1)\tau_{j'}}$$

to $\|N\|_{j'}$ along the ancestry line of $\|N\|_j$. The new dynamic part of the kernel function for $\|N\|_j$ for $j \le p-k$ after this adjustment becomes

$$e^{-R_j\tau_j}$$

where

$$R_j = \sum_{j' \in T(j)} (m_1(j') + n_1(j')),$$

and we have

$$R_j \ge 1$$

from Lemma 3.6. Now all node in τ-variable in $\|\mathbb{T}\|(k,\omega)$ can be factored out and separately evaluated.

For all $j \in \hat{S}(k) \setminus P(p-k)$, we have $I_j \subset \mathbb{R}_+$ if $Q_j > 0$, but $I_j \subset \mathbb{R}_-$ if $Q_j < 0$. This is because the interval I_j always starts at 0, and ends at

$$\sigma_j(\pi/2-\omega^{-1})-\sum_{j'\in P(j)} s_{j'}.$$

According to Lemma 5.3,

$$\left|\sum_{j'\in P(j)} s_{j'}\right| \le \pi/2-\omega^{-1}.$$

Consequently, the sign of the end point of this integral interval is the same as σ_j.

From this claim, we have $Q_j s_j > 0$ on I_j. This is to imply

$$\left|e^{-\omega Q_j s_j}\right| \le 1,$$

which in turn implies

$$\left|\int_{I_j} e^{-\omega Q_j s_j} ds_j\right| < \pi$$

for all $j \in \hat{S}(k) \setminus P(p-k)$.

By taking the factor $(K\omega)^{27}$ out of all nodes, we obtain in total a factor $(K\omega)^{27p}$. What is left are the integrals defined by the variables along the ancestry line of $\|N\|_{p-k}$. It is precisely the integral $\mathbb{P}$ explicitly defined as in the above. □

Lemma 8.4. *We have* $|e^{-\sigma_{p-k}Q_{p-k}\omega\pi/2}\mathbb{P}| \le (3\pi)^m e^{-\omega\pi/2}$.

Proof. The proof of this lemma is technically involved partly because we have no control on the total index of the subtrees along the ancestry line of N_{p-k} other than Q_p and Q_{p-k} are not zero: we have to consider combination of all possible cases. What we do here, in essence, is nothing more than evaluating an explicitly defined definite integral, and going over all admissible parameters involved. To facilitate our presentation, we start by re-indexing $j_1, \cdots, j_m$ as $1, \cdots m$, and re-writing the variables $s_{j_1}, \cdots, s_{j_m}$ as $s_1, \cdots, s_m$. Similarly, we write $Q_{j_1}, \cdots, Q_{j_m}$ as $Q_1, \cdots Q_m$. We also write Q_{p-k} as Q_{m+1}. We emphasize that this change of notation is strictly local, used only inside of this proof. We also let $\sigma_j = sgn(Q_j)$, and drop ω^{-1} in all integral bounds because its presence here is inconsequential. This is to say we write $\mathbb{P}$ as

$$\mathbb{P} = \int_0^{\sigma_1\pi/2} e^{-\omega(Q_1-Q_{m+1})s_1} \int_0^{\sigma_2\pi/2-s_1} e^{-\omega(Q_2-Q_{m+1})s_2} \cdots$$
$$\left[\int_0^{\sigma_{m-1}\pi/2-s_1-s_2-\cdots-s_{m-2}} e^{-\omega(Q_{m-1}-Q_{m+1})s_{m-1}}\right.$$

$$\left(\int_0^{\sigma_m\pi/2-s_1-s_2-\cdots-s_{m-1}} e^{-\omega(Q_m-Q_{m+1})s_m}ds_m\right)ds_{m-1}\Bigg]\cdots ds_2ds_1,$$

and we assume

$$Q_1\neq 0,\quad Q_{m+1}\neq 0.$$

In what follows, we let j,k be such that $1\le j<k\le m+1$, and denote

$$\mathbb{P}(j,k)=\int_0^{\sigma_1\pi/2}e^{-\omega(Q_1-Q_k)s_1}\int_0^{\sigma_2\pi/2-s_1}e^{-\omega(Q_2-Q_k)s_2}\cdots$$
$$\Bigg[\int_0^{\sigma_{j-1}\pi/2-s_1-s_2-\cdots-s_{j-2}}e^{-\omega(Q_{j-1}-Q_k)s_{j-1}}$$
$$\left(\int_0^{\sigma_j\pi/2-s_1-s_2-\cdots-s_{j-1}}e^{-\omega(Q_j-Q_k)s_j}ds_j\right)ds_{j-1}\Bigg]\cdots ds_1.$$

In particular, we have

$$\mathbb{P}=\mathbb{P}(m,m+1).$$

Sublemma 8.1. *We have, assuming $j<k$,*

$$\left|e^{-\sigma_kQ_k\omega\pi/2}\mathbb{P}(j,k)\right|\le\pi^je^{-\omega\pi/2}+\pi\left|e^{-\sigma_kQ_k\omega\pi/2}\mathbb{P}(j-1,k)\right|$$
$$+|\sigma_j|\cdot\left|e^{-\sigma_jQ_j\omega\pi/2}\mathbb{P}(j-1,j)\right|. \tag{8.5}$$

We first assume this sublemma to inductively prove that, for all j,k satisfying $1\le j<k\le m+1$ and $\sigma_k\neq 0$, we have

$$\left|e^{-\sigma_kQ_k\omega\pi/2}\mathbb{P}(j,k)\right|\le(3\pi)^je^{-\omega\pi/2}. \tag{8.6}$$

This is a two-index induction. First, we let k_0 be the smallest $k>1$ satisfying $\sigma_k\neq 0$. The integer $k_0\le m+1$ is well-defined because $\sigma_{m+1}\neq 0$. We prove (8.6) holds for all $j<k_0$. We start with $j=1$ to prove

$$\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(1,k_0)\right|\le 3\pi e^{-\omega\pi/2}. \tag{8.7}$$

This estimate obviously hold if $Q_{k_0}=Q_1$. We further verify (8.7) assuming $Q_1\neq Q_{k_0}$. In this case, we have

$$\begin{aligned}\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(1,k_0)\right| &= e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\left|\int_0^{\sigma_1\pi/2} e^{-\omega(Q_1-Q_{k_0})s_1}ds_1\right| \\ &\le e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\frac{1}{\omega}\left(e^{-\sigma_1\omega(Q_1-Q_{k_0})\pi/2}+1\right) \\ &\le \frac{1}{\omega}\left(e^{-\sigma_1 Q_1\omega\pi/2}+e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\right) \\ &\le e^{-\omega\pi/2}\end{aligned}$$

where, to obtain the second inequality, we need to consider two cases: when $\sigma_1 = \sigma_{k_0}$ and when $\sigma_1 = -\sigma_{k_0}$. This proves (8.7).

Let us now assume

$$\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(j,k_0)\right| \le (3\pi)^j e^{-\omega\pi/2}$$

for $1 < j < k_0 - 1$. We have, by using (8.5),

$$\begin{aligned}\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(j+1,k_0)\right| &\le \pi^{j+1}e^{-\omega\pi/2} + \pi\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(j,k_0)\right| \\ &\quad + |\sigma_{j+1}|\cdot\left|e^{-\sigma_j Q_j\omega\pi/2}\mathbb{P}(j,j+1)\right|.\end{aligned}$$

Observe that, since $1 < j+1 < k_0$, we have $\sigma_{j+1} = 0$. It then follows that

$$\left|e^{-\sigma_{k_0}Q_{k_0}\omega\pi/2}\mathbb{P}(j+1,k_0)\right| \le \pi^{j+1}e^{-\omega\pi/2} + \pi(3\pi)^j e^{-\omega\pi/2} \le (3\pi)^{j+1}e^{-\omega\pi/2}.$$

We are now at the next stage of the intended induction. Let $k > k_0$ be such that $\sigma_k \neq 0$. We assume inductively that for all $k' \le k$ satisfying $\sigma_{k'} \neq 0$, and for all j, k' satisfying $j < k'$,

$$\left|e^{-\sigma_{k'}Q_{k'}\omega\pi/2}\mathbb{P}(j,k')\right| \le (3\pi)^j e^{-\omega\pi/2}. \tag{8.8}$$

Let k_1 be the smallest integer $> k$, so that $\sigma_{k_1} \neq 0$. Again, $k_1 \le m+1$ is well-defined. We need to prove that for all j, $1 \le j < k_1$,

$$\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j,k_1)\right| \le (3\pi)^j e^{-\omega\pi/2}. \tag{8.9}$$

