More on The Heteroclinic Orbits for The Monotone Twist Maps

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Abstract

We continue our study of heteroclinic connections for monotone twist maps in this paper. We show that if the Percival's barrier function is not identically zero, and the energy difference for an adjacent pair of a local minimal and a global minimal fixed point is sufficiently small, heteroclinic connections from one fixed point to the other exist. The connections we construct in this paper are local minimal energy orbits.

1 The Statement

We continue our study of heteroclinic connections for monotone twist map in this paper. Take a function $h(x,x') : R^2 \mapsto R$. We assume that h is C^2, $h(x+1,x'+1) = h(x,x')$, $\partial_{12}h(x,x') < -\delta$. Where δ is a positive constant. The function h generates a monotone twist diffeomorphism of the infinite cylinder $(R/Z) \times R$.

Let $f(x) = h(x,x)$. The critical points of the function $f(x)$ corresponds to the fixed points of the monotone twist map generated by $h(x,x')$. Denote the set of all the locally minimum critical points of $f(x)$ as S and the set of all the globally minimum critical points of $f(x)$ as $\hat{S}$. $\hat{S}$ is a subset of S. Take two elements, $c_1, c_2 \in S$, $c_1 < c_2$. We introduced the concept of "adjacent pair" in [8] as the following:

Definition 1 *$c_1, c_2 \in S$ are an adjacent pair if there is a d, $c_1 < d < c_2$, such that $f(x) > f(c_1)$ if $x \in (c_1, d]$, and $f(x) > f(c_2)$ if $x \in [d, c_2)$. We call d a separating point for this pair.*

Assume that there are three points, c_1, c_2, c; $c_1, c_2 \in \hat{S}$ and $c \in S \setminus \hat{S}$. $c_1 < c < c_2$ and (c_1, c_2), (c, c_2) are adjacent pairs. We study the possible heteroclinic connections between c_1, c_2 and c. We will show that, when Percival's Barrier function P_{0^-} is not identically zero between c_1 and c_2, there exists heteroclinic connections from c_2 to c if $f(c) - f(c_2)$ is sufficiently small, and the heteroclinic connection is a local minimal orbit. For the definition of Percival's barrier function, refer to [4].

Without loss of generality, in the rest of this paper we set $f(c_1) = f(c_2) = 0$. To precisely state our result, we need the following assumptions and parameters.

- Let $P_{0^-}(x)$ be the Percival's barrier function with rotation number 0^-. We assume that $P_{0^-}(x)$ is not identically zero on $[c, c_2]$. Further, denote
$$P = max\{P_{0^-}(x); \quad c \le x \le c_2\}$$
- We assume that c, c_2 are both non-degenerate minimal points of $f(x)$. So there are positive constants K_1, K_2 and δ_1, such that
$$K_1(x-c_2)^2 \le f(x) \le K_2(x-c_2)^2 \qquad | x - c_2 | \le \delta_1$$
and
$$K_1(x-c)^2 \le f(x) - f(c) \le K_2(x-c)^2 \qquad | x - c | \le \delta_1$$
- There are no other locally minimal fixed points in $[c, c_2]$.
- For all (x,x'), $| x' - x | < 1$, there is a positive constant θ, such that
$$| \partial_1 h(x,x') |, | \partial_1 h(x,x') |, | h(x,x') | < \theta.$$
$$| \partial_{12} h(x,x') |, | \partial_{11} h(x,x') |, | \partial_{22} h(x,x') | < \theta.$$

Under all these assumptions, we have

Theorem 1 *Assume that*

$$f(c) \leq min\{\frac{\delta^2\delta_1^2}{4(K_2+\delta)},\ \frac{1}{2}K_1K_5^2,\ \frac{K_6P}{20\theta+2P},\ \frac{P^2\delta^2}{1600(K_1+\delta)}\}$$

where

$$K_5 = min\{\delta_1,\ \frac{P}{2\theta}\}\quad K_6 = min\{\frac{K_1P^2}{400\theta^2},\ \frac{1}{2}K_1K_5^2\}$$

Then there exists heteroclinic connection from c_2 to c.

