ARTICLE IN PRESS
JID:YJDEQ AID:9729 /FLA [m1+; v1.295; Prn:12/02/2019; 14:10] P.1 (1-34)
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ScienceDirect
Journal of Differential Equations
J. Differential Equations ••• (••••) •••–•••
www.elsevier.com/locate/jde

High order Melnikov method: Theory and application

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Received 15 August 2018; revised 8 February 2019

Abstract

Let $D(t_0, \varepsilon)$ be the splitting distance of the stable and unstable manifold of a time-periodic second order equation. We expand $D(t_0, \varepsilon)$ as a formal power series in ε as

$$D(t_0, \varepsilon) = E_0(t_0) + \varepsilon E_1(t_0) + \cdots + \varepsilon^n E_n(t_0) + \cdots .$$

In this paper we derive an explicit integral formula for $E_1(t_0)$. We also evaluate $E_1(t_0)$ to prove the existence of homoclinic tangles for an equation to which the Poincaré/Melnikov method fails to apply.

Keywords: High order Melnikov method; Homoclinic intersection; Time periodic equation

It is commonly acknowledged that Poincaré's discovery of homoclinic tangles in periodically perturbed ordinary differential equations marked the beginning of the chaos theory in modern times [4], [5]. After making the observation that a transversal homoclinic intersection of the stable and unstable manifold induces into existence a homoclinic tangle, Poincaré moved on to tackle the problem on how to rigorously prove the existence of transversal homoclinic intersections in time-periodic equations [6]. He worked on the equation of a periodically perturbed pendulum, expanding the splitting distance of $D(t_0, \varepsilon)$ into a formal power series in ε as

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https://doi.org/10.1016/j.jde.2019.02.003
0022-0396/

$$D(t_0, \varepsilon) = E_0(t_0) + \varepsilon E_1(t_0) + \cdots + \varepsilon^n E_n(t_0) + \cdots .$$

He then evaluated $E_0(t_0)$ for this example to conclude that $E_0(t_0) = 0$ has a solution such that $E_0'(t_0) \neq 0$. By a simple application of the implicit function theorem, he further concluded that, for all sufficiently small $\varepsilon \neq 0$, $D(t_0, \varepsilon) = 0$ also admits a non-tangential solution. The computational method employed by Poincaré on $E_0(t_0)$ was later generalized by Melnikov ([3]). The Poincaré/Melnikov method has served as a main venue, through which the modern theory of dynamic systems is applied to the study of differential equations ([2]).

We note that Poincaré/Melnikov method is based entirely on the derivation and the evaluation of an explicit integral for $E_0(t_0)$. After obtaining a neat formula for $E_0(t_0)$, it is nature to ask if we can also compute $E_1(t_0)$, $E_2(t_0)$ and so on. Melnikov gave an affirmative answer to this question in [3] by proposing an inductive scheme to calculate $E_n(t_0)$ for all n in ascending orders of ε starting with $E_0(t_0)$. However, a proposed computational scheme in theory is not quite the same as a practical computational method that can be used to prove the existence of homoclinic tangles for equation to which Poincaré/Melnikov method fails to apply. The computational scheme proposed by Melnikov has never been developed into a practical computational method in the last fifty-five years: we are not aware anywhere in the existing literature, an explicit integral formula for any time-periodic equation for $E_1(t_0)$ is acquired based on the method Melnikov proposed.

The objective of this paper is to develop a computational method that can be applied to time-periodic equations to which the Poincaré/Melnikov method fails to apply. We introduce a new method to derive $E_1(t_0)$. We also evaluate $E_1(t_0)$ to prove the existence of homoclinic tangles for a time-periodic equation for which $E_0(t_0) = 0$ for all t_0. The method presented in this paper can be used in computing higher order terms of the splitting functions, see [7].

1. Statement of results

We start with an unperturbed equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x + g(x) \tag{1.1}$$

where $g(x)$ is such that $g(0) = g'(0) = 0$. Let $\ell = (a(t), b(t))$ be the homoclinic solution of the saddle fixed point $(0, 0)$ of equation (1.1) satisfying $b(0) = 0$. We have $a(t) = a(-t)$, $b(t) = -b(-t)$ because equation (1.1) is an invertible system. Let D_ℓ be a small neighborhood of $\ell \cup (0, 0)$ in the (x, y)-plane. In this paper, we study the dynamics of the periodically perturbed second order equation

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x + g(x) + \varepsilon P(t, x, y). \tag{1.2}$$

We assume

(A1) the function $g(x)$, satisfying $g(0) = g'(0) = 0$, is real analytic on an open interval of x that covers $[0, a(0)]$;
(A2) the function $P(t, x, y)$ is such that $P(t, 0, 0) = 0$, and it is a periodic function of period T in t. This is to say that there exists a constant $T > 0$ so that

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$$P(t+T,x,y)=P(t,x,y)$$

for all $(x,y)\in D_\ell$; and

(A3) the function $P(t,x,y)$ is C^1 in t, real analytic in (x,y) on D_ℓ for all $t\in[0,T)$.