This again holds for $j = 1$ by (8.7). Assuming (8.9) holds for all j, $1 \le j < k_1 - 2$, we have, by using (8.5),

$$\begin{aligned}\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j+1,k_1)\right| &\le \pi^{j+1}e^{-\omega\pi/2} + \pi\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j,k_1)\right| \\ &\quad + |\sigma_{j+1}|\cdot\left|e^{-\sigma_{j+1}Q_{j+1}\omega\pi/2}\mathbb{P}(j,j+1)\right|\end{aligned}$$

We have two cases to consider. The first is when $\sigma_{j+1} = 0$. In this case, we have

$$\begin{aligned}\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j+1,k_1)\right| &\le \pi^{j+1}e^{-\omega\pi/2}+\pi\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j,k_1)\right| \\ &\le \pi^{j+1}e^{-\omega\pi/2}+\pi(3\pi)^j e^{-\omega\pi/2} \\ &\le (3\pi)^{j+1}e^{-\omega\pi/2}\end{aligned}$$

where, for the second, we used (8.9). The second case is when $\sigma_{j+1}\neq 0$. Note that by assumption, $j+1<k_1$, we have in this case

$$\begin{aligned}\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j+1,k_1)\right| &\le \pi^{j+1}e^{-\omega\pi/2}+\pi\left|e^{-\sigma_{k_1}Q_{k_1}\omega\pi/2}\mathbb{P}(j,k_1)\right| \\ &\quad+\left|e^{-\sigma_{j+1}Q_{j+1}\omega\pi/2}\mathbb{P}(j,j+1)\right| \\ &\le \pi^{j+1}e^{-\omega\pi/2}+\pi(3\pi)^j e^{-\omega\pi/2}+(3\pi)^j e^{-\omega\pi/2} \\ &\le (3\pi)^{j+1}e^{-\omega\pi/2}.\end{aligned}$$

Here, we again used (8.9) for the second and the third term for the second inequality. This finished the inductive proof for (8.6). In particular, we have

$$\left|e^{-\sigma_{m+1}Q_{m+1}\omega\pi/2}\mathbb{P}(m,m+1)\right| \le (3\pi)^m e^{-\omega\pi/2}.$$

This proves Lemma 8.4. □

Proof of Sublemma 8.1. First, we observe that

$$|\mathbb{P}(j,k)| \le \pi^j \tag{8.10}$$

assuming $Q_k=0$. We again use the fact that $Q_{j'}s_{j'}>0$ for all $j'\le j$, which is reasoned out in the proof of Lemma 8.3, to prove (8.10).

Assume $Q_k\neq 0$. To study $\mathbb{P}(j,k)$, we have two cases. The first case is $Q_j=Q_k$. In this case, we have

$$|\mathbb{P}(j,k)| < \pi|\mathbb{P}(j-1,k)|. \tag{8.11}$$

The second case is $Q_j\neq Q_k$. For the inner most integral, we have

$$\begin{aligned}\mathcal{P} &:= \int_0^{\sigma_j\pi/2-s_1-s_2-\cdots-s_{j-1}} e^{-\omega(Q_j-Q_k)s_j}ds_j \\ &= \frac{1}{-\omega(Q_j-Q_k)}\left[e^{-\omega(Q_j-Q_k)(\sigma_j\pi/2-s_1-s_2-\cdots-s_{j-1})}-1\right].\end{aligned}$$

It then follows that

$$\left|e^{-\sigma_k Q_k\omega\pi/2}\mathbb{P}(j,k)\right| < |(I)|+|(II)| \tag{8.12}$$

where

$$(I) = \frac{-e^{-\sigma_k \omega Q_k \pi/2 - \sigma_j \omega (Q_j - Q_k)\pi/2}}{\omega(Q_j - Q_k)} \mathbb{P}(j-1, j)$$

$$(II) = \frac{e^{-\sigma_k \omega Q_k \pi/2}}{\omega(Q_j - Q_k)} \mathbb{P}(j-1, k).$$

We have three sub-case for (I):
(a) $\sigma_k = \sigma_j$. In this sub-case,

$$(I) = \frac{e^{-\sigma_j \omega Q_j \pi/2}}{\omega(Q_j - Q_k)} \mathbb{P}(j-1, j).$$

(b) $\sigma_j = -\sigma_k$. In this sub-case,

$$(I) = \frac{e^{-2\sigma_k \omega Q_k \pi/2 - \sigma_j \omega Q_j \pi/2}}{\omega(Q_k - Q_j)} \mathbb{P}(j-1, j).$$

(c) $\sigma_j = 0$. In this sub-case,

$$|(I)| = \frac{e^{-\sigma_k \omega Q_k \pi/2}}{\omega |Q_j - Q_k|} |\mathbb{P}(j-1, j)| \le \pi^j e^{-\sigma_k \omega Q_k \pi/2}.$$

We note that for the last inequality in (c), we used (8.10).
In summary, we have, assuming $Q_k \neq 0$,

(1) If $Q_j = Q_k$, then

$$\left| e^{-\sigma_k Q_k \omega \pi/2} \mathbb{P}(j, k) \right| < \pi \left| e^{-\sigma_k Q_k \omega \pi/2} \mathbb{P}(j-1, k) \right|.$$

This is from (8.11).
(2) If $Q_j \neq Q_k$, and $Q_j \neq 0$, then

$$|e^{-\sigma_k Q_k \omega \pi/2} \mathbb{P}(j, k)| \le |e^{-\sigma_j \omega Q_j \pi/2} \mathbb{P}(j-1, j)| + |e^{-\sigma_k \omega Q_k \pi/2} \mathbb{P}(j-1, k)|.$$

This follows from items (a), (b) and (8.12) in the above.
(3) If $Q_j \neq Q_k$, and $Q_j = 0$, then

$$|e^{-\sigma_k Q_k \omega \pi/2} \mathbb{P}(j, k)| \le \pi^j e^{-\sigma_k \omega Q_k \pi/2} + |e^{-\sigma_k \omega Q_k \pi/2} \mathbb{P}(j-1, k)|.$$

This follows from item (c) and (8.12) in the above.

Putting (1)-(3) together we obtain, assuming $j < k$,

$$\left|e^{-\sigma_k Q_k \omega\pi/2}\mathbb{P}(j,k)\right| \le \pi^j e^{-\omega\pi/2} + \pi\left|e^{-\sigma_k Q_k \omega\pi/2}\mathbb{P}(j-1,k)\right| + |\sigma_j|\cdot\left|e^{-\sigma_j Q_j \omega\pi/2}\mathbb{P}(j-1,j)\right|.$$

Note that the factor $|\sigma_j|$ in front of the third term signifies that this term is only there when $Q_j \neq 0$. □

Proof of Proposition 6.1. Follows directly from Lemmas 8.2, 8.3 and 8.4. □

9. Proof of Proposition 6.2

In this section, we prove Proposition 6.2. In writing this proof, we face a common challenge in presenting a rather involved combinatorial proof. For mathematical rigor, we need to introduce detailed indexing and carry out tedious counting, but such long details are sometimes burdensome, making it harder for the reader to extract the main ideas of the proof. To get around this problem, we start with a special case, gradually introducing the combinatorial details of a proof that applies to the general case. In Sect. 9.1, we consider the case in which the structure tree has only one zero subtree. In Sect. 9.2, we allow more than one zero subtree but assume all zero subtrees are mutually exclusive. A formal proof of Proposition 6.2 is presented in Sect. 9.3.

9.1. Integrals with one zero-subtree

Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral of order p. We start with the simplest non-trivial case by making the following assumptions on $\mathcal{T}_{\mathbf{q}}(0)$:

(A1) The total index of $\mathcal{T}_{\mathbf{q}}(0)$, which we denote as Q_p, is *not zero*.
(A2) The integral $\mathcal{T}_{\mathbf{q}}(0)$ contains only one zero subtree, the root of which we denote as N_{k_0}. This is to say that we assume $Q_j = 0$ if and only if $j = k_0$.

Assume $\mathcal{T}_{\mathbf{q}}(0)$ satisfies (A1) and (A2). We first obtain $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ in τ variables. We then decompose $\hat{\mathcal{T}}_{\mathbf{q}}(0)$ to obtain $\mathbb{T}_{\mathbf{q}}(p-1,v)$ and $\mathbb{T}_{\mathbf{q}}(k,\omega)$ for $k = 0,\cdots,p-1$. See (6.1). Here, at the center of our consideration is $\mathbb{T}(p-k_0,\omega)$.

All tree nodes in $\mathbb{T}(p-k_0,\omega)$ are put into three groups.