Our study of heteroclinic connections was motivated by Mather's work on the connecting orbit inside the Birkhorff's region of instability [7]. We intend to modify Mather's construction to get a connecting orbit crossing around local minimal fixed point. Theorem 1 is the first step toward such a construction. The rest of this paper is for the proof of this theorem.

2 Some Properties on Aubry-Mather Orbits

2.1 The Gap Size

First we recall that, for a given sequence $x = \{x_i\}_{i=i_1}^{i_2}$, by the energy of x we mean the value

$$H(x) = \sum_{i=i_1}^{i_2-1} h(x_i, x_{i+1}).$$

Let $x = \{x_i\}_{i=-\infty}^{\infty}$ be a minimal energy orbit with rotation number 0^-, $\lim_{i\to-\infty} x_i = c_2$, $\lim_{i\to\infty} x_i = c_1$. Such an orbit always exist [2], and $\{x_i\}_{i=-\infty}^{\infty}$ is a strictly decreasing sequence. x is a minimal energy orbit also in the sense that, changing x by adding or dropping one of its points does not decrease the total energy of the orbit.

For the given minimal orbit x, let j be the index such that

$$x_{j+1} \leq c \leq x_j$$

we have

Lemma 1 *If $f(c) < \frac{\delta^2\delta_1^2}{4(K_2+\delta)}$, then*

$$\mid x_j - x_{j+1} \mid < 2\hat{\varepsilon}$$

where

$$\hat{\varepsilon} = \frac{2\sqrt{K_1+\delta}}{\delta}\sqrt{f(c)}$$

Proof: Like most of the arguments given in [8] , we first assume that $\mid x_j - x_{j+1} \mid \geq 2\hat{\varepsilon}$. then try to add a point in the middle of x_{j+1} and x_j to reduce the energy, therefore, to induce a contradiction.

The first step of the proof is to show that $\mid x_j - c \mid < \delta_1$. Assume otherwise, take

$$t_0 = \frac{\delta(x_j - c) - \sqrt{\delta^2(x_j - c)^2 - 4f(c)(K_2 + \delta)}}{K_2 + \delta} \tag{1}$$

By our assumption on $f(c)$, we have $t_0 \geq 0$, $t_0 < x_j - c$. Furthermore, we have

$$t_0 = \frac{4f(c)(K_2 + \delta)}{(K_2 + \delta)(\delta(x_j - c) + \sqrt{\delta^2(x_j - c)^2 - 4f(c)(K_2 + \delta)})}$$

$$\leq \frac{4f(c)}{\delta(x_j - c)} \leq \frac{4f(c)}{\delta\delta_2} < \delta_1.$$

Therefore, we have

$$f(t_0 + c) < K_2 t_0^2 + f(c)$$

Now add the point $t_0 + c$ in the middle of x_j, x_{j+1} to form a new sequence, say, y. We have

$$H(y) - H(x) \leq K_2 t_0^2 - \delta(t_0 + c - x_{j+1})(x_j - (t_0 + c)) + f(c)$$
$$< (K_2 + \delta)t_0^2 - \delta(x_j - c)t_0 + f(c)$$

Notice that t_0 is exactly a root of the last formula. we conclude

$$H(y) < H(x)$$

which contradicts to the definition of x.

Now for the proof of the lemma. If $\mid x_j - x_{j+1} \mid > 2\hat{\varepsilon}$, we either have $\mid x_j - c \mid > \hat{\varepsilon}$ or $\mid c - x_{j+1} \mid > \hat{\varepsilon}$. Without loss of generality, we assume the former. Now take t_0 as in (1). The formula inside the the square root sign is positive under our assumption so t_0 is a real positive number. Notice $t_0 + c < x_j < \delta_1$, we have

$$f(t_0 + c) < K_2 t_0^2 + f(c)$$

Again, the sequence y obtained by adding $t_0 + c$ in the middle of x_j, x_{j+1} reduces the energy therefore gives a contradiction. This finishes the proof of the lemma.

