The Diffusion Time of The Connecting Orbit Around Rotation Number Zero for The Monotone Twist Maps

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Abstract

We improve Mather's proof on the existence of the connecting orbit around rotation number zero (Proposition 8.1 in [7]) in this paper. Our new proof not only assures the existence of the connecting orbit, but also gives a quantitative estimation on the diffusion time.

1 Introduction

We improve Mather's proof on the existence of the connecting orbit around rotation number zero (Proposition 8.1 in [7]) in this paper. Our new proof not only assures the existence of the connecting orbit, but also gives a quantitative estimation on the diffusion time, a problem that motivated this, and other studies of this author on the monotone twist maps. [8, 9]

By diffusion time, we mean various quantities (L_-, L_+, K, K_+, K_-) in the statement of the proposition 8.1 in [7]. Mather proved that, if all these quantities are big enough, the local minimal configuration constructed in this proposition is J-free. His proof is constructive. So, in addition to an existence argument, Mather also provided many partial estimations throughout his proof. However, the difficulties related to the existence of the local minimal fixed points prevent the writing of a nice quantitative estimation for all these quantities.

The time it takes for a connecting orbit shifting its frequencies from one to another depends on many factors. It is clear that, for a quantitative estimation of the diffusion time, we have to include the twist constant δ and the convexity constant θ. Also, we have to include a barrier parameter P according to Percival's barrier functions. As an invariant set, Aubry-Mather orbits act as barriers which prevent the frequency shifting in Birkhoff's region of instability. Percival's barrier function is essentially a quantitative measure on the size of the rooms that is left for the orbit diffusion.

Another seemingly unavoidable factor which affects the speed of the frequency shifting is related to the occurrence of the periodic orbits inside the Birkhoff's region of instability. If a connecting orbit gets into a very small neighborhood of a given periodic point(as it will inevitably), how fast it shifts away depends on the property of this specific periodic point. In general, it takes longer if the periodic point is degenerate. A new parameter is necessary to represent the influence of this additional factor. We will introduce this parameter D and call it as the **degeneracy constant**.

The goal of this paper is to show that, by modifying Mather's construction, one can obtain lower bound estimation for L_-, L_+, K, K_+, K_-. Furthermore, one can write these estimations in terms of δ, θ, P, D. In fact, according to our version of the proof, we have the following estimations

$$L_+ = L_- = 68[\frac{\theta}{P}] + 3,$$

$$\frac{K}{2} = K_+ = K_- = [\frac{2\theta}{\hat{D}}(\frac{5\theta^2}{P})^{(\frac{20\theta}{P})}] + 1,$$

where

$$\hat{D} = min\{D, \ \frac{\delta^4 P^2}{80000\theta^5}, \ \frac{\delta^2}{4(\theta+\delta)}(\frac{P}{20\theta})^2, \ \frac{P^2}{64\theta^2}\}.$$

Again, δ is the twist parameter, θ is the convexity constant, P is the barrier parameter and D is the degeneracy constant.

We will start our discussion with the definition of the first two parameters and that of the Mather's generating function in the second section. This section also includes many basic, but very useful facts on Mather's theory. At the end, we will give some technical estimations that will be frequently used in the rest of the paper.

The precise definition of the barrier parameter P is presented in the third section. Our choice of the barrier points a, and b is a little different from that of Mather's. The heart of this section is to show that there exist two points a, b, such that (i). $P_{0^-}(a) \neq 0, P_{0^+}(b) \neq 0$ (ii). There is a lower bound estimation of the value of the function $f(x) = h(x,x)$ on the interval $[b,a]$ in the term of δ, θ and $P = \min\{P_{0^-}(a), P_{0^+}(b)\}$. This is the first major step to get the desired estimation.

We will start the forth section with the precise definition of the degeneracy constant D. What we will do is to first pinpoint all the local minimal periodic points the connecting orbit might gets close to, and to define the degeneracy constant according to this set of points. After this parameter is defined, we will go forward to define the "turning point" of our shifting orbit and give a modified version of the proposition 8.1. In Mather's construction, the turning point is always one of the global minimal fixed points. If there exists local minimal point with extremely small energy, the shifting orbit Mather constructed have to stay in the neighborhood of this local minimal point for a long period of time before it reaches to the neighborhood of the turning point. Instead of being forced to go towards the global minimum, the orbit we construct will turn around at this local minimal point. We will explain how to modify Mather's proof to make it work for our construction. The key of the modification is to construct heteroclinic connections between a global minimal and the turning point. Combining what is in this section and in the section three, we will obtain the lower bound estimations presented earlier in this introduction.

Up to the completion of this paper, this author is acknowleged by Sen Hu that he also obtained similar results on the same problem in the special case of geodesic flow. His approach is completely different from ours.

2 Some Preliminaries

2.1 Basic concepts and strategy

2.1.1 The generating function

Assume that $h(x,x')$ is a continues function defined on R^2. We call $h(x,x')$ a **Mather's generating function** [5] if it satisfies the following:

(M1) For all $x, x' \in R$

$$h(x,x') = h(x+1, x'+1).$$

(M2) Uniformly in x, we have

$$\lim_{|\xi|\to\infty} h(x, x+\xi) = \infty.$$

(M3) There exists a positive number δ, such that if $x < \xi$, $x' < \xi'$

$$h(\xi,x') + h(x,\xi) - h(x,x') - h(\xi,\xi') \geq \delta(\xi - x)(\xi' - x').$$

(M4) There is a constant θ, such that the function

$$x \mapsto \frac{\theta x^2}{2} - h(x,x')$$

is convex for any given x', and

$$x' \mapsto \frac{\theta x'^2}{2} - h(x,x')$$

is convex for any given x.

We call δ the **twist constant** and θ the **convexity constant**. From now on, without loss generality, we assume that $\delta \leq \theta$. This additional restriction is not necessary for the purpose of this paper. However, it will helps to reduce the complexity of the resulted formulas.

2.1.2 The Aubry-Mather set

Now we recall some of the basic terminologies in Aubry-Mather theory. By a **configuration** we mean a real bi-infinite sequences $x = \{x_i\}_{i=-\infty}^{\infty}$. For any segment $\hat{x} = \{x_i\}_{i=j}^{k}$ of a given configuration x, we call the value

$$H(\hat{x}) = \sum_{i=j}^{k-1} h(x_i, x_{i+1})$$

the **energy** of $\hat{x}$. If $H(x) = \sum_{i=-\infty}^{\infty} h(x_i, x_{i+1})$ converges, we call $H(x)$ the energy of the configuration x.

We call a segment $\hat{x} = \{x_i\}_{i=j}^{k}$ a **minimal energy segment** if for any other segment $\hat{x}^* = \{x_i^*\}_{i=j}^{k}$ with $x_j = x_j^*, x_k = x_k^*$, we have $H(\hat{x}) \leq H(\hat{x}^*)$. If for a given configuration x, all of its finite segments are minimal energy segment, we call x a **global minimal energy configuration**. The **Aubry-Mather set** for the generating function h is the set of all the global minimal energy configurations. For the background of this setting and the basics of the theory on the Aubry-Mather set, refer to Bangert [2].

We also need the concept of local minimal energy configuration. A given configuration $x = \{x_i\}_{i=-\infty}^{\infty}$ is a **local minimal energy configuration** if for any index $i_0 \in \mathbf{Z}$, there is a neighborhood $I_{i_0} \in R$ of x_{i_0}, such that for any $\xi \in I_{i_0}$

$$h(x_{i_0-1}, x_{i_0}) + h(x_{i_0}, x_{i_0+1}) \leq h(x_{i_0-1}, \xi) + h(\xi, x_{i_0+1}).$$

In the rest of the paper, we will save the word **energy** and call global minimal energy configurations as **global minimal configurations** or **global minimal orbit**, and so on.