Group (a): (Top part of the tree in s variables) Group (a) is the collection of N_j so that $k_0 < j \le p$. It occupies the top portion of the structure tree. These are tree nodes in s variables. For N_j in this group, the integral variable is s_j, the interval of integration is

$$I_j = \left[0,\ \ \sigma_j(\pi/2-\omega^{-1}) - \sum_{j'\in P(j)} s_j\right],$$

the trigonometric part of the kernel function is

$$e^{-\omega Q_j s_j},$$

and the dynamic part of the kernel function is

$$b^{m_1}(iS_j)\tilde{H}^{m_2}(iS_j)a^{n_1}(iS_j)A^{n_2}(iS_j)H^{n_3}(iS_j)$$

where

$$S_j = s_j + \sum_{j' \in P(j)} s_{j'}$$

is obtained by adding all s-variables along its ancestry line to s_j. We note that the trigonometric part of the kernel function for N_j on the ancestry line of N_{k_0} is $e^{-\omega Q_j s_j}$ because $Q_{k_0} = 0$.

Group (b): (Outside of the subtree rooted at N_{k_0} but not yet converted to s-variables) Group (b) is the collection of N_j such that $j < k_0$ but $j \notin T(k_0)$. The induction that converts τ variables to s variables has not reached N_j in this group. For N_j in group (b), the integral variable is τ_j, the interval of integration is $[0, +\infty)$, the trigonometric part of the kernel function is $e^{i\omega Q_j \tau_j}$, and the dynamic part of the kernel function is

$$b^{m_1}(w_j)\tilde{H}^{m_2}(w_j)a^{n_1}(w_j)A^{n_2}(w_j)H^{n_3}(w_j)$$

where w_j is a mix of τ and s variables: it is obtained by adding all integral variables along the ancestry line of N_j to τ_j. We also multiply i to all s variables to signify that they are the imaginary part of a set of complex variables. This is to say we have

$$w_j = \tau_j + \sum_{j' \in P(j),\ j' < k_0} \tau_{j'} + i \sum_{j' \in P(j),\ j' > k_0} s_{j'}.$$

Group (c): (Subtree rooted at N_{k_0}) Group (c) is all nodes in the subtree rooted at N_{k_0}. For N_j in this subtree, the variable of integration is τ_j, the interval of integration is $[0, +\infty)$, the trigonometric part of the kernel function is $e^{i\omega Q_j \tau_j}$, and the dynamical part of kernel function is again in the form of

$$b^{m_1}(w_j)\tilde{H}^{m_2}(w_j)a^{n_1}(w_j)A^{n_2}(w_j)H^{n_3}(w_j).$$

However, here w_j is different from that in Group (b). We have, from (5.10),

$$w_j = \tau_j + \sum_{j' \in T(k_0) \cap P(j)} \tau_{j'}.$$

This is to say that w_j is obtained by adding to τ_j all τ variables along the ancestry line of N_j *inside of this subtree*. We observe that since $Q_{k_0} = \sigma_{k_0} = 0$, the term $\sigma_{p-k}(\pi/2 - \omega^{-1})$ for $k = p - k_0$ in (5.10) goes away.

Let the integral rooted at N_{k_0} in $\mathcal{T}_{\mathbf{q}}(0)$ be denoted as $\mathcal{T}_{k_0} = \mathcal{T}_{k_0}(t_{j'})$ where j' is such that $k_0 \in C(j')$. From the description in the above on Group (c), we conclude that the subtree rooted at N_{k_0} in $\mathbb{T}(p - k_0, \omega)$ is $\mathcal{T}_{k_0}(0)$.

Denote the structure tree obtained by deleting from $\mathcal{T}_{\mathbf{q}}(0)$ the subtree $\mathcal{T}_{k_0}$ as $\mathcal{R}_{k_0}(0)$. We divide the defining vector $\mathbf{q}$ for $\mathcal{T}_{\mathbf{q}}(0)$ into two. The first is the part of $\mathbf{q}$ that defines $\mathcal{T}_{k_0}(0)$, which we denote as

$$\mathbf{q}_T := \mathbf{q}(\mathcal{T}_{k_0}).$$

The second is the part that defines $\mathcal{R}_{k_0}(0)$, which we denote as

$$\mathbf{q}_R := \mathbf{q}(\mathcal{R}_{k_0}).$$

We write $\mathbf{q}$ as

$$\mathbf{q} = (\mathbf{q}_R, \mathbf{q}_T).$$

The integral $\mathcal{T}_{k_0}(0)$ is defined by using $\mathbf{q}_T$ and the integral $\mathcal{R}_{k_0}(0)$ is defined by using $\mathbf{q}_R$. We also denote $\mathbb{T}(p-k_0,\omega)$ as $\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega)$, $\mathcal{T}_{k_0}(0)$ as $\mathcal{T}_{\mathbf{q}_T}$, and $\mathcal{R}_{k_0}(0)$ as $\mathcal{R}_{\mathbf{q}_R}$.

Lemma 9.1. *We have*
(a) $\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega) = \mathcal{T}_{\mathbf{q}_T} \cdot \mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v)$*; and*
(b) there exists $K > 0$ *so that*

$$|\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega) - \mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega)| < (K\omega)^{27p}e^{-\pi/2}, \tag{9.1}$$

$$|\mathbb{T}_{(\mathbf{q}_R,-\mathbf{q}_T)}(p-k_0,\omega) - \mathbb{T}_{(-\mathbf{q}_R,-\mathbf{q}_T)}(p-k_0,\omega)| < (K\omega)^{27p}e^{-\pi/2}. \tag{9.2}$$

Proof. Item (a) follows directly from the fact that the subtree rooted at N_{k_0} in $\mathbb{T}(p-k_0,\omega)$ is $\mathcal{T}_{k_0}(0)$, which we have also denoted as $\mathcal{T}_{\mathbf{q}_T}$. Factor $\mathcal{T}_{\mathbf{q}_T}$ out of $\mathbb{T}(k_0,\omega)$, and denote the remainder tree as $\mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v)$. An alternative way to obtain $\mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v)$ is to apply the decomposition process of Section 5 to $\mathcal{R}_{\mathbf{q}_R} = \mathcal{R}_{k_0}(0)$. The integral $\mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v)$ is the one that is passed forward at the end of step $p-k_0-1$.

To prove (9.1) we start with

$$\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega) = \mathcal{T}_{\mathbf{q}_T} \cdot \mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v),$$
$$\mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega) = \mathcal{T}_{\mathbf{q}_T} \cdot \mathbb{R}_{-\mathbf{q}_R}(p-k_0-1,v)$$

to obtain

$$\begin{aligned}&\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega) - \mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p-k_0,\omega)\\ &\quad= \mathcal{T}_{\mathbf{q}_T} \cdot \left[\mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v) - \mathbb{R}_{-\mathbf{q}_R}(p-k_0-1,v)\right].\end{aligned}$$

Because no subtree of $\mathcal{R}_{\mathbf{q}_R}$ is of total index zero from (A2), we conclude, by using Proposition 5.2 and Proposition 6.1, that $\mathbb{R}_{\mathbf{q}_R}(p-k_0-1,v)$ is a sum of

$$i^{k_0-p_{k_0}-1}\mathbb{R}_{\mathbf{q}_R}(p-p_{k_0}-1,v)$$

and a collection of exponentially small terms. Similarly, $\mathbb{R}_{-\mathbf{q}_R}(p-k_0-1,v)$ is also a sum of

$$i^{k_0-p_{k_0}-1}\mathbb{R}_{-\mathbf{q}_R}(p-p_{k_0}-1,v)$$

and a collection of exponentially small terms. Note that both $\mathbb{R}_{\mathbf{q}_R}(p-p_{k_0}-1,v)$ and $\mathbb{R}_{-\mathbf{q}_R}(p-p_{k_0}-1,v)$ are completely in s-variables. We again use Lemma 6.1 to obtain

$$\mathbb{R}_{\mathbf{q}_R}(p - p_{k_0} - 1, v) = \mathbb{R}_{-\mathbf{q}_R}(p - p_{k_0} - 1, v).$$

This proves (9.1). Proof for (9.2) is similar. □

For a $\mathbf{q} \in \mathcal{S}^+$ satisfying (A1) and (A2), the basic blocks are $\mathbf{q}_T$ and $\mathbf{q}_R$. We have

$$\mathbb{S}_{\mathbf{q}} = \{(\mathbf{q}_R, \mathbf{q}_T);\ (\mathbf{q}_R, -\mathbf{q}_T)\},$$

and by definition,

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = \mathcal{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(0) - \mathcal{T}_{(-\mathbf{q}_R,-\mathbf{q}_T)}(0) + \mathcal{T}_{(\mathbf{q}_R,-\mathbf{q}_T)}(0) - \mathcal{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(0).$$