2.2 minimal configuration that reaches ξ

Now recall some basic facts of the Aubry-Mather theory. Let

$$\Lambda = \{x : \quad x \in (c_1, c_2); P_{0^-}(x) = 0\}$$

Λ is a closed set. Let ξ be a point in (c, c_2) such that

$$P_{0^-}(\xi) = P$$

and (η^-, η^+) be the complement interval of Λ including ξ. The twist map generated by $h(x, x')$ induces a map $\phi : R \mapsto R$. [3, 4].

Denote $I_i = [\phi^i(\eta^-), \phi^i(\eta^+)]$, we form a space

$$X = \prod_{i=-\infty}^{-1} I_i \times \{\xi\} \times \prod_{i=1}^{\infty} I_i$$

A configuration $y = \{y_i\}_{i=-\infty}^{\infty} \in X$ if and only if $y_i \in I_i$ when $i \neq 0$ and $y_0 = \xi$.

For any given configuration $y \in X$, the energy of y is given by

$$H(y) = \sum_{i=-\infty}^{\infty} h(x_i, x_{i+1})$$

Denote $x^+ = \{x_i^+\}_{i=-\infty}^{\infty} = \{\phi^i(\eta^+)\}_{i=-\infty}^{\infty}$, $x^- = \{x_i^-\}_{i=-\infty}^{\infty} = \{\phi^i(\eta^-)\}_{i=-\infty}^{\infty}$. Let y be the configuration which minimizes the energy $H(y)$ over X. By the definition of the Percival's Barrier function, we have

$$H(y) - H(x^+) = P, \quad H(y) - H(x^-) = P$$

Remember that x^+, x^- are both minimal heteroclinic orbits from c_2 to c_1.

Now take any configuration $\hat{y} = \{\hat{y}_i\}_{i=-\infty}^{\infty}$ such that $\hat{y}_0 = \xi$, $\hat{y}_i \in (c_1, c_2)$ and $\lim_{i \to -\infty} \hat{y}_i = c_2$, $\lim_{i \to +\infty} \hat{y}_i = c_1$, We have

Lemma 2 *For any configuration $\hat{y}$ given as above, we have*

$$H(\hat{y}) \geq H(y)$$

Proof. This lemma is either easy or hard depending on how much one knows Mather's work. By the Aubry's inequality

$$H(x^+ \bigvee \hat{y}) + H(x^+ \bigwedge \hat{y}) \leq H(x^+) + H(\hat{y})$$

where

$$\{x^+ \bigvee \hat{y}\}_i = max\{x_i^+, \ \hat{y}_i\} \quad \{x^+ \bigwedge \hat{y}\}_i = min\{x_i^+, \ \hat{y}_i\}.$$

Since

$$\{x^+ \bigvee \hat{y}\}_0 = max\{x_0^+, \ \hat{y}_0\} = x_0^+,$$

$x^+ \bigvee \hat{y}$ and x^+ are asymptotic, we have

$$H(x^+ \bigvee \hat{y}) \geq H(x^+)$$

Therefore we have

$$H(x^+ \bigwedge \hat{y}) \le H(\hat{y})$$

Now apply the inequality again on $x^+ \bigvee \hat{y}$ and x^- we have

$$H((x^+ \bigwedge \hat{y}) \bigvee x^-) + H((x^+ \bigwedge \hat{y}) \bigwedge x^-) \le H(x^+ \bigwedge \hat{y}) + H(x^-)$$

Again since $(x^+ \bigwedge \hat{y}) \bigwedge x^-$ and x^- are asymptotic, and

$$\{(x^+ \bigwedge \hat{y}) \bigwedge x^-\}_0 = x_0^-,$$

We conclude

$$H((x^+ \bigwedge \hat{y}) \bigwedge x^-) \ge H(x^-)$$

therefore

$$H((x^+ \bigwedge \hat{y}) \bigvee x^-) \le H(x^+ \bigwedge \hat{y}) \le H(\hat{y})$$

Notice that $(x^+ \bigwedge \hat{y}) \bigvee x^- \in X$, we finally conclude that

$$H(y) \le H(\hat{y})$$

This finishes the proof of the lemma.