2.1.3 The constraints and J-free configuration

Take a bi-infinite sequence $J = \{J_i\}_{i=-\infty}^{\infty}$. Assume that each J_i is a closed interval of the real line R. We call J as a **constraint**. For a given constraint J, $x = \{x_i\}_{i=-\infty}^{\infty}$ is a **J-configuration** if $x_i \in J_i$ for all the integers i. We also call a finite segment $\hat{x} = \{x_i\}_{i=j}^{k}$ of a J-configuration $x = \{x_i\}_{i=-\infty}^{\infty}$ as a **J-segment**.

For any given J-segment $\hat{x}$, we appropriately define the energy function $H(\hat{x})$. A segment $\hat{x} = \{x_i\}_{i=j}^{k}$ is a **J-minimal segment** if for any J-segment $\hat{x}^* = \{x_i^*\}_{i=j}^{k}$ with $x_j = x_j^*, x_k = x_k^*$, we have $H(\hat{x}) \leq H(\hat{x}^*)$. A **J-minimal configuration** is a J-configuration such that all its finite segments are J-minimal segments. Furthermore, we say that a J-minimal configuration is **J-free** if $x_i \in int(J_i)$ for all i. It is obvious that a J-free configuration represents a local minimal energy orbit of the twist map created by $h(x, x')$.

Mather's proof on the existence of a local minimal orbit of a given kind consists two steps. First he formulates an appropriate constraint J such that the corresponding J-minimal configuration has the desired properties, then he tries to show that this J-minimal configuration is J-free.

According to Mather, J-minimal configuration exists for a given constraint J if there exist arbitrarily small i and arbitrarily large j such that J_i, J_j are bounded. This restriction on J-constraint is so weak that one can virtually obtain all kinds of J-minimal configurations by properly formulating the constraint. The hard part of an existence proof usually lies in the second step. i.e. to show that the constructed J-minimal configuration is J-free.

2.1.4 Percival's barrier function

In order to get a J-free minimal configuration, one has to appropriately chose all the end points of the intervals J_i in the constraint. These end points should be among those with **higher energy potential** so that if a point in a J-configuration touches them, we can always reduce the energy of the configuration by moving it to the interior of the interval. **Percival's barrier function** is a precise and quantitative representation of the **energy potential** for all the points on R. In the discussion follows, we again assume the theory on the structure of the Aubry-Mather set presented in Bangert [2].

For a given rotation number ω, let M_ω be all the points $x \in R$, such that there is a global minimal configuration with rotation number ω passing through x. In general, M_ω is a closed subset of R. According to Mather [4], the monotone twist map generated by $h(x, x')$ has invariant curve with rotation number ω if and only if $M_\omega = R$.

Now pick a point $\xi \in R$, we denote the value of the Percival's barrier function with respect to the rotation number ω as $P_\omega(\xi)$. Let $P_\omega(\xi) = 0$ if there is a global minimal energy orbit with rotation number ω that passing through ξ. In the case that such a global minimal energy orbit does not exist, we consider all the configurations that passing through x with rotation number ω, and let $P_\omega(\xi)$ be the minimum of the energy differences between these configurations and a corresponding global minimal energy configuration. We easily see that, in this case, $P_\omega(\xi) > 0$. These points with positive barrier values will be appropriately called as energy barriers. They are the candidates of the end points for the desired constraint J.

To formally define the Percival's barrier function, we take $\xi \in R$. If $\xi \in M_\omega$, $P_\omega(\xi) = 0$. If $\xi \in R \setminus M_\omega$, let (η^-, η^+) be the open interval of the complement of M_ω which includes ξ. Let $x^+ = \{x_i^+\}_{i=-\infty}^{\infty}$ be the global minimal configuration with rotation number ω and $x_0 = \eta^+$, and $x^- = \{x_i^-\}_{i=-\infty}^{\infty}$ be that with $x_0 = \eta^-$. Take all the configurations $y = \{\xi_i\}_{i=-\infty}^{\infty}$ with the restriction $\xi_i \in [x_i^-, x_i^+]$ when $i \neq 0$ and $\xi_0 = \xi$. Define

$$P_\omega(\xi) = \min_y \{ \sum_{i=-\infty}^{\infty} h(\xi_i, \xi_{i+1}) - h(x_i, x_{i+1}) \}$$

where the minimal sign is over all the possible configurations y.

2.2 Mather's energy formula

One sees that, for the construction of the J-free configurations, it is fundamentally important to be able to analyze the change of the energy according to the change of a given configuration. Mather's energy decomposition [6] provides a very power technical tool for this analysis. Our discussion relies on this energy decomposition, and we will repeatedly recall it in the rest of this paper.

For a given segment $\hat{x} = \{x_i\}_{i=j}^{k}$, recall that

$$H(\hat{x}) = \sum_{i=j}^{k-1} h(x_i, x_{i+1}).$$

Mather shows that we can write the energy of $\hat{x}$ as

$$H(\hat{x}) = \sum_{i=j}^{k-1} h(x_i, x_i) - \int_{x_j}^{x_k} \partial_2 h(y, y+) + \sum_{i=j}^{k-1} \mu_H(\Delta_i) \qquad (1)$$

where in the third term, Δ_i is the planar region defined by

$$\Delta_i = \{(x, x') : \quad x_i \le x \le x' \le x_{i+1}\} \quad if \ x_i \le x_{i+1}$$

or

$$\Delta_i = \{(x, x') : \quad x_i \ge x \ge x' \ge x_{i+1}\} \quad if \ x_i \ge x_{i+1}$$

and $\mu_H(\Delta_i)$ is the measure of the region Δ_i. The positive definite measure μ_H is defined by

$$\mu_H([x, \xi] \times [x', \xi']) = h(\xi, x') + h(x, \xi) - h(x, x') - h(\xi, \xi').$$

As a set function, μ_H is always positive according to (M3), and it is easy to check that the formula defined above indeed induces a Borel measure on R^2.

According to (1), the energy of the segment $\hat{x}$ consists of three components. The term in the middle is the **conservative part** of the energy. It only depends on the end points of the segment. So one can indeed ignore the contribution of this term in the process of changing the segment as far as the end points are untouched.

Denote $f(x) = h(x, x)$. For every point x on the real line, we will call $f(x)$ as the **intrinsic energy** of the point x. The first term in (1) is the intrinsic energy of the given segment. It treats all the points involved in $\hat{x}$ as isolated points, and simply summarizes the intrinsic energies of all these points. Note that, according to the assumptions on $h(x, x')$, $f(x)$ is bounded from below so one could always assume that $f(x) \ge 0$.

The third term of this formula represents, dynamically, the potential of the system which couples all the points of $\hat{x}$ together. We will call it the **transitive energy**. Mather's formula claims that, conceptually, one can think the size of this part of the energy as two dimensional areas.

Here is an example for how to use this formula to analyze the energy change. If one drops a point, say, x_0, from the segment $\hat{x}$. It does nothing to the conservative part of the energy. The intrinsic energy of x_0 drops out so that part of the energy is decreased by $f(x_0)$. The transitive energy is increased by this change, and the increment is $\mu_H([x_{-1}, x_0] \times [x_0, x_1])$, the measure of the planar region $[x_{-1}, x_0] \times [x_0, x_1]$. So the total change of the energy is

$$\mu_H([x_{-1}, x_0] \times [x_0, x_1]) - f(x_0).$$

Similarly, one knows the change of the energy if a point is added into the given segment.