We have from the decomposition formula (6.1), Lemma 6.1 and Proposition 6.1 that

$$\begin{aligned}\left|\mathbb{D}_{\mathbb{S}_{\mathbf{q}}}\right| \le (K\omega)^{27p} e^{-\omega\pi/2} + \big|&\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p - k_0, \omega) - \mathbb{T}_{(-\mathbf{q}_R,-\mathbf{q}_T)}(p - k_0, \omega)\\ &+ \mathbb{T}_{(\mathbf{q}_R,-\mathbf{q}_T)}(p - k_0, \omega) - \mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p - k_0, \omega)\big|\end{aligned}$$

By switching the position of $\mathbb{T}_{(-\mathbf{q}_R,-\mathbf{q}_T)}(p - k_0, \omega)$ and $\mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p - k_0, \omega)$ on the right-hand side, we obtain

$$\begin{aligned}\left|\mathbb{D}_{\mathbb{S}_{\mathbf{q}}}\right| \le (K\omega)^{27p} e^{-\omega\pi/2} + \big|&\mathbb{T}_{(\mathbf{q}_R,\mathbf{q}_T)}(p - k_0, \omega) - \mathbb{T}_{(-\mathbf{q}_R,\mathbf{q}_T)}(p - k_0, \omega)\big|\\ &+ \big|\mathbb{T}_{(\mathbf{q}_R,-\mathbf{q}_T)}(p - k_0, \omega) - \mathbb{T}_{(-\mathbf{q}_R,-\mathbf{q}_T)}(p - k_0, \omega)\big|.\end{aligned}$$

We can now apply (9.1) and (9.2) to conclude that

$$\left|\mathbb{D}_{\mathbb{S}_{\mathbf{q}}}\right| \le (K\omega)^{27p} e^{-\omega\pi/2}.$$

9.2. *Pure integrals with mutually exclusive zero subtrees*

In this subsection, we deal with a less restricted case. Let $\mathcal{T}_{\mathbf{q}}(0)$ be a pure integral of order p. We again assume $Q_{\mathbf{q}} \neq 0$. We allow $\mathcal{T}_{\mathbf{q}}(0)$ to have more zero subtrees but *assume all zero subtrees are mutually exclusive*.

Assume that $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees rooted at $N_{j_1}, \cdots, N_{j_n}$ where

$$j_1 > j_2 > \cdots > j_n.$$

We use $\mathcal{T}_{j_k}$ to denote the subtree rooted at N_{j_k} and $\mathcal{R}_{j_k}(0)$ to denote the remainder tree obtained by deleting $\mathcal{T}_{j_k}$ from $\mathcal{T}_{\mathbf{q}}(0)$. In general, let

$$k_1 < k_2 < \cdots < k_m \le n.$$

We use $\mathcal{R}_{j_{k_1},\cdots,j_{k_m}}(0)$ to denote the remainder tree obtained by deleting from $\mathcal{T}_{\mathbf{q}}(0)$ the zero subtrees $\mathcal{T}_{j_{k_1}}, \cdots, \mathcal{T}_{j_{k_m}}$. By the assumption that all zero subtrees are mutually exclusive, we know that zero subtrees in $\mathcal{R}_{j_{k_1},\cdots,j_{k_m}}(0)$ are all $\mathcal{T}_{j_k}$ so that $k \notin \{k_1, \cdots, k_m\}$.

From decomposition formula (6.1) and Proposition 6.1, we have, modulo exponentially small terms,

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}_{\mathbf{q}}(p-1, v) + \sum_{k=1}^{n} i^{p-j_k} \mathcal{T}_{j_k}(0) \cdot \mathbb{R}_{j_k}(p - j_k - 1, v). \tag{9.3}$$

Note that to obtain $\mathbb{R}_{j_k}(p - j_k - 1, v)$, we delete $\mathcal{T}_{j_k}$ from $\mathcal{T}_{\mathbf{q}}(0)$ to obtain $\mathcal{R}_{j_k}(0)$ first. We then decompose $\mathcal{R}_{j_k}(0)$ by using (6.1), and $\mathbb{R}_{j_k}(p - j_k - 1, v)$ is the integral we pass forward in this decomposition process at step $p - j_k - 1$. This integral is partially in s and partially in τ variables, which we need to decompose further by using (5.11). We have, modulo exponentially small terms,

$$\begin{aligned} \mathbb{R}_{j_k}(p - j_k - 1, v) = {} & i^{j_k - p_{j_k}} \mathbb{R}_{j_k}(p - p_{j_k} - 1, v) \\ & + \sum_{w=k+1}^{n} i^{j_{kw}} \mathcal{T}_{j_w}(0) \cdot \mathbb{R}_{j_k j_w}(p - j_k + j_{kw} - 1, v) \end{aligned} \tag{9.4}$$

where

$$j_{kw} = \#\{j' \in [j_w, j_k] \setminus T(j_k)\}$$

is the number of steps it takes for this induction to go from N_{j_k+1} to N_{j_w} in $\mathcal{R}_{j_k}(0)$. We note that in $\mathbb{R}_{j_k}(p - j_k - 1, v)$, the integral variables of $N_p, \cdots, N_{j_k+1}$ have been converted to s variables, and the roots of zero subtrees yet to be converted are N_{j_w} where $w = k+1, k+2, \cdots, n$. This is why the sum on the right of (9.4) is from $w = k+1$ to n. The number of steps it takes for this induction to go from N_{j_k+1} to N_{j_w} in $\mathcal{R}_{j_k}(0)$ is j_{kw}, and $p - j_k + j_{kw}$ is the total number of steps it takes to reach N_{j_w} starting from N_p in decomposing $\mathcal{R}_{j_k}(0)$. The power on i for the first term on the right is $j_k - p_{j_k}$ because, from N_{j_k+1} to reach the end of this induction for $\mathcal{R}_{j_k}(0)$, it takes $j_k - p_{j_k}$ steps.

Let us note that, here, the indexing becomes fairly detailed. However, $\mathbb{R}_{j_k j_w}(p - j_k + j_{kw} - 1, v)$ remains an integral partially in s and partially in τ variables, for which we need to further convert τ to s variables. This decomposition process eventually ends, and the end product is as follows:

We let, for $m \le n$,

$$\mathcal{K}_m = \{\mathbf{k} = (k_1, \cdots, k_m), \quad k_1 < \cdots < k_m \le n\}$$

and

$$\mathcal{K} = \cup_{m=1}^{n} \mathcal{K}_m.$$

We use the index set $\mathcal{K}_m$ to represent the action of picking m zero subtrees with the freedom of skipping $n - m$ of them at will in $\mathcal{T}_{\mathbf{q}}(0)$. We have

Lemma 9.2. *For $\mathbf{k} = (k_1, \cdots, k_m)$, let $p_{\mathbf{k}} = p - p_{j_{k_1}} - \cdots - p_{j_{k_m}}$ be the order of the remainder tree $\mathcal{R}_{j_{k_1} \cdots j_{k_m}}(0)$. We have, modulo exponentially small terms,*

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}_{\mathbf{q}}(p-1, v) + \sum_{m=1}^{n} \left(\sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{k_m}}(0) \cdot \mathbb{R}_{j_{k_1} \cdots j_{k_m}}(p_{\mathbf{k}} - 1, v) \right). \quad (9.5)$$

Proof. We start by replacing $\mathbb{R}_{j_k}(p - j_k - 1, v)$ in (9.3) using (9.4) to obtain

$$\begin{aligned}\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) &+ \sum_{k=1}^{n} i^{p-p_{j_k}} \mathcal{T}_{j_k}(0) \cdot \mathbb{R}_{j_k}(p - p_{j_k} - 1, v) \\ &+ \sum_{k=1}^{n} \sum_{w=k+1}^{n} i^{p-j_k+j_{kw}} \mathcal{T}_{j_k}(0) \cdot \mathcal{T}_{j_w}(0) \cdot \mathbb{R}_{j_k j_w}(p - j_k + j_{kw} - 1, v).\end{aligned}$$

The first sum obviously matches the sum in (9.5) over all $\mathbf{k} \in \mathcal{K}_1$. The second sum in the above can be rewritten as

$$\sum_{(k_1,k_2) \in \mathcal{K}_2} i^{p-j_k+j_{k_1k_2}} \mathcal{T}_{j_{k_1}}(0) \cdot \mathcal{T}_{i_{k_2}}(0) \cdot \mathbb{R}_{j_{k_1} j_{k_2}}(p - j_k + j_{k_1k_2} - 1, v)$$

regarding $k = k_1$, $w = k_2$. This is to say we have

$$\begin{aligned}\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) &+ \sum_{k=1}^{n} i^{p-p_{j_k}} \mathcal{T}_{j_k}(0) \cdot \mathbb{R}_{j_k}(p - p_{j_k} - 1, v) \\ &+ \sum_{(k_1,k_2) \in \mathcal{K}_2} i^{p-j_k+j_{k_1k_2}} \mathcal{T}_{j_{k_1}}(0) \cdot \mathcal{T}_{i_{k_2}}(0) \cdot \mathbb{R}_{j_{k_1} j_{k_2}}(p - j_k + j_{k_1k_2} - 1, v). \quad (9.6)\end{aligned}$$

To continue, we further decompose $\mathbb{R}_{j_{k_1} j_{k_2}}(p - j_k + j_{k_1k_2} - 1, v)$ by using (5.11) in Proposition 5.2, and so on. At this point it should become clear that, to formally prove (9.5), we need to do another local induction.