Now pick any positive number j. According to this lemma, the segment $\{y_i\}_{i=0}^j$ is also a minimal segment connecting $y_0 = \xi$ and y_i in the sense given in [8]. i.e. replacing the segment $\{y_i\}_{i=1}^{j-1}$ by any finite segment of points will always increase the energy of the orbit. We will need this fact later on in our proof.

3 The Connecting Configuration

3.1 The modified energy function

Before introducing the new energy function for the construction of the connecting orbit, we first assure that ξ is also a separating point for (c, c_2).

Lemma 3 *If $f(c) < \frac{1}{2}K_1K_5^2$, then*

$$f(\xi) - f(c) > \frac{1}{2}K_1K_5^2,$$

where

$$K_5 = min\{\delta_1, \frac{P}{2\theta}\}$$

Proof: For any two points $\xi_1, \xi_2 \in (c_1, c_2)$, we have

$$\mid P_{0^-}(\xi_1) - P_{0^-}(\xi_2) \mid \le 2\theta \mid \xi_1 - \xi_2 \mid$$

according to Mather [5]. So for any $\xi_1 > \xi$, we have

$$\mid \xi_1 - \xi \mid \ge \frac{1}{2\theta} \mid P_{0^-}(\xi_1) - P_{0^-}(\xi) \mid$$

Now take $\xi_1 = \eta^+$, we have

$$| c_2 - \xi | > | \eta^+ - \xi | \geq \frac{P}{2\theta}.$$

Therefore

$$f(\xi) \geq K_1 K_5^2 \geq f(c) + \frac{1}{2} K_1 K_5^2$$

This proves the lemma.

Since we assumed that there are no other local minimal point in (c, c_2), This lemma implies that ξ is a separating point for (c, c_2).

Now let $x = \{x_i\}_{i=-\infty}^{\infty}$. We assume that $x_i \in [c, \xi]$ for $i > 0$ and $x_i \in [\xi, c_2]$ for $i \leq 0$. Call the collection of all the x's so defined as $X(c, c_2)$. For any give $x \in X(c, c_2)$ define

$$\hat{H}(x) = \sum_{i=-\infty}^{-1} h(x_i, x_{i+1}) + \sum_{i=0}^{\infty} (h(x_i, x_{i+1}) - f(c))$$

We have

Lemma 4 *There is an element $\hat{x} = \{\hat{x}_i\} \in X(c, c_2)$, such that*

(a). For any other element $x \in X(c, c_2)$

$$\hat{H}(x) \geq \hat{H}(\hat{x})$$

(b). $\hat{x}$ is a strictly decreasing sequence. Furthermore

$$\lim_{i \to -\infty} \hat{x}_i = c_2 \qquad \lim_{i \to \infty} \hat{x}_i = c.$$

Proof: (a) First note that, since ξ is a separating point for (c, c_2), the value of the functional $\hat{H}$ is bounded from below. Therefore for any real number $K > h(c_2, c)$, let

$$\hat{\Lambda} = \{x \in X(c, c_2) : \quad \hat{H}(x) \leq K\}$$

The set $\hat{\Lambda}$ is non-empty and compact. This assures the existence of $\hat{x}$.

(b). By Mather's energy formula, $\hat{x}$ is monotonic. It is strictly decreasing since dropping any duplicated points would reduce the energy $\hat{H}$.

The sequence can not actually reach c or c_2. i.e. there is no integer j, such that $\hat{x}_j = c$ (or c_2). Otherwise we can adding a point $c + \varepsilon$ (or $c_2 - \varepsilon$), with ε appropriately chosen, to reduce the energy of the sequence. Refer to the proof of Lemma 1.

Both limits must be true since otherwise $\hat{H}(\hat{x}) = \infty$ which is impossible.

This finishes the proof of the lemma.