To further show that the measure μ_H indeed behaves like two dimensional area, Mather introduced two other one-dimensional positive measures

$$\nu^1(y,z) = \theta(z-y) + \partial_1 h(y+,y) - \partial_1 h(z+,z),$$

$$\nu^2(y,z) = \theta(z-y) + \partial_2 h(y,y+) - \partial_2 h(z,z+),$$

and proved that $\nu^i(y,y+1) = \theta,\ i = 1,2$, and

$$\mu_H([x,\xi]^2) \leq (\xi - x)\nu^i(x,\xi), \quad i = 1,2. \tag{2}$$

According to (2), $\mu_H([x,\xi]^2) = o(\xi - x)$ as ξ approaches to x.

2.3 Some technical remarks

2.3.1 Minimal orbit with rotation number 0^-

We only study the property of the twist map around the rotation number zero in this paper. So from now on we further assume that

(M5) Let $f(x) = h(x,x)$,

$$\min_{x \in R} f(x) = 0.$$

We can make this normalization since adding a constant to the generating function does not change the problem. We list (M5) as an additional assumption since, technically, we will need it all the times in this paper.

Let c', c be adjacent global minimal fixed points of $f(x)$. By adjacent we mean $f(x) > 0$ for all $x \in (c', c)$. Now take $x = \{x_i\}_{i=-\infty}^{\infty}$ and assume that x is a global minimal orbit with rotation number 0^- between c' and c. According to the standard theory on Aubry-Mather set, such a minimal orbit exists. Furthermore, the configuration is strictly decreasing and

$$\lim_{i \to +\infty} x_i = c' \qquad \lim_{i \to -\infty} x_i = c.$$

Under the assumption (M5),

$$H(x) = \sum_{i=-\infty}^{\infty} h(x_i, x_{i+1})$$

converges absolutely. It is not hard to see that x is also a minimal configuration in the following sense:

Lemma 1 *For any monotone decreasing sequence $y = \{y_i\}_{i=-\infty}^{\infty}$, such that*

$$\lim_{i\to+\infty} y_i = c' \qquad \lim_{i\to-\infty} y_i = c,$$

We have

$$H(y) \geq H(x).$$

Note that, as a direct consequence of this lemma, adding or dropping one point from the minimal configuration x, or replacing a finite segment of the configuration x by another finite segment (even with different length) will increases its energy. This is a crucial property of the configuration x we will use repeatedly. It is correct only under the assumption (M5).

2.3.2 On the intrinsic energy

Recall that the intrinsic energy $f(x) = h(x, x)$. We will need the next technical estimation later on.

Lemma 2 *If c is a local minimal fixed point of the function $f(x)$, we will always have*

$$f(x) - f(c) \leq \theta(x-c)^2.$$

Proof: We first assume that $x > c$. Since c is a local minimal point of $f(x)$, the first order partial derivatives of $h(x, x')$ exist at (c, c), and

$$f'(c) = \partial_1 h(c, c) + \partial_2 h(c, c) = 0.$$

According to (M3),

$$h(x, c) + h(c, x) - h(c, c) - h(x, x) \geq 0.$$

We have

$$h(x, x) - h(c, c) \leq h(x, c) - h(c, c) + h(c, x) - h(c, c)$$

$$\leq \int_c^x (\partial_1 h(y+, c) + \partial_2 h(c, y+))dy$$

$$= \int_c^x (\partial_1 h(y+,c) - \partial_1 h(c,c) + \partial_2 h(c,y+) - \partial_2 h(c,c))dy.$$

According to Mather(refer to lemma 5 in this paper),

$$h(x,x) - h(c,c) \le \int_c^x 2\theta(y-c)dy = \theta(x-c)^2.$$

The proof for the case of $x < c$ is similar.

2.3.3 A lower bound

For a given rotation number ω, Mather proved that the Percival's barrier function is a Lipschitz function with the Lipschitz constant 2θ. So we have

Lemma 3

$$| P_\omega(\xi') - P_\omega(\xi) | \le 2\theta \mid \xi' - \xi \mid .$$

The proof of this lemma is not hard at all. Refer to Mather [5] for the details.

Here we only consider the configurations with rotation number 0^-. Assume that there is a point $\xi \in (c', c)$, such that $P = P_{0^-}(\xi) \neq 0$, we have

Lemma 4

$$f(\xi) \ge \frac{\delta P^2}{4\theta^2}.$$

Proof: Assume (η^-, η^+) is the complement interval of M_{0^-} such that $\xi \in (\eta^-, \eta^+)$. We have $P_{0^-}(\eta^-) = P_{0^-}(\eta^+) = 0$. Let $x = \{x_i\}_{-\infty}^{\infty}$ be the global 0^- configuration with $x_0 = \eta^+$, we have $x_1 \le \eta^- \le \xi \le x_0$.

Now according to Lemma (3),

$$| \xi - \eta^+ |, \quad | \xi - \eta^- | \ge \frac{P}{2\theta}.$$

Recall that x_0, x_1 are consecutive points on a global minimal orbit with rotation number 0^-, adding the point ξ into this orbit will increase its energy. So according to Mather's energy formula,

$$0 \le f(\xi) - \mu_H([x_1, \xi] \times [\xi, x_0]).$$

So

$$f(\xi) \geq \mu_H([x_1, \xi] \times [\xi, x_0]) \geq \delta(x_0 - \xi)(\xi - x_1) \geq \frac{\delta P^2}{4\theta^2}.$$

This proves the lemma.

2.3.4 More on Mather's generating function

We finish this section by listing some easy properties for the Mather's generating function that will be used later in this paper. First, according to (M4), all the side derivatives of the first order

$$\partial_1 h(x\pm, x'), \quad \partial_2 h(x, x'\pm)$$

exist for $h(x, x')$. In addition, we have

$$\partial_1 h(x-, x') \geq \partial_1 h(x+, x'), \quad \partial_2 h(x, x'-) \geq \partial_2 h(x, x'+). \tag{3}$$

It is not hard to see from (3) that, if $x = \{x_i\}_{i=-\infty}^{\infty}$ is a local minimal orbit, then the first partial derivatives $\partial_1 h(x_i, x_{i+1}), \partial_2 h(x_{i-1}, x_i)$ exist at all x_i, $i \in \mathbf{Z}$. We further have the next estimation on the size of these partial derivatives according to Mather.

Lemma 5 *For $y < z$, we have*

$$\partial_1 h(z\pm, z) \leq \partial_1 h(z\pm, y) \leq \partial_1 h(y\pm, y) + \theta(z - y).$$

$$\partial_2 h(z, z\pm) \leq \partial_2 h(y, z\pm) \leq \partial_2 h(y, y\pm) + \theta(z - y).$$

It is easy to see from this lemma that, if $\mid \xi - \xi' \mid < 1$, $\mid x - x' \mid < 1$,

$$\mid \partial_1 h(x\pm, \xi) - \partial_1 h(x'\pm, \xi') \mid < 2\theta \quad \mid \partial_2 h(x, \xi\pm) - \partial_2 h(x', \xi'\pm) \mid < 2\theta.$$

We also have ([5], Lemma 7.10)

Lemma 6 *Consider a minimal segment (x_{i-1}, x_i, x_{i+1}) and let $y \in R$, then*

$$0 \leq H(\{x_{i-1}, x_i, x_{i+1}\}) - H(\{x_{i-1}, y, x_{i+1}\}) < \theta(x_i - y)^2.$$

We will only need its weaker version that if $\mid x_i - y \mid \leq 1$, then

$$0 \leq H(\{x_{i-1}, x_i, x_{i+1}\}) - H(\{x_{i-1}, y, x_{i+1}\}) < \theta \mid x_i - y \mid .$$

3 The Barrier Parameter P

In this section, we discuss the selection of the barrier points and the definition of the barrier parameter P. We will assume that $c' < c$ are two adjacent global minimal points of the intrinsic energy function $f(x) = h(x,x)$, and both of the Percival's barrier function $P_{0^-}(\xi)$ and $P_{0^+}(\xi)$ are not identically zero on the interval $[c', c]$.