First, we work on formulating an inductive assumption. Here, the problem is again less about mathematical contents, but more about detailed indexing, as often is the case in a rather involved combinatoric proof. Let

$$\mathbf{k} = (k_1, \cdots, k_m) \in \mathcal{K}_m.$$

(i) We denote $\mathcal{R}_{j_{k_1} \cdots j_{k_m}}(0)$ as $\mathcal{R}_{\mathbf{k}}(0)$, and correspondingly, $\mathbb{R}_{j_{k_1} \cdots j_{k_m}}$ as $\mathbb{R}_{\mathbf{k}}$; and the order of $\mathcal{R}_{\mathbf{k}}(0)$ as $p_{\mathbf{k}}$.
(ii) The number of steps it takes from the starting point of the decomposition process of Section 5 to reach $N_{j_{k_m}+1}$ in decomposing $\mathcal{R}_{\mathbf{k}}(0)$, is denoted as J_{pm}. The subscript pm indicates that it is from N_p to $N_{j_{k_m}+1}$
(iii) The number of steps it takes from $N_{j_{k_m}+1}$ to reach N_{j_w} for a given $w > k_m$ in decomposing $\mathcal{R}_{\mathbf{k}}(0)$, is denoted as J_{mw}.
(iv) The number of steps it takes from $N_{j_{k_m}+1}$ to reach the end in decomposing $\mathcal{R}_{\mathbf{k}}(0)$ is denoted as J_{me}. The subscript me indicates that it is from $N_{j_{k_m}+1}$ to the end of the tree $\mathcal{R}_{\mathbf{k}}(0)$.

Note that in writing items (ii)-(iv) in the above, we tacitly implied that $N_{j_{k_m}+1}$ is in $\mathcal{R}_{\mathbf{k}}(0)$. In the case where $N_{j_{k_m}+1}$ is not in $\mathcal{R}_{\mathbf{k}}(0)$, we need to replace $N_{j_{k_m}+1}$ in items (ii)-(iv) by the tree node immediately proceeding $N_{j_{k_m}}$ in $\mathcal{R}_{\mathbf{k}}(0)$.

We start by using (5.11) repeatedly to obtain, for $\mathbf{k} \in \mathcal{K}_m$,

$$\mathbb{R}_{\mathbf{k}}(J_{pm}-1,v) = i^{J_{me}}\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v) + \sum_{w=k_m+1}^{n} i^{J_{mw}}\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v). \tag{9.7}$$

This is a generalized version of (9.4).

Our proof of (9.5) is inductive on m, and the inductive assumption is as follows

Inductive Assumption.

$$\begin{aligned}\mathcal{T}_{\mathbf{q}}(0) = i^p\mathbb{T}(p-1,v) &+ \sum_{m'=1}^{m}\left(\sum_{\mathbf{k}\in\mathcal{K}_{m'}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m'}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\right)\\ &+ \sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{J_{p(m+1)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(J_{p(m+1)}-1,v).\end{aligned} \tag{9.8}$$

We note that $\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$ in the first line are all completely in s variables. They are the end product of a completed decomposition process (on $\mathcal{R}_{\mathbf{k}}(0)$). In the second line, however, $\mathbb{R}_{\mathbf{k}}(J_{p(m+1)}-1,v)$ is still an integral of mixed in s and τ variables, yet to be completely converted to that in s variables.

The initial step is $m=1$. In this case, (9.8) is reduced to (9.6). To advance the induction, we need (9.7) for $m+1$. This is to say that we have, for $\mathbf{k}\in\mathcal{K}_{m+1}$,

$$\begin{aligned}\mathbb{R}_{\mathbf{k}}(J_{p(m+1)}-1,v) &= i^{J_{(m+1)e}}\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\\ &\quad+\sum_{w=k_{m+1}+1}^{n} i^{J_{(m+1)w}}\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v).\end{aligned} \tag{9.9}$$

Assuming (9.8) holds for m, we prove it holds for $m+1$ by using (9.9) for $\mathbb{R}_{\mathbf{k}}(J_{p(m+1)}-1,v)$ in (9.8). This is to say we have

$$\begin{aligned}\mathcal{T}_{\mathbf{q}}(0) = i^p\mathbb{T}(p-1,v) &+ \sum_{m'=1}^{m}\left(\sum_{\mathbf{k}\in\mathcal{K}_{m'}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m'}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\right)\\ &+\sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{J_{p(m+1)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot i^{J_{(m+1)e}}\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\\ &+\sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{J_{p(m+1)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\left(\sum_{w=j_{k_{m+1}}+1}^{n} i^{J_{(m+1)w}}\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v)\right).\end{aligned}$$

The second line in the above is

$$\sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{J_{p(m+1)}+J_{(m+1)e}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$$
$$=\sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$$

because by definition $p_{\mathbf{k}}=J_{p(m+1)}+J_{(m+1)e}$. It is added to the first line, lifting the upper bound for m' from m to $m+1$.

What is left is

$$\sum_{\mathbf{k}\in\mathcal{K}_{m+1}} i^{J_{p(m+1)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\left(\sum_{w=k_{m+1}+1}^{n} i^{J_{(m+1)w}}\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v)\right)$$
$$=\sum_{\mathbf{k}\in\mathcal{K}_{m+1}}\sum_{w=k_{m+1}+1}^{n} i^{J_{p(m+1)}+J_{(m+1)w}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+1}}}(0)\cdot\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v)$$
$$=\sum_{\mathbf{k}\in\mathcal{K}_{m+2}} i^{J_{p(m+2)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m+2}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(J_{p(m+2)}-1,v)$$

where, to obtain the last equality, we recognize that for

$$\mathbf{k}=(k_1,\cdots,k_{m+1})\in\mathcal{K}_{m+1},$$

we have $(\mathbf{k},k_w)\in\mathcal{K}_{m+2}$ if and only if $k_{m+1}+1\le w\le n$. In conclusion, we have

$$\mathcal{T}_{\mathbf{q}}(0)=i^p\mathbb{T}(p-1,v)+\sum_{m'=1}^{m+1}\left(\sum_{\mathbf{k}\in\mathcal{K}_{m'}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m'}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\right)$$
$$+\sum_{\mathbf{k}\in\mathcal{K}_{m+2}} i^{J_{p(m+2)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{m+2}}(0)\cdot\mathbb{R}_{\mathbf{k}}(J_{p(m+2)}-1,v).$$

This induction is completed. Finally, to obtain (9.5), we set $m=n$ in (9.8). □

We move on to prove Proposition 6.2 assuming all zero-subtrees of $\mathcal{T}(0)$ are mutually exclusive. Letting $\mathbf{q}_k$ be the part of $\mathbf{q}$ that defines $\mathcal{T}_{j_k}$, we have

$$\mathbf{q}=\mathbf{q}_R\cup\mathbf{q}_1\cup\cdots\cup\mathbf{q}_n,$$

and

$$\mathbb{S}_{\mathbf{q}}=\{\hat{\mathbf{q}}:=\mathbf{q}_R\cup\pm\mathbf{q}_1\cup\cdots\cup\pm\mathbf{q}_n\}. \tag{9.10}$$

Also recall that

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}}=\sum_{\hat{\mathbf{q}}\in\mathbb{S}_{\mathbf{q}}}\left(\mathcal{T}_{\hat{\mathbf{q}}}(0)-\mathcal{T}_{-\hat{\mathbf{q}}}(0)\right).$$

We prove that, under the assumption that the total index of $\mathbf{q}$ is not zero and all zero subtrees of $\mathcal{T}_{\mathbf{q}}(0)$ are mutually exclusive, we have, modulo all terms that are exponentially small,

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = 0. \tag{9.11}$$

In this proof, all exponentially small terms produced in the inductive process of Section 5 are ignored. This is to say that all equalities presented in this proof hold modulo these terms. We will have to count how many of these are dropped, but this will come later. Here we are focused entirely on the issue of cancellation.