We want to show that $\hat{x}$ is indeed an orbit. According to this lemma, it is enough for us to prove $\hat{x}_{-1} \neq \xi$ and $\hat{x}_0 \neq \xi$. For the rest of the paper, we will assume $\hat{x}_0 = \xi$, then try to induce contradiction. Same argument applies if $\hat{x}_{-1} = \xi$.

3.2 More on the connecting configuration $\hat{x}$

In this subsection, we study the configuration $\hat{x}$ more closely under the assumption that $\hat{x}_0 = \xi$.

Take $\varepsilon = \frac{P}{20\theta}$. Let i_0 be the index of $\hat{x}$ such that

$$\hat{x}_{i_0} \leq c + \varepsilon \leq \hat{x}_{i_0 - 1}$$

Replacing $\hat{x}_{i_0}$ by $c + \varepsilon$ in $\hat{x}$, we obtain a new sequence, say, $\hat{x}^\varepsilon$. It is easy to see that

$$| \hat{H}(\hat{x}^\varepsilon) - \hat{H}(\hat{x}) | \leq \frac{P}{10}.$$

We have

Lemma 5

$$\sum_{i=i_0}^{\infty} (h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) < \frac{P}{20}.$$

proof: Take another sequence $u = \{u_i\}_{i=-\infty}^{\infty} \in X(c, c_2)$. $u_i = \hat{x}_i$ for $i \leq i_0$ and $u_i = c$ for $i > i_0$. We see

$$\hat{H}(u) - \hat{H}(\hat{x}) = h(\hat{x}_{i_0}, c) - f(c) - \sum_{i=i_0}^{\infty} (h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) \geq 0.$$

Therefore

$$\sum_{i=i_0}^{\infty} (h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) \leq h(\hat{x}_{i_0}, c) - f(c) \leq \frac{P}{20}.$$

We further have

Lemma 6 *Assume $f(c) < \frac{1}{2} K_1 K_5^2$. For any $x \in [c + \varepsilon, \xi]$, we have*

$$f(x) - f(c) \geq K_6$$

where

$$K_6 = min\{\frac{K_1 P^2}{400\theta^2}, \quad \frac{1}{2} K_1 K_5^2\}.$$

Proof: $f(x)$ reaches its minimal value either at $c + \varepsilon$ or ξ on $[c + \varepsilon, \xi]$. By Lemma 3

$$f(\xi) - f(c) \geq \frac{1}{2} K_1 K_5^2,$$

and at $c + \varepsilon$

$$f(c + \varepsilon) - f(c) \geq K_1 \varepsilon^2 = \frac{K_1 P^2}{400\theta^2}$$

This proves the lemma.

Our next lemma is the key for the construction.

Lemma 7 *Assume $f(c) < \frac{1}{2}K_1K_5^2$, we have*

$$i_0 < \frac{20\theta + P}{10K_6}$$

Proof: Construct another sequence $v = \{v_i\}_{i=-\infty}^{\infty}$. $v_i = \hat{x}_i$ for $i \le 0$ and $v_i = c$ for $i > 0$. We see

$$0 \le \hat{H}(v) - \hat{H}(\hat{x}) = h(\xi, c) - f(c) - \sum_{i=0}^{\infty}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c))$$

$$= h(\xi, c) - f(c) - \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) - \sum_{i=i_0+1}^{\infty}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c))$$

According to Lemma 5, we have

$$\sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) \le \theta + \frac{P}{10}.$$

.

Now according to Mather's energy formula

$$\sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) > \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_i) - f(c)) - \int_{\xi}^{\hat{x}_{i_0+1}} \partial_2 h(y, y)$$

$$\ge i_0 K_6 - \int_{\xi}^{\hat{x}_{i_0+1}} \partial_2 h(y, y).$$

Therefore

$$i_0 K_6 < \theta + \frac{P}{10} + \int_{\xi}^{\hat{x}_{i_0+1}} \partial_2 h(y, y) < 2\theta + \frac{P}{10}.$$

This finishes the proof of the lemma.