3.1 The barrier points

3.1.1 Barrier points in Mather's proof

The first step in Mather's construction is to formulate constraints by the help of the barrier points. To start the construction, we first take two barrier points a and b such that $P_{0^-}(a) \neq 0$, $P_{0^+}(b) \neq 0$. a, b will be fixed once for all for the rest of the construction. We will call a the 0^- **barrier point** and b the 0^+ **barrier point**.

For the purpose of the existence theorem, the choice of barrier points a and b are quite arbitrary. There is no dependency on the selection of a and that of b. Any pair of points in $[c', c]$ will do as far as

$$P = min\{P_{0^-}(a), P_{0^+}(b)\} \neq 0.$$

After the selection of a and b, the quantity P will be used as a parameter in the construction. We call P the **barrier parameter**. For instance, one may chose a, b such that

$$P_{0^-}(a) = \max_{\xi \in [c', c]} \{P_{0^-}(\xi)\} \qquad P_{0^+}(b) = \max_{\xi \in [c', c]} \{P_{0^+}(\xi)\}.$$

However, this selection of the barrier points does not work for the purpose of this paper. In order to overcome the difficulties caused by local minimal fixed points, the barrier points a and b should be chosen in a way that also allows us to control the lower bound of the value of the function $f(x)$ on the interval $[a, b]$ (or $[b, a]$ if $b < a$). It is the purpose of this section to show the existence of such barrier points and provide the desired lower bound estimation for $f(x)$.

3.1.2 The set of feasible pairs

We start with the definition of the set of feasible pairs. Let a, b be two real numbers, $a, b \in (c', c)$. We say that (a, b) is a feasible pair if $P_{0^-}(a) \neq 0, P_{0^+}(b) \neq 0$ AND $f(x) = h(x,x)$ reaches its minimal at a or b on the interval bounded by a and b. ($[a,b]$ if $a \leq b$ or $[b,a]$ otherwise). We denote the set of all the feasible pairs as Λ.

If the set Λ is not empty, denote

$$P_{ab} = min\{P_{0^+}(b), P_{0^-}(a)\}.$$

Let

$$\hat{P} = \sup_{(a,b)\in\Lambda} \{P_{ab}\}$$

and $P = \frac{\hat{P}}{2}$.

By the definition, there exists a pair $(a,b) \in \Lambda$, such that $P_{ab} > P$. We will take a as our 0^- barrier point and b as the 0^+ barrier point. According to Lemma 4, we have the lower bound estimation of $f(x)$ on the interval bounded by a and b as

$$f(x) \geq \frac{\delta P^2}{4\theta^2}.$$

3.1.3 The lower bound

Now we turn to the case that the set Λ is empty. We first remark that the question on the non-emptiness of the set Λ seems to be a hard one to us. So far, we can neither prove the non-emptiness nor we can find an example with an empty Λ.

Anyway, in the case that Λ is empty, we take a point $a \in [c', c]$, such that

$$P_{0^-}(a) = \max_{\xi\in(c',c)} P_{0^-}(\xi)$$

and let $P = P_{0^-}(a)$. Further, let

$$\hat{\varepsilon} = \frac{\delta P}{64\theta^2}.$$

We have

Proposition 1 *Assume that Λ is empty, then there exists a point $b \in (c', a)$, such that $P_{0^+}(b) \neq 0$, and for all $x \in [b, a]$, we have*

$$f(x) > \frac{\delta^2}{4(\theta+\delta)}\hat{\varepsilon}^2.$$

We will take a as our 0^- barrier point, and a point b which satisfies proposition 1 as our 0^+ barrier point.

3.2 The proof of proposition 1

The proof of this proposition relies on a sequence of lemmas on the properties of the global 0^+ minimal configurations around a given local minimal fixed point. So, for the moment, we assume that $c' < a < c$ are given as usual, that $\{x_i\}_{i=-\infty}^{\infty}$ is a 0^+ global minimal orbit connecting c' and c, and that $\hat{c}$ is a local minimal fixed point for $f(x)$, $c' < \hat{c} < a$. Furthermore, we assume

$$f(\hat{c}) < \frac{\delta^2}{4(\theta+\delta)}\hat{\varepsilon}^2.$$

3.2.1 On the local minimal point

First we claim:

Lemma 7 *Let*

$$\varepsilon_1 = \frac{P}{8\theta},$$

we have

$$\hat{c} < \hat{c} + \varepsilon_1 < a - 2\varepsilon_1 < a - \varepsilon_1 < a.$$

Proof: All the inequalities are trivial except the second one. By the assumption on $f(\hat{c})$, we have

$$f(\hat{c}) < \frac{\delta P^2}{64\theta^2}.$$

According to lemma 3, for any $x \in [a - 3\varepsilon_1, a]$,

$$| P_{0^-}(x) - P | < 6\theta\varepsilon_1 = \frac{3P}{4}.$$

So,

$$P_{0^-}(x) > \frac{P}{4}.$$

Therefore we have

$$f(x) > \frac{\delta P^2}{64\theta^2}$$

according to the proof of lemma 4. We conclude that $\hat{c} \notin [a - 3\varepsilon_1, a]$. Therefore

$$\hat{c} < a - 3\varepsilon_1.$$

$$\hat{c} + \varepsilon_1 < a - 2\varepsilon_1.$$

This proves the lemma.

3.2.2 On the 0^+ orbit

Recall that we assumed that

$$f(\hat{c}) < \frac{\delta^2}{4(\theta+\delta)}\hat{\varepsilon}^2$$

and $\{x_i\}_{i=-\infty}^{\infty}$ is a 0^+ minimal energy orbit from c' to c. We also have

Lemma 8 *For the given 0^+ minimal energy orbit, there is an index i, such that*

$$\hat{c} < x_i < \hat{c} + \hat{\varepsilon}.$$

Proof: For the given minimal 0^+ configuration x, let j be the index such that

$$x_{j-1} \le \hat{c} \le x_j.$$

If the lemma is false, then

$$x_j \ge \hat{c} + \hat{\varepsilon}.$$

Now take

$$t_0 = \frac{\delta(x_j - \hat{c}) - \sqrt{\delta^2(x_j-\hat{c})^2 - 4f(\hat{c})(\theta+\delta)}}{\theta+\delta}.$$

According to the assumption on $f(\hat{c})$, in this case, t_0 is a well defined real number and $0 < t_0 < x_j - \hat{c}$. So $x_{j-1} < \hat{c} + t_0 < x_j$. By adding $\hat{c} + t_0$ into the sequence x, we obtain a new sequence y. We see

$$H(y) - H(x) \le \theta t_0^2 - \delta(t_0 + \hat{c} - x_{j-1})(x_j - (t_0 + \hat{c})) + f(\hat{c}).$$

$$< (\theta+\delta)t_0^2 - \delta(x_j - \hat{c})t_0 + f(\hat{c})$$

Notice that t_0 is exactly a root of the last formula, we have

$$H(y) < H(x).$$

which is a contradiction. So we must have $x_i - \hat{c} \le \hat{\varepsilon}$. This proves the lemma.