Take a term for $\mathcal{T}_{\mathbf{q}}(0)$ in (9.5), say,

$$\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_m}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v) \tag{9.12}$$

where $\mathbf{k}=(k_1,\cdots,k_m)\in\mathcal{K}_m$. We define a $\hat{\mathbf{q}}$ by changing, in $\mathbf{q}$, the sign of $\mathbf{q}_{k_1},\cdots\mathbf{q}_{k_m}$ to $-\mathbf{q}_{k_1},\cdots,-\mathbf{q}_{k_m}$. Now, in (9.5) for $\mathcal{T}_{-\hat{\mathbf{q}}}(0)$, we have a corresponding term

$$\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_m}}(0)\cdot\hat{\mathbb{R}}_{\mathbf{k}}(p_{\mathbf{k}}-1,v), \tag{9.13}$$

where $\hat{\mathbb{R}}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$ is obtained by reverting the defining $\mathbf{q}$ vector, say $\mathbf{q}_{R_{\mathbf{k}}}$, for $\mathcal{R}_{\mathbf{k}}$ in $\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$ to $-\mathbf{q}_{R_{\mathbf{k}}}$. We note that both $\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$ and $\hat{\mathbb{R}}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)$ are completely in s-variables, and we have

$$\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)=\hat{\mathbb{R}}_{\mathbf{k}}(p_{\mathbf{k}}-1,v). \tag{9.14}$$

Equality (9.14) shares the same proof with Lemma 6.1. Consequently, the term in (9.13) cancels the term in (9.12).

We now present a formal proof. All equalities are as usual modulo exponentially small terms. By using (9.5), we have

$$\begin{aligned}
\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} &= \sum_{\hat{q}\in\mathbb{S}_{\mathbf{q}}}\sum_{m=0}^{n}\sum_{\mathbf{k}\in\mathcal{K}_m} i^{p_{\mathbf{k}}}\Big(\mathcal{T}_{\hat{\mathbf{q}}_{k_1}}\cdots\mathcal{T}_{\hat{\mathbf{q}}_{k_m}}\cdot\mathbb{R}_{\mathbf{q}_R\cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)}\hat{\mathbf{q}}_j}(p_{\mathbf{k}}-1,v)\\
&\quad -\mathcal{T}_{-\hat{\mathbf{q}}_{k_1}}\cdots\mathcal{T}_{-\hat{\mathbf{q}}_{k_m}}\cdot\mathbb{R}_{-[\mathbf{q}_R\cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)}\hat{\mathbf{q}}_j]}(p_{\mathbf{k}}-1,v)\Big)\\
&= \sum_{\pm}\sum_{m=0}^{n}\sum_{\mathbf{k}\in\mathcal{K}_m} i^{p_{\mathbf{k}}}\Big(\mathcal{T}_{\pm\mathbf{q}_{k_1}}\cdots\mathcal{T}_{\pm\mathbf{q}_{k_m}}\cdot\mathbb{R}_{\mathbf{q}_R\cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)}\pm\mathbf{q}_j}(p_{\mathbf{k}}-1,v)\\
&\quad -\mathcal{T}_{\mp\mathbf{q}_{k_1}}\cdots\mathcal{T}_{\mp\mathbf{q}_{k_m}}\cdot\mathbb{R}_{-[\mathbf{q}_R\cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)}\pm\mathbf{q}_j]}(p_{\mathbf{k}}-1,v)\Big)
\end{aligned}$$

where on the right hand side of the last equality, $\pm$ and $\mp$ assume opposite signs, and $\sum_{\pm}$ means the sum over all admitted choices of $\pm$. We now switch the order of $\sum_{\pm}$ and $\sum_{m=0}^{n}\sum_{\mathbf{k}\in\mathcal{K}_m}$ to obtain

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = \sum_{m=0}^{n} \sum_{\mathbf{k}\in\mathcal{K}_m} \sum_{\pm} i^{p_{\mathbf{k}}} \Big(\mathcal{T}_{\pm\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\pm\mathbf{q}_{k_m}} \cdot \mathbb{R}_{\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j}(p_{\mathbf{k}}-1, v)$$
$$- \mathcal{T}_{\mp\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\mp\mathbf{q}_{k_m}} \cdot \mathbb{R}_{-[\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j]}(p_{\mathbf{k}}-1, v) \Big).$$

Note that this switch of order is allowed because the definition of the index set $\mathcal{K}$ is independent of $\hat{\mathbf{q}}$. To continue, we have

$$\begin{aligned}
\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} &= \sum_{m=0}^{n} \sum_{\mathbf{k}\in\mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\pm\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\pm\mathbf{q}_{k_m}} \right) \cdot \left(\sum_{\pm} \mathbb{R}_{\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j}(p_{\mathbf{k}}-1, v) \right) \\
&\quad - \sum_{m=0}^{n} \sum_{\mathbf{k}\in\mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\mp\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\mp\mathbf{q}_{k_m}} \right) \cdot \left(\sum_{\pm} \mathbb{R}_{-[\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j]}(p_{\mathbf{k}}-1, v) \right) \\
&= \sum_{m=0}^{n} \sum_{\mathbf{k}\in\mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\pm\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\pm\mathbf{q}_{k_m}} \right) \cdot \sum_{\pm} \Big(\mathbb{R}_{\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j}(p_{\mathbf{k}}-1, v) \\
&\quad - \mathbb{R}_{-[\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j]}(p_{\mathbf{k}}-1, v) \Big) \\
&= 0.
\end{aligned}$$

In the first equality, we break one $\sum_{\pm}$ into two because the assignment of $\pm$ to all individual basic $\mathbf{q}$-block is independent of each other. The second equality holds because

$$\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\mp\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\mp\mathbf{q}_{k_m}} = \sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\pm\mathbf{q}_{k_1}} \cdots \mathcal{T}_{\pm\mathbf{q}_{k_m}}.$$

The last equality follows from

$$\mathbb{R}_{\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j}(p_{\mathbf{k}}-1, v) = \mathbb{R}_{-[\mathbf{q}_R \cup_{j\in[1,n]\setminus(k_1,\cdots,k_m)} \pm\mathbf{q}_j]}(p_{\mathbf{k}}-1, v).$$

This equality shares the same proof as that of Lemma 6.1. This proves (9.11).

9.3. *Proof of the general case*

In this subsection we assume $\mathcal{T}_{\mathbf{q}}(0)$ is a pure integral and $Q_{\mathbf{q}} \neq 0$. We prove Proposition 6.2 *without* assuming all zero subtrees are mutually exclusive. Assuming $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees, we have in total, counting $\mathbf{q}_R$, $n+1$ basic $\mathbf{q}$-block which we denote as

$$\mathbf{q}_R,\ \mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n.$$

These are, by definition, a mutually disjoint subset of $\mathbf{q}$ and we have

$$\mathbf{q} = \mathbf{q}_R \cup \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_n.$$

Recall that, for $\mathbf{q} \in \mathcal{S}_{\mathbf{q}}^{+}$,

$$\mathbb{S}_{\mathbf{q}} = \{\hat{\mathbf{q}} = \mathbf{q}_R \cup \pm\mathbf{q}_1 \cup \cdots \cup \pm\mathbf{q}_n\} \tag{9.15}$$

where $\pm$ is either $+$ or $-$ if the basic $\mathbf{q}$-block it marks is not a zero vector. We set $\pm = 0$ if the basic $\mathbf{q}$-block it marks is a zero vector. Also recall that

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = \sum_{\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}} \left(\mathcal{T}_{\hat{\mathbf{q}}}(0) - \mathcal{T}_{-\hat{\mathbf{q}}}(0)\right).$$

Proposition 9.1. *We have, modulo exponentially small terms, that*

$$\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = 0.$$

Proof. We follow closely the proof of the same identity presented in the previous subsection, introducing necessary adjustment along the way. Assume that $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees rooted at $N_{j_1}, \cdots, N_{j_n}$ respectively where

$$j_1 > j_2 > \cdots > j_n.$$

Denote the subtree rooted at N_{j_k} as $\mathcal{T}_{j_k}$. We denote the remainder tree obtained by dropping $\mathcal{T}_{j_k}$ from $\mathcal{T}_{\mathbf{q}}(0)$ as $\mathcal{R}_{j_k}(0)$. By using (5.13), we again have, modulo exponentially small terms,

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}_{\mathbf{q}}(p-1, v) + \sum_{k=1}^{n} i^{p-j_k} \mathcal{T}_{j_k} \cdot \mathbb{R}_{j_k}(p - j_k - 1, v). \tag{9.16}$$

The integral $\mathbb{R}_{j_k}(p - j_k - 1, v)$ is in a mix of s and τ variables. We can think of it as the one obtained in the middle of decomposing $\mathcal{R}_{j_k}(0)$. The top portion of $\mathbb{R}_{j_k}(p - j_k - 1, v)$ up to N_{j_k+1} is in s-variables and the rest is in τ-variable.

We continue to decompose $\mathbb{R}_{j_k}(p - j_k - 1, v)$. There are two complications in our current situation.