4 The contradiction

First recall all the sequences we have introduced so far. $x^+ = \{x_i^+\}_{i=-\infty}^{\infty}$ is a minimal energy orbit, $y = \{y_i\}_{i=-\infty}^{\infty}$ is the minimal energy configuration in X, and $\hat{x} = \{\hat{x}_i\}_{i=-\infty}^{\infty}$ is the configuration in $X(c, c_2)$ which minimizes the modified energy function $\hat{H}$. Note that we assumed that $\hat{x}_0 = y_0 = \xi$.

Let j be the index of x^+ such that

$$x_j^+ < c + \varepsilon < x_{j-1}^+$$

By definition we also have

$$x_j^+ \ge y_j \ge x_{j+1}^+$$

Now let

$$x^u = \{x_i^+\}_{i=-\infty}^{j} \qquad y^u = \{y_i\}_{i=-\infty}^{j}$$

We have

Lemma 8 *If*

$$f(c) < \frac{\delta^2 P^2}{1600\theta^2(K_1+\delta)}$$

then

$$H(y^u) \geq H(x^u) + \frac{19P}{20}$$

Proof: Take a new sequence $v = \{v_i\}_{i=-\infty}^{\infty}$. $v_i = x_i^+$ for all $i \geq j$ and $v_i = y_i$ for $i < j$. We see that $v_0 = \xi$ so

$$H(v) - H(x) = H(y^u) - H(x^u) + h(x_j^+, x_{j+1}^+) - h(y_j, x_{j+1}^+) \geq P$$

By the restriction on $f(c)$ and Lemma 1, we have

$$\mid y_j - x_j^+ \mid < \frac{P}{20\theta}$$

therefore

$$H(y^u) \geq H(x^u) + \frac{19P}{20}.$$

This proves the lemma.

Now we are at the final stage of the proof. We see

$$\hat{H}(\hat{x}) = \sum_{i=-\infty}^{-1} h(\hat{x}_i, \hat{x}_{i+1}) + \sum_{i=0}^{\infty}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c))$$

$$= \sum_{i=-\infty}^{-1} h(\hat{x}_i, \hat{x}_{i+1}) + \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) + \sum_{i=i_0}^{\infty}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c))$$

$$> \sum_{i=-\infty}^{-1} h(\hat{x}_i, \hat{x}_{i+1}) + \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1}) - f(c)) - \frac{P}{10}$$

according to Lemma 5. So

$$\hat{H}(\hat{x}) > \sum_{i=-\infty}^{-1} h(\hat{x}_i, \hat{x}_{i+1}) + \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1})) - i_0 f(c) - \frac{P}{10}$$

By the remark following Lemma 2,

$$\sum_{i=-\infty}^{-1} h(\hat{x}_i, \hat{x}_{i+1}) + \sum_{i=0}^{i_0}(h(\hat{x}_i, \hat{x}_{i+1})) > H(y^u) - \frac{P}{10}$$

So

$$\hat{H}(\hat{x}) > H(y^u) - \frac{P}{5} - i_0 f(c)$$

$$\geq H(x^u) + \frac{19P}{20} - \frac{P}{5} - i_0 f(c)$$

by Lemma 8. According to our restriction on $f(c)$, we have

$$i_0 f(c) < \frac{P}{10}$$

Therefore

$$\hat{H}(\hat{x}) > H(x^u) + \frac{P}{5}$$

Now form a new sequence $T = \{t_i\}_{i=-\infty}^{\infty}$. $t_i = x_i$ for $i < j$ and $t_i = c$ for $i > j$. We see $T \in X(c, c_2)$. Furthermore

$$\hat{H}(T) < H(x^u) - (h(x_j, c) - f(c))$$

therefore

$$\hat{H}(\hat{x}) > \hat{H}(T)$$

This contradict to the definition of $\hat{x}$. So we proved that $\hat{x}_0 \neq \xi$. Therefore the configuration $\hat{x}$ is indeed an heteroclinic orbit from c_2 to c

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