Furthermore, we have

Lemma 9 *For the given minimal 0^+ orbit x, there is an index i, such that*

$$\hat{c} + \hat{\varepsilon} < x_i < a - \varepsilon_1$$

where $\varepsilon_1 = \frac{P}{8\theta}$.

Proof: If the point x_i claimed in this lemma does not exist, there will be an index j such that

$$\hat{c} < x_j < \hat{c} + \hat{\varepsilon} < a - \varepsilon_1 < x_{j+1}.$$

We are going to show that, in this case, we can reduce the energy of the orbit by appropriately adding a point $x_j + t$ into the minimal orbit x. This induces a contradiction therefore shows that the lemma is true.

Assume that $t < x_{j+1} - x_j$, the change of the energy by adding $x_j + t$ into the configuration x is

$$\Delta H \leq f(\hat{c}) + \theta(x_j + t - \hat{c})^2 - \delta t(x_{j+1} - x_j - t).$$

Since $x_j - \hat{c} < \hat{\varepsilon}$, we have

$$\Delta H \leq f(\hat{c}) + \theta(\hat{\varepsilon} + t)^2 - \delta t(x_{j+1} - x_j - t).$$

It is easy to see that $\hat{\varepsilon} < \varepsilon_1$ therefore $x_{j+1} - x_j > \varepsilon_1$ according to lemma 7. So we have

$$\Delta H < f(\hat{c}) + \theta(\hat{\varepsilon} + t)^2 - \delta t(\frac{P}{8\theta} - t)$$

$$= f(\hat{c}) + \theta\hat{\varepsilon}^2 + (2\hat{\varepsilon} - \delta\frac{P}{8\theta})t + (\theta + \delta)t^2.$$

There exists a real number t_0 such that the last formula equals to zero if and only if

$$(2\hat{\varepsilon} - \delta\frac{P}{8\theta})^2 \geq 4(\theta + \delta)(f(\hat{c}) + \theta\hat{\varepsilon}^2).$$

According to the definition for $\hat{\varepsilon}$, we easily see that

$$\hat{\varepsilon} < \frac{\delta P}{32\theta}.$$

Therefore

$$(2\hat{\varepsilon} - \delta\frac{P}{8\theta})^2 > (\frac{\delta P}{16\theta})^2$$

So we only need to show that

$$(\frac{\delta P}{16\theta})^2 > (\theta+\delta)(f(\hat{c})+\theta\hat{\varepsilon}^2).$$

Recall our assumption on $f(\hat{c})$, it is enough to have

$$(\frac{\delta P}{16\theta})^2 > (\theta+\delta)(\frac{\delta^2}{4(\theta+\delta)}+\theta)\hat{\varepsilon}^2.$$

The last inequality is indeed true since our definition for $\hat{\varepsilon}$ was originated by this inequality.

We still need to check that the real number t_0 we obtained here is indeed smaller than $x_{j+1}-x_j$. This is the case since, by quadratic formula,

$$t_0 < \frac{\delta P}{16\theta(\theta+\delta)} < \frac{P}{16\theta}$$

and according to lemma 7, $x_{j+1}-x_j > \varepsilon_1 = \frac{P}{8\theta}$.

This proves the lemma. Note that it is crucial for the proof of the proposition (1) that the upper bound for $f(\hat{c})$ and the value $\hat{\varepsilon}$ are expressed explicitly in δ, θ and P.

3.2.3 The proof

Now we turn to the proof of proposition 1. Let M_{0^+} be the Aubry-Mather set with rotation number 0^+, and C be its complement in $[c', c]$. M_{0^+} is closed and C is open. For a given point $b \in C$, we denote the connected component of C that includes b as $I_b = (\eta_b^-, \eta_b^+)$.

Let ∂C be the boundary of the set C. For any point p in ∂C, there is a 0^+ minimal configuration passing through p. According to the basic Aubry-Mather theory, all the points of this minimal configuration is in ∂C. One of the direct consequences of this property of ∂C is that, in any small neighbourhood of c', there exists b, such that $b \in C$. We certainly can take this neighbourhood so small such that $b < a$.

Since the set Λ is empty by assumption, the function $f(x)$ reaches its minimal value on $[b, a]$ at a point $\hat{c} \in (b, a)$. If we have

$$f(\hat{c}) \geq \frac{\delta^2}{4(\theta+\delta)}\hat{\varepsilon}^2.$$

We already find the points a and b this proposition claimed. So our only worry is the otherwise situation.

Now we are right in the situation assumed at the beginning of this subsection. Take the minimal configuration x as the 0^+ configuration passing through η_b^+, the proceeding lemmas guarantees the following:

(i). $\hat{c} < a - 3\varepsilon_1$. (by Lemma 7).

(ii). There is an point x_i in the configuration x, such that $x_i \in [\hat{c} + \hat{\varepsilon}, a - \varepsilon_1]$. (by lemma 9).

According to (ii) and the fact that $x_i \in \partial C$, we obtain a point b_1, such that $b_1 \in C$ and $b_1 \in [\hat{c} + \hat{\varepsilon}, a - \varepsilon_1]$. Note that we have

$$b_1 > b + \hat{\varepsilon}.$$

Now study the function $f(x)$ on the interval $[b_1, a]$. Same analysis apply. If the proposition is false, we will have a local minimal point $\hat{c}_1$ and another point b_2 such that

(i). $\hat{c}_1 < a - 3\varepsilon_1$. (again by Lemma (7)).

(ii). $b_2 \in [\hat{c}_1 + \hat{\varepsilon}, a - \varepsilon_1]$, and $b_2 \in C$.

Note we also have $b_2 > b_1 + \hat{\varepsilon}$.

One see that this process can not be repeated forever, every time we repeat it, we move the end point b_i towards a by a distance $\hat{\varepsilon}$, so we can not keep (i) forever. Therefore, there will be an integer i, such that $b_i \in C$ and

$$f(\hat{c}_i) \geq \frac{\delta^2}{4(\theta + \delta)} \hat{\varepsilon}^2.$$

Where $f(x)$ reaches its minimal at $\hat{c}_i$ on $[b_i, a]$. This finishes the proof of the proposition.

3.2.4 The barrier points

Here we sum up what we have concluded so far. Assume that $c' < c$ are two adjacent global minimal fixed point of the function $f(x)$, and that both of the Percival's barrier functions P_{0^-} and P_{0^+} are not identically zero on $[c', c]$. We can take a, b such that

(i). $P_{0^-}(a) \neq 0$ and $P_{0^+}(b) \neq 0$.

(ii). Let

$$P = \min\{P_{0^-}(a), P_{0^+}(b)\}.$$

We have

$$f(x) > \frac{\delta^2}{4(\theta + \delta)} \hat{\varepsilon}^2$$

for all $x \in [b, a]$ (or $[a, b]$ if $a < b$). Where

$$\hat{\varepsilon} = \frac{\delta P}{64\theta^2}.$$

We can further simplify the expression on the lower bound of $f(x)$ by making it smaller. It is easy to see that we have

$$f(x) > \frac{\delta^4 P^2}{80000\theta^5}.$$

We will take a, b as our barrier points and the P so defined as our **barrier parameter**.