(1) We can not assume that $\mathcal{T}_{j_{k'}}$ is in $\mathcal{R}_{j_k}(0)$ for all $k' > k$ because $\mathcal{T}_{j_{k'}}$ might be a subtree of $\mathcal{T}_{j_k}$. In this case by deleting $\mathcal{T}_{j_k}$ from $\mathcal{T}_{\mathbf{q}}(0)$, we also delete $\mathcal{T}_{j_{k'}}$.

(2) A zero-subtree of $\mathcal{R}_{j_k}(0)$ is no longer necessarily a zero subtree of $\mathcal{T}_{\mathbf{q}}(0)$. Assuming $k < k'$, and that $\mathcal{T}_{j_{k'}}$ is a subtree of $\mathcal{T}_{j_k}$, then $\mathcal{T}_{j_k} \setminus \mathcal{T}_{j_{k'}}$ is a zero subtree in $\mathcal{R}_{j_{k'}}(0)$.

With these issues in mind, we proceed to obtain a correspondence of (9.5). We start with

Definition 9.1. Let $\mathbf{k} = (k_1, \cdots, k_m)$ be such that

$$k_1 < k_2 < \cdots < k_m \le n.$$

We say $\mathbf{k}$ is *admissible* if $\mathcal{T}_{j_{k_1}}, \cdots, \mathcal{T}_{j_{k_m}}$ are mutually exclusive.

Let $\mathcal{K}_m$ be the set of all admissible $\mathbf{k}$ of m components, and

$$\mathcal{K} = \cup_{m=1}^{n} \mathcal{K}_m.$$

For $\mathbf{k} = (k_1, \cdots, k_m) \in \mathcal{K}_m$, let $\mathcal{R}_{\mathbf{k}}(0) = \mathcal{R}_{j_{k_1} \cdots j_{k_m}}(0)$ be the remainder tree obtained by deleting from $\mathcal{T}_{\mathbf{q}}(0)$ the subtrees $T_{j_{k_1}}, \cdots, T_{j_{k_m}}$. Also let $p_{\mathbf{k}} = p - p_{j_{k_1}} - \cdots - p_{j_{k_m}}$ be the order of $\mathcal{R}_{\mathbf{k}}(0)$. We now use this new $\mathcal{K}$ to replace the one in (9.5) to obtain

$$\mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) + \sum_{m=1}^{n} \left(\sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}} \cdots \mathcal{T}_{j_{k_m}} \cdot \mathbb{R}_{j_{k_1} \cdots j_{k_m}}(p_{\mathbf{k}} - 1, v) \right). \tag{9.17}$$

To prove (9.17), we repeat the proof of Lemma 9.2, except we change all sums over w from $k_m + 1$ to n to sums over all w so that $(k_1, \cdots, k_m, w)$ is admissible. The details of this proof in the current case are as follows. Let $\mathbf{k} = (k_1, \cdots, k_m) \in \mathcal{K}_m$ where $\mathcal{K}_m$ is the set of admissible $\mathbf{k}$ of size m. We again define the integers J_{pm}, J_{mw}, and J_{me} in the same way as in the previous subsection. For a correspondence of (9.7), we use (5.11) repeatedly to obtain

$$\mathbb{R}_{\mathbf{k}}(J_{pm} - 1, v) = i^{J_{me}} \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) + \sum_{w:(\mathbf{k},w) \in \mathcal{K}_{m+1}} i^{J_{mw}} \mathcal{T}_{j_w}(0) \cdot \mathbb{R}_{(\mathbf{k},w)}(J_{pw} - 1, v). \tag{9.18}$$

Our proof of (9.17) is inductive on m, and the inductive assumption is

$$\begin{aligned} \mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) &+ \sum_{m'=1}^{m} \left(\sum_{\mathbf{k} \in \mathcal{K}_{m'}} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{k_{m'}}}(0) \cdot \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) \right) \\ &+ \sum_{\mathbf{k} \in \mathcal{K}_{m+1}} i^{J_{p(m+1)}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{m+1}}(0) \mathbb{R}_{\mathbf{k}}(J_{p(m+1)} - 1, v). \end{aligned} \tag{9.19}$$

To prove (9.19), we need (9.18) for $m+1$ for $\mathbf{k} = (k_1, \cdots, k_{m+1}) \in \mathcal{K}_{m+1}$, which is

$$\begin{aligned} \mathbb{R}_{\mathbf{k}}(J_{p(m+1)} - 1, v) &= i^{J_{(m+1)e}} \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) \\ &+ \sum_{w:(\mathbf{k},w) \in \mathcal{K}_{m+2}} i^{J_{(m+1)w}} \mathcal{T}_{j_w}(0) \cdot \mathbb{R}_{(\mathbf{k},w)}(J_{pw} - 1, v). \end{aligned} \tag{9.20}$$

Assume (9.19) for m. We advance this induction by replacing $\mathbb{R}_{\mathbf{k}}(J_{p(m+1)} - 1, v)$ in (9.19) by using (9.20) to obtain

$$\begin{aligned} \mathcal{T}_{\mathbf{q}}(0) = i^p \mathbb{T}(p-1, v) &+ \sum_{m'=1}^{m} \left(\sum_{\mathbf{k} \in \mathcal{K}_{m'}} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{k_{m'}}}(0) \cdot \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) \right) \\ &+ \sum_{\mathbf{k} \in \mathcal{K}_{m+1}} i^{J_{p(m+1)} + J_{(m+1)e}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{m+1}}(0) \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) \\ &+ \sum_{\mathbf{k} \in \mathcal{K}_{m+1}} i^{J_{p(m+1)}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{m+1}}(0) \left(\sum_{w:(\mathbf{k},w) \in \mathcal{K}_{m+2}} i^{J_{(m+1)w}} \mathcal{T}_{j_w}(0) \cdot \mathbb{R}_{(\mathbf{k},w)}(J_{pw} - 1, v) \right). \end{aligned}$$

Note that $p_{\mathbf{k}} = J_{p(m+1)} + J_{(m+1)e}$ so the second line is added to the first sum to lift the upper index for m' from m to $m+1$. This implies

$$\begin{aligned}\mathcal{T}_{\mathbf{q}}(0) &= i^p \mathbb{T}(p-1,v) + \sum_{m'=1}^{m+1}\left(\sum_{\mathbf{k}\in\mathcal{K}_{m'}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m'}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\right)\\ &\quad+\sum_{\mathbf{k}\in\mathcal{K}_{m+1}}\sum_{w:(\mathbf{k},w)\in\mathcal{K}_{m+2}} i^{J_{pw}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{m+1}}(0)\cdot\mathcal{T}_{j_w}(0)\cdot\mathbb{R}_{(\mathbf{k},w)}(J_{pw}-1,v)\\ &= i^p\mathbb{T}(p-1,v)+\sum_{m'=1}^{m+1}\left(\sum_{\mathbf{k}\in\mathcal{K}_{m'}} i^{p_{\mathbf{k}}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{k_{m'}}}(0)\cdot\mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}}-1,v)\right)\\ &\quad+\sum_{\mathbf{k}\in\mathcal{K}_{m+2}} i^{J_{p(m+2)}}\mathcal{T}_{j_{k_1}}(0)\cdots\mathcal{T}_{j_{m+2}}(0)\cdot\mathbb{R}_{\mathbf{k}}(J_{p(m+2)}-1,v).\end{aligned}$$

To obtain the last equality, we make the following switch of notation. For $\mathbf{k}=(k_1,\cdots,k_{m+1})\in\mathcal{K}_{m+1}$, and $(\mathbf{k},w)\in\mathcal{K}_{m+2}$, we write $(\mathbf{k},w)=(k_1,\cdots,k_{m+1},k_{m+2})$ so $k_{m+2}=w$.

The last equality is (9.19) for $m+1$. To obtain (9.17), we set $m=n$ in (9.19).