4 Degeneracy Constant and Heteroclinic Connection

Assume that $c' < c$ are two adjacent global minimal fixed points of $f(x)$. Also assume that both $P_{0-}(\xi), P_{0+}(\xi)$ are not identically zero on $[c', c]$. Let b, a be the barrier points selected in the last section and P be the barrier parameter. In this section, we always let

$$\varepsilon = \frac{P}{20\theta}.$$

4.1 The degeneracy constant

We start this section with the definition of the **non-ignorable local minimal set**. Then we introduce the **degeneracy constant D** and the **turning point** c_{i_0}. The degeneracy constant is the last parameter we mentioned in the introduction which will be involved in our lower bound estimation of connecting time. The purpose of all these technical definitions is to identify the point where our connecting orbit should turn around(c_{i_0}), and gain control on the value of $f(x)$ on $[c_{i_0} + \varepsilon, b]$.

4.1.1 The non-ignorable minimal set

Let Λ be the set of all locally minimal points of the function $f(x)$ in $[c', b]$. We take a subset Φ of Λ inductively as the following:

(i) $c_1 = c' \in \Phi$.

(ii) For $c_i \in \Phi$, let S be the set of absolute minimal points of the function $f(x)$ on $[c_i+\varepsilon, b]$. S is a closed subset of the interval $[c_i+\varepsilon, b]$. Denote the maximal point of S as x'.

(iii). If x' is an interior point of the interval $[c_i+\varepsilon, b]$, pick it as our next element $c_{i+1} \in \Phi$. In the case that $x' = c_i + \varepsilon$ or $x' = b$, stop the inductive process. Also, stop the inductive process whenever we have $x' > b - \frac{P}{4\theta}$.

The set Φ is not empty and it contains only finitely many points. For the convenience of our later on discussion, we now re-index the points in Φ as

$$\Phi = \{c' = c_n < c_{n-1} < \cdots < c_1 < b\}$$

where

$$n < \varepsilon^{-1} = \frac{20\theta}{P} = n_0.$$

We call Φ the **non-ignorable local minimal set**.

We have

Definition 1 *The degeneracy constant D is defined by*

$$D = \min_{d\in\Phi}\{f(d+\varepsilon) - f(d)\}.$$

According to the definition of Φ, $D > 0$.

Now, also let

$$\hat{D} = min\{D, \quad \frac{\delta^4 P^2}{80000\theta^5}, \quad \frac{\delta^2}{4(\theta+\delta)}(\frac{P}{20\theta})^2, \quad \frac{P^2}{64\theta^2}\}.$$

4.1.2 The turning point

Remember that

$$\Phi = \{c' = c_n < c_{n-1} < \cdots < c_1 < b\}$$

is the non-ignorable local minimal set. We have:

Lemma 10 *On $[c_1+\varepsilon, b]$, $f(x) > \hat{D}$.*

Proof: According to the definition of Φ, there are three possibilities for c_1:

(a). $f(c_1+\varepsilon)$ is the absolute minimal value of $f(x)$ on $[c_1+\varepsilon, b]$.

(b). $f(b)$ is that absolute minimal.

(c). The minimal point x' of $f(x)$ on $[c_1+\varepsilon, b]$ satisfy $x' > b - \frac{P}{4\theta}$.

The lemma is clearly true for (a). For (b), we have

$$f(x) > f(b) > \frac{\delta P^2}{4\theta^2} > \hat{D}$$

according to Lemma 4.

For (c), we have

$$P_{0^+}(x') > P_{0^+}(b) - \theta \mid x' - b \mid > \frac{P}{2}.$$

Now repeat the argument in Lemma 4, we have

$$f(x) > \frac{\delta P^2}{64\theta^2} > \hat{D}.$$

This finishes the proof.

Now let $k_0 = \hat{D}$ and

$$k_i = (\frac{2P}{5\theta^2})^i \hat{D}$$

for $i = 1$ to n. We also have

Lemma 11 *There exist an index $i_0 > 0$, such that*

$$f(c_{i_0}) < k_{i_0}$$

and for all $0 < i < i_0$

$$f(c_i) \geq k_i.$$

Proof: Let A be the set of all the local minimal points $c_i \in \Phi$ such that $f(c_i) < k_i$. A is not empty because $f(c_n) = f(c') = 0 < k_n$. Take c_{i_0} as the point in A with the smallest index. The lemma is proved.

Now we are ready to define the turning point.

Definition 2 *We call the local minimal point c_{i_0} described in the last lemma the* **turning point**.

The next proposition is crucial. It is actually the motivation of all these definitions in this subsection.

Proposition 2 *For the turning point c_{i_0}, we have*

$$f(c_{i_0}) < k_{i_0}$$

and

$$f(x) \geq k_{i_0-1}$$

for all $x \in (c_{i_0} + \varepsilon, a)$.

Proof: The case of $i_0 > 1$ is according to the definition of Φ and i_0. The case of $i_0 = 1$ is according to Lemma 10.

4.2 The heteroclinic connection

Here we show the existence of heteroclinic connections between our turning point c_{i_0} and the global minimal fixed point c. We will construct the heteroclinic connection from c to c_{i_0} using a as our barrier point. The heteroclinic orbit from c_{i_0} to c can be obtained in exactly the same way. Of course there one has to change the barrier point to b.

4.2.1 The modified energy function

Let $x = \{x_i\}_{i=-\infty}^{\infty}$. We assume that $x_i \in [c_{i_0}, a]$ for $i > 0$ and $x_i \in [a, c]$ for $i \leq 0$. We call the set of all the $x's$ so defined as $X(c_{i_0}, c)$. For any given $x \in X(c_{i_0}, c)$ define

$$\hat{H}(x) = \sum_{i=-\infty}^{-1} h(x_i, x_{i+1}) + \sum_{i=0}^{\infty}(h(x_i, x_{i+1}) - f(c_{i_0})).$$

In this subsection, we will prove:

Proposition 3 *There is an element $\hat{x} \in X(c_{i_0}, c)$ such that*

(a). For any other element $x \in X(c_{i_0}, c)$

$$\hat{H}(x) \geq \hat{H}(\hat{x}).$$

(b). $\hat{x}$ is a strictly decreasing sequence. Furthermore

$$\lim_{i\to\infty} \hat{x}_i = c_{i_0}, \quad \lim_{i\to-\infty} \hat{x}_i = c.$$

(c). For any $y = \{y_i\}_{i=-\infty}^{\infty} \in X(c_{i_0}, c)$ with $y_0 = a$, we have

$$\hat{H}(y) - \hat{H}(\hat{x}) > \frac{P}{4}.$$

Note that (c) implies that $\hat{x}$ is indeed a stationary configuration therefore representing a heteroclinic connection from c to c_{i_0}.

(a) and (b) in this proposition are somehow trivial. (See Lemma 4 in [9]). To prove part (c), we will use Mather's standard "by contradiction" argument. i.e. we will first assume that $\hat{x}_0 = a$, then construct another configuration in $X(c_{i_0}, c)$ such that the energy of this new configuration is smaller than that of $\hat{x}$. In fact, we will also show that the energy difference is bigger that $\frac{P}{4}$. The rest of this subsection is devoted to this argument. From now on, we assume $\hat{x}_0 = a$.

4.2.2 The gap size

We will need other sequences to construct the new configuration with smaller energy value. So before we work further on $\hat{x}$, we go back to the Aubry-Mather orbits with rotation number 0^-. Let $x = \{x_i\}_{i=-\infty}^{\infty}$ be such an orbit, we have $\lim_{i\to\infty} x_i = c', \lim_{i\to-\infty} x_i = c$. Recall that, for the configuration x, adding or dropping points will not reduce its energy value.

For the configuration x, let j be the index such that

$$x_{j+1} \leq c_{i_0} \leq x_j.$$

We have

Lemma 12

$$| x_j - x_{j+1} | < 2\varepsilon.$$

Proof: Recall the proof of Lemma 8. Exactly the same argument works here. Note that according to the definition, we have

$$f(c_{i_0}) < \hat{D} < \frac{\delta^2}{4(\theta+\delta)}(\frac{P}{20\theta})^2.$$

The lemma is proved.

4.2.3 Minimal configuration that reaches a

We need further details on Percival's barrier function. Let

$$A = \{x : \quad x \in (c', c); P_{0^-}(x) = 0\}.$$

A is a closed set. $a \notin A$. As usual we let (η^-, η^+) be the interval of the complement of A which contains a, and use $x^- = \{x_i^-\}_{i=-\infty}^{\infty}$, $x^+ = \{x_i^+\}_{i=-\infty}^{\infty}$ for the minimal configurations passing through η^-, η^+ respectively. According to the definition of the function $P_{0^-}(a)$, there is a configuration $y = \{y_i\}_{i=-\infty}^{\infty}$ with $y_0 = a$ and $y_i \in [x_i^-, x_i^+]$, such that

$$H(y) - H(x^+) = P, \quad H(y) - H(x^-) = P.$$

We again have:

Lemma 13 *For any configuration $\hat{y} = \{\hat{y}_i\}_{i=-\infty}^{\infty}$ such that $\hat{y}_0 = a$, $\hat{y}_i \in (c', c)$ and $\lim_{i\to-\infty} \hat{y}_i = c$, $\lim_{i\to+\infty} \hat{y}_i = c'$, we have*

$$H(\hat{y}) \geq H(y).$$

Note that according to this lemma, replacing the segment $\{y_i\}_{i=1}^{j-1}$ by certain finite segment of points, even with different length, will always increase the energy of the orbit.

Proof: Somehow trivial. Refer to [9] for the details.

4.2.4 The related configurations

Our goal here is to construct a configuration $x \in X(c_{i_0}, c)$ such that $\hat{H}(x) < \hat{H}(\hat{x})$ therefore induces a contradiction. To form such a configuration, we have to somehow mix the sequences $\hat{x}$, y, and x^+. Remember that $\hat{x}$ is the sequence which minimizes the energy function $\hat{H}$ over $X(c_{i_0}, c)$, y is the sequence minimizes the energy function H over all the elements in $X(c', c)$ with $y_0 = a$, and x^+ is that minimal energy configuration with rotation number 0^- passing through η^+.

To make our presentation easier, we introduce one more notation. Let $x = \{x_i\}_{i=-\infty}^{\infty}$. i.e.

$$x = \{\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots\}.$$

For any given integer j, denote

$$\underline{jx} = \{x_i\}_{i=j+1}^{\infty} = \{x_{j+1}, x_{j+2}, \cdots\}$$

and

$$\underline{xj} = \{x_i\}_{i=-\infty}^{j-1} = \{\cdots, x_{j-2}, x_{j-1}\}.$$

We can write x as

$$x = \{\underline{xj}, x_j, \underline{jx}\}.$$

Now take another configuration y, and another index $\hat{j}$,

$$\{\underline{xj}, y_{\hat{j}}, \underline{jx}\}$$

will be the new configuration obtained from x by changing x_j to $y_{\hat{j}}$. Further

$$\{\underline{xj}, x_j, \underline{\hat{j}y}\}$$

will be the new configuration

$$\{\cdots, x_{j-2}, x_{j-1}, x_j, y_{\hat{j}+1}, y_{\hat{j}+2}, \cdots\},$$

which is created by combining x and y.

4.2.5 the contradiction

Recall the definitions of the sequence x^+ and the constant ε. Let j be the index such that

$$x_j^+ \le c_{i_0} + \varepsilon < x_{j-1}^+.$$

With the same j, we also have

$$y_j \le c_{i_0} + \varepsilon.$$

Furthermore, according to the definition of c_{i_0} and the estimation on gap size, we also have

$$c_{i_0} - \varepsilon < y_j.$$

We also have an corresponding index j_0 for the configuration $\hat{x}$. i.e. there is an index j_0, such that

$$\hat{x}_{j_0} \le c_{i_0} + \varepsilon < \hat{x}_{j_0-1}.$$

Note that, in general, $j \ne j_0$. First we provide an estimation for j_0.

Lemma 14

$$j_0 \le \frac{\theta}{k_{i_0-1} - f(c_{i_0})}.$$

Proof: Form a new sequence $\{\underline{\hat{x}0}, \hat{x}_0, \hat{x}_{j_0}, \underline{j_0\hat{x}}\}$ by deleting the segment $\{\hat{x}_i\}_{i=1}^{j_0-1}$ from $\hat{x}$. This change will increase the energy. So we have

$$0 < \hat{H}(\{\underline{\hat{x}0}, \hat{x}_0, \hat{x}_{j_0}, \underline{j_0\hat{x}}\}) - \hat{H}(\hat{x}) \leq \theta - \sum_{i=1}^{j_0-1}(f(\hat{x}_i) - f(c_{i_0})).$$

So

$$\theta > \sum_{i=1}^{j_0-1}(f(\hat{x}_i) - f(c_{i_0})) > j_0(k_{i_0-1} - f(c_{i_0})).$$

The second inequality is by the fact that $f(x) > k_{i_0-1}$ in $(c_{i_0} + \varepsilon, a)$. This proves the lemma.

We further have

Lemma 15

$$H(\{\underline{yj}, y_j\}) - H(\{\underline{x^+j}, y_j\}) \geq \frac{19P}{20}.$$

Proof: According to Lemma 2,

$$H(\{\underline{x^+j}, y_j, \underline{jx^+}\}) - H(\{\underline{x^+j}, x_j^+, \underline{jx^+}\}) < \theta \mid y_i - x_j^+ \mid < \frac{P}{20}.$$

We also have

$$H(\{\underline{x^+j}, y_j, \underline{jx^+}\}) - H(\{\underline{x^+j}, x_j^+, \underline{jx^+}\})$$

$$= H(\{\underline{x^+j}, y_j, \underline{jx^+}\}) - H(y) + H(y) - H(\{\underline{x^+j}, x_j^+, \underline{jx^+}\})$$
$$\geq P + H(\{\underline{x^+j}, y_j, \underline{jx^+}\}) - H(y).$$

By the definition of the function P_{0^-}. However,

$$H(\{\underline{x^+j}, y_j, \underline{jx^+}\}) - H(y)$$

$$= H(\{\underline{x^+j}, y_j\}) - H(\{\underline{yj}, y_j\}) + H(\{y_j, \underline{jx^+}\}) - H(\{y_j, \underline{jy}\})$$
$$\geq H(\{\underline{x^+j}, y_j\}) - H(\{\underline{yj}, y_j\})$$

since $H(\{y_j, \underline{jx^+}\}) - H(\{y_j, \underline{jy}\}) \geq 0$. Therefore, we have

$$P + H(\{\underline{x^+j}, y_j\}) - H(\{\underline{yj}, y_j\}) < \frac{P}{20}.$$

So

$$H(\{\underline{yj}, y_j\}) - H(\{\underline{x^+j}, y_j\}) \geq \frac{19P}{20}.$$

This proves the lemma.