We are now ready to prove Proposition 9.1. All equalities are as usual modulo exponentially small terms. Assume $\mathcal{T}_{\mathbf{q}}(0)$ has n zero subtrees, which we denote as

$$\mathcal{T}_{j_1},\cdots,\mathcal{T}_{j_n}$$

where $j_1>j_2\cdots>j_n$. We start by writing $\mathbf{q}$ as

$$\mathbf{q}=\mathbf{q}_R\cup\mathbf{q}_1\cup\cdots\cup\mathbf{q}_n$$

where $\mathbf{q}_R,\mathbf{q}_1,\cdots,\mathbf{q}_n$ are basic $\mathbf{q}$-block. Let $\mathbf{q}(j_k)$ be the $\mathbf{q}$-vector that defines the zero subtree $\mathcal{T}_{j_k}$. We have

$$\mathbf{q}(j_k)=\cup_{k'\in B(j_k)}\mathbf{q}_{k'}.$$

Note that if $\mathcal{T}_{j_{k'}}\subset\mathcal{T}_{j_k}$, then $B(j_{k'})\subset B(j_k)$. Let $\mathbf{k}=(k_1,\cdots,k_m)\in\mathcal{K}_m$ be admissible. Since $\mathcal{T}_{j_{k_1}},\cdots,\mathcal{T}_{j_{k_m}}$ are mutually exclusive,

$$B(j_{k_1}),\cdots,B(j_{k_m})$$

are mutually disjoint. Let

$$B(\mathbf{k})=B(j_{k_1})\cup\cdots\cup B(j_{k_m}).$$

The $\mathbf{q}$-vector that defines $\mathcal{R}_{\mathbf{k}}(0)$ is

$$\mathbf{q}(R_{\mathbf{k}})=\mathbf{q}_R\cup_{k'\in[1,n]\setminus B(\mathbf{k})}\mathbf{q}_{k'}.$$

Let $\hat{\mathbf{q}}$ be such that $\hat{\mathbf{q}}\in\mathbb{S}_{\mathbf{q}}$. We write $\hat{\mathbf{q}}$ as

$$\hat{\mathbf{q}}=\mathbf{q}_R\cup\pm\mathbf{q}_1\cup\cdots\cup\pm\mathbf{q}_n$$

where $\pm$ assumes either $+$ or $-$ if the basic $\mathbf{q}$-block it marks is not a zero vector, and $\pm = 0$ if the basic $\mathbf{q}$-block it marks is a zero vector. For a fixed $\hat{\mathbf{q}}$, let $\mathbf{k} = (k_1, \cdots, k_m) \in \mathcal{K}_m$. We write $\hat{\mathbf{q}}$ as

$$\hat{\mathbf{q}} = \hat{\mathbf{q}}(R_{\mathbf{k}}) \cup \hat{\mathbf{q}}(j_{k_1}) \cup \cdots \cup \hat{\mathbf{q}}(j_{k_m})$$

where

$$\hat{\mathbf{q}}(j_k) = \cup_{k' \in B(j_k)} \pm \mathbf{q}_{k'},$$

and

$$\hat{\mathbf{q}}(R_{\mathbf{k}}) = \mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}.$$

We have, by using (9.17),

$$\begin{aligned}
\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = & \sum_{\hat{q} \in \mathbb{S}_{\mathbf{q}}} \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \Big(\mathcal{T}_{\hat{\mathbf{q}}(j_{k_1})} \cdots \mathcal{T}_{\hat{\mathbf{q}}(j_{k_m})} \cdot \mathbb{R}_{\hat{\mathbf{q}}(R_{\mathbf{k}})}(p_{\mathbf{k}} - 1, v) \\
& - \mathcal{T}_{-\hat{\mathbf{q}}(j_{k_1})} \cdots \mathcal{T}_{-\hat{\mathbf{q}}(j_{k_m})} \cdot \mathbb{R}_{-\hat{\mathbf{q}}(R_{\mathbf{k}})}(p_{\mathbf{k}} - 1, v) \Big) \\
= & \sum_{\pm} \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \Big(\mathcal{T}_{\cup_{k' \in B(j_1)} \pm \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \pm \mathbf{q}_{k'}} \cdot \mathbb{R}_{\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}}(p_{\mathbf{k}} - 1, v) \\
& - \mathcal{T}_{\cup_{k' \in B(j_1)} \mp \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \mp \mathbf{q}_{k'}} \cdot \mathbb{R}_{-\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \mp \mathbf{q}_{k'}}(p_{\mathbf{k}} - 1, v) \Big)
\end{aligned}$$

where in the second equality, (i) $\pm$ can take either $+$ or $-$ if the $\mathbf{q}$-vector it marks is not a zero vector, and $\pm = 0$ if the $\mathbf{q}$-vector it marks is a zero vector; (ii) $\pm$ and $\mp$ assume opposite signs; and (iii) $\sum_{\pm}$ means the sum over all possible $\pm$ as laid out in (i). We have

$$\begin{aligned}
\mathbb{D}_{\mathbb{S}_{\mathbf{q}}} = & \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\cup_{k' \in B(j_1)} \pm \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \pm \mathbf{q}_{k'}} \right) \\
& \cdot \left(\sum_{\pm} \mathbb{R}_{\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}}(p_{\mathbf{k}} - 1, v) \right) \\
& - \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\cup_{k' \in B(j_1)} \mp \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \mp \mathbf{q}_{k'}} \right) \\
& \cdot \left(\sum_{\pm} \mathbb{R}_{-\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \mp \mathbf{q}_{k'}}(p_{\mathbf{k}} - 1, v) \right) \\
= & \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} \left(\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\cup_{k' \in B(j_1)} \pm \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \pm \mathbf{q}_{k'}} \right)
\end{aligned}$$

$$\cdot \sum_{\pm} \left(\mathbb{R}_{\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}} (p_{\mathbf{k}} - 1, v) - \mathbb{R}_{-[\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}]} (p_{\mathbf{k}} - 1, v) \right)$$
$$= 0$$

where in the first equality we are allowed to break one $\sum_{\pm}$ into two because the assignment of $\pm$ to all individual basic $\mathbf{q}$-blocks are independent of each other. The second equality holds because

$$\sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\cup_{k' \in B(j_1)} \pm \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \pm \mathbf{q}_{k'}} = \sum_{\pm} i^{p_{\mathbf{k}}} \mathcal{T}_{\cup_{k' \in B(j_1)} \mp \mathbf{q}_{k'}} \cdots \mathcal{T}_{\cup_{k' \in B(j_m)} \mp \mathbf{q}_{k'}}.$$

The last equality follows again from

$$\mathbb{R}_{[\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}]} (p_{\mathbf{k}} - 1, v) = \mathbb{R}_{-[\mathbf{q}_R \cup_{k' \in [1,n] \setminus B(\mathbf{k})} \pm \mathbf{q}_{k'}]} (p_{\mathbf{k}} - 1, v).$$

This proves Proposition 9.1. □

Proof of Proposition 6.2. We are finally ready to prove

$$\left| \mathbb{D}_{\mathbb{S}_{\mathbf{q}}} \right| \leq (K\omega)^{27p} e^{-\omega\pi/2}.$$

We start with the correspondence of (9.18), which we now write as

$$\mathbb{R}_{\mathbf{k}}(J_{pm} - 1, v) = i^{J_{me}} \mathbb{R}_{\mathbf{k}}(p_{\mathbf{k}} - 1, v) + \sum_{w:(\mathbf{k},w) \in \mathcal{K}_{m+1}} i^{J_{mw}} \mathcal{T}_{j_w}(0) \mathbb{R}_{(\mathbf{k},w)}(J_{pm} + J_{mw} - 1, v) + E_{\mathbf{k}}.$$

It follows directly from Proposition 6.1 that

$$|E_{\mathbf{k}}| < \left(K\omega^{27} \right)^p e^{-\omega\pi/2}.$$

We then obtain the correspondence of (9.5), which we write as

$$\mathcal{T}_{\mathbf{q}}(0) = \sum_{m=0}^{n} \left(\sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{k_m}}(0) \cdot \mathbb{R}_{j_{k_1} \cdots j_{k_m}}(p_{\mathbf{k}} - 1, v) \right) + E_{\mathbf{q}}. \tag{9.21}$$

We have

$$|E_{\mathbf{q}}| = \left| \sum_{m=0}^{n} \sum_{\mathbf{k} \in \mathcal{K}_m} i^{p_{\mathbf{k}}} \mathcal{T}_{j_{k_1}}(0) \cdots \mathcal{T}_{j_{k_m}}(0) E_{\mathbf{k}} \right| \leq \left(\sum_{m=0}^{p} \binom{p}{m} K^{p-m} \right) \left(K\omega^{27} \right)^p e^{-\omega\pi/2}$$
$$\leq ((K+1)\omega^{27})^p e^{-\omega\pi/2}.$$

The same estimate applies to $E_{\pm\hat{\mathbf{q}}}$ for all $\hat{q} \in \mathbb{S}_{\mathbf{q}}$. We have

$$\left| \mathbb{D}_{\mathbb{S}_{\mathbf{q}}} \right| = \left| \sum_{\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}} \left(\mathcal{T}_{\hat{\mathbf{q}}}(0) - \mathcal{T}_{-\hat{\mathbf{q}}}(0) \right) \right| \leq \sum_{\hat{\mathbf{q}} \in \mathbb{S}_{\mathbf{q}}} \left(|E_{\hat{\mathbf{q}}}| + |E_{-\hat{\mathbf{q}}}| \right) \leq 2^{p+1} ((K+1)\omega^{27})^p e^{-\omega\pi/2}$$

where Proposition 9.1 is used to obtain the first inequality.

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