Now we are ready to induce the contradiction. We get back to the configuration $\hat{x}$ and the modified energy function $\hat{H}$.

$$\hat{H}(\hat{x}) = \hat{H}(\{\underline{\hat{x}j_0}, \hat{x}_{j_0}, \underline{j_0\hat{x}}\}) > \hat{H}(\{\underline{\hat{x}j_0}, y_j, \underline{j_0\hat{x}}\}) - \frac{P}{20}$$

$$= \hat{H}(\{\underline{\hat{x}0}, \hat{x}_0\}) + \sum_{i=0}^{j_0-2} (h(\hat{x}_i, \hat{x}_{i+1}) - f(c_{i_0}))$$

$$+h(\hat{x}_{j_0-1}, y_j) - f(c_{i_0}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) - \frac{P}{20}$$

$$= \hat{H}(\{\underline{\hat{x}0}, \hat{x}_0\}) + \sum_{i=0}^{j_0-2} h(\hat{x}_i, \hat{x}_{i+1}) + h(\hat{x}_{j_0-1}, y_j)$$

$$-j_0 f(c_{i_0}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) - \frac{P}{20}$$

$$\geq \hat{H}(\{\underline{\hat{x}0}, \hat{x}_0\}) + \sum_{i=0}^{j-1} h(y_i, y_{i+1}) - j_0 f(c_{i_0}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) - \frac{P}{20}.$$

Since $\hat{x}_0 = y_0 = a$, we have

$$H(\{\underline{\hat{x}0}\}) = H(\{\underline{y0}\}).$$

Therefore

$$\hat{H}(\hat{x}) > H(\{\underline{yj}, y_j\}) - j_0 f(c_{i_0}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) - \frac{P}{20}$$

$$> H(\{\underline{x^+j}, y_j\}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) + \frac{19P}{20} - \frac{P}{20} - j_0 f(c_{i_0}).$$

According to the Lemma 14 and the definition of c_{i_0}, we have

$$j_0 f(c_{i_0}) < \frac{8P}{20}.$$

Therefore,

$$\hat{H}(\hat{x}) > H(\{\underline{x^+j}, y_j\}) + \hat{H}(\{y_j, \underline{j_0\hat{x}}\}) + \frac{P}{2}$$

$$= \hat{H}(\{\underline{x^+j}, y_j, \underline{j_0\hat{x}}\}) + \frac{P}{2}.$$

Now what left is to change y_j in this configuration back to $\hat{x}_{j+1}$. The estimation for the corresponding change of energy is

$$| \ h(x^+_{j-1}, y_j) + h(y_j, \hat{x}_{j+1}) - h(x^+_{j-1}, \hat{x}_{j+1}) - h(\hat{x}_{j+1}, \hat{x}_{j+1}) \ |$$

$$=| \int_{\hat{x}_{j+1}}^{y_j} \partial_2 h(x^+_{j-1}, y+) + \partial_1 h(y+, \hat{x}_{j+1}) \ |$$

$$=| \int_{\hat{x}_{j+1}}^{y_j} \partial_2 h(x^+_{j-1}, y+) - \partial_2 h(c_{i_0}, c_{i_0}) + \partial_1 h(y+, \hat{x}_{j+1}) - \partial_1 h(c_{i_0}, c_{i_0}) \ |$$

$$< 4\theta \mid \hat{x}_{j+1} - y_j \mid .$$

The last inequality is because that all these functions of first partial derivatives have total variation less than 2θ. This finally gives

$$\hat{H}(\hat{x}) > \hat{H}(\{\underline{x^+j}, \hat{x}_{j+1}, \underline{j_0\hat{x}}\}) + \frac{P}{4}.$$

This is the desired contradiction since $\{\underline{x^+j}, \hat{x}_{j+1}, \underline{j_0\hat{x}}\}$ is an element in $X(c_{i_0}, c)$.

4.3 The connecting orbit and diffusion time

4.3.1 A brief statement

For anyone who is familiar with Mather's construction, it should be clear that we have already finished all the major steps to get the desired quantitative estimation on diffusion time. The appropriate selection of barrier points and the estimation on $f(x)$ presented in the third section enable us to remove the minimal value of $f(x)$ on the interval $[b, a]$, which is one of the key parameters occurred in Mather's proof. The existence of the heteroclinic connection between the turning point c_{i_0} and c enables us to replace the global minimal point c' in Mather's proof by c_{i_0}, and construct the connecting orbit in exactly the same way. Finally, Since we have the lower bound for the value of $f(x)$ on $[c_{i_0}+\varepsilon, a]$, We can now take appropriat value of $i_+ + i_-$ to push $\xi_{\hat{i}}$ into $[c_{i_0}, c_{i_0}+\varepsilon]$, which makes $\mu_H([c_{i_0}, \xi_{\hat{i}}]^2)$ sufficiently small.

We will present our version for the proposition 8.1 in Mather's paper [6]. The major difference between our statement and that of

Mather's is that, instead of assuming that L_+, L_-, i_+, i_- and K sufficiently large, lower bound estimations on all these quantities are presented. However, the presentation of our new proposition will become the end of our discussion. From that point on, one can copy Mather's proof line by line to prove the new proposition, and make couple of trivial computation to make sure all these estimated lower bounds are big enough. We will simply skip that part.

4.3.2 The constraint J

We start by properly formulating the constraint J for the connecting orbit. We will use the barrier points a and b introduced in the second section to form the constraint J. Without loss of generality, let us assume that $b \leq a$.

To make the J-mininal configuration initially behaves like an orbit with a small negative rotation number, we first take two sufficiently large positive integers L_-, i_-. For all the index $i < -i_-$, let

$$J_i = [a + \beta - 1, a + \beta] \;\; for \; -i_- - L_-\beta \leq i < -i_- - L_-\beta + L_-$$

where β is any positive integer. We see that according to this definition, a J-configuration must have rotation number $-L_-^{-1}$ in the negative direction of the index i.

Since the connecting orbit has to change its rotation number from negative to positive, it has to change its direction of motion somewhere. So let us keep the whole orbit on one side of the turning point c_{i_0}, and give some room for this orbit to turn around. We define

$$J_i = [c_{i_0}, a] \;\; for \; -i_- \leq i \leq 0.$$

Once the orbit changed its direction of motion, we have to replace the barrier point a for 0^- by the barrier point b for 0^+. Therefore, take another sufficiently large integer i_+ and let

$$J_i = [c_{i_0}, b] \;\; for \; 0 < i < i_+.$$

By the time the index reaches i_+, the orbit is already turned around. Now it should behave like an orbit with small positive rotation number. So take a large integer L_+, and let

$$J_i = [a + \beta, a + \beta + 1] \;\; for \; i_+ + L_+\beta - L_+ \leq i < i_+ + L_+\beta$$

for all positive β.

Put all these J_i together, we define a constraint $J = \{J_i\}_{i=-\infty}^{\infty}$.

4.3.3 The theorem

Here come our final statement that will conclude this paper:

Theorem 1 *For the constraint J as defined, let*

$$L_+ = L_- > 48[\frac{\theta}{P}] + 3,$$

$$i_+ = i_- > [\frac{2\theta}{\hat{D}}(\frac{5\theta^2}{P})^{(\frac{20\theta}{P})}] + 1,$$

where

$$\hat{D} = min\{D, \quad \frac{\delta^4 P^2}{80000\theta^5}, \quad \frac{\delta^2}{4(\theta+\delta)}(\frac{P}{20\theta})^2, \quad \frac{P^2}{64\theta^2}\}.$$

then, *the J-minimal configuration for the given constraint is J-free, therefore representing a local minimal energy orbit. Again, δ is the twist parameter, θ is the convexity constant, P is the barrier parameter and D is the degeneracy constant.*

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