1.1. High order Melnikov method

For $p=(x,y)\in D_\ell$, the unperturbed energy function is

$$E(p)=\frac{1}{2}y^2-\frac{1}{2}x^2-G(x)$$

where

$$G(x)=\int_0^x g(u)du.$$

For a given $t_0\in[0,T)$, there exists a unique stable solution $(x^s(t,\varepsilon),y^s(t,\varepsilon))$ of equation (1.2) in D_ℓ so that $y^s(t_0,\varepsilon)=0$. In parallel, there exists also a unique unstable solution $(x^u(t,\varepsilon),y^u(t,\varepsilon))$ in D_ℓ so that $y^u(t_0,\varepsilon)=0$. Let $p^+=(x^s(t_0,\varepsilon),0)$ be for the stable solution and $p^-=(x^u(t_0,\varepsilon),0)$ be for the unstable solution.

Definition 1.1. We define

$$D(t_0,\varepsilon)=\varepsilon^{-1}\left(E(p^+(t_0,\varepsilon))-E(p^-(t_0,\varepsilon)\right)$$

as the **splitting distance** of the stable and unstable manifold.

We expand $D(t_0,\varepsilon)$ into a power series of ε to write

$$D(t_0,\varepsilon)=E_0(t_0)+\varepsilon E_1(t_0)+\cdots+\varepsilon^n E_n(t_0)+\cdots.$$

We start with the following proposition, the proof of which is a simple application of the implicit function theorem.

Proposition 1.1. *There exists a constant $K>0$, so that if (i) $E_0(t_0)\equiv 0$, (ii) there exists a t_0 so that $E_1(t_0)=0$, $E_1'(t_0)\neq 0$, then for all $0<|\varepsilon|<K^{-1}E_1'(t_0)$, there exists a homoclinic solution of equation (1.2), over which the stable and unstable manifold intersect transversally.*

In what follows we let m be the smallest integer such that $g^{(m)}(0)\neq 0$ and

$$k=(m+1)/2.$$

We also recall that $(a(t),b(t))$ is a homoclinic solution of the unperturbed equation satisfying $b(0)=0$. Denote $a=a(t)$, $b=b(t)$. Let

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$$R(t) = kb^2 - ab'; \quad A(t) = \frac{a^2 b''}{a'} - kab' - k(k-1)b^2. \tag{1.3}$$

We also let

$$h(t) = -R(t)\int_0^t A(\tau)R^{-2}(\tau)d\tau; \quad H(t) = \frac{1}{\sqrt{R(t)}}(bh(t) + a). \tag{1.4}$$

Finally, we denote

$$\mathcal{P}(t, t_0) := P(t + t_0, a, b); \quad \mathcal{P}_x(t, t_0) := P_x(t + t_0, a, b); \quad \mathcal{P}_y(t, t_0) := P_y(t + t_0, a, b). \tag{1.5}$$

Theorem 1. *We have*

(i) *(Integral Formula for E_0)*

$$E_0(t_0) = -\int_{-\infty}^{+\infty} b(\tau)P(\tau, t_0)d\tau;$$

(ii) *(Integral Formula for E_1)*

$$\begin{aligned} E_1(t_0) = &\frac{H(0)\mathcal{P}(0, t_0)}{\sqrt{R(0)}}E_0(t_0) + \int_{-\infty}^{+\infty}\int_0^{\tau_2} b(\tau_2)\mathcal{P}(\tau_2, t_0)\mathcal{P}_y(\tau_1, t_0)d\tau_1 d\tau_2 \\ &- \int_{-\infty}^{+\infty}\int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}}\left[\mathcal{P}_t(\tau_2, t_0)\mathcal{P}(\tau_1, t_0) + \mathcal{P}_t(\tau_1, t_0)\mathcal{P}(\tau_2, t_0)\right]d\tau_1 d\tau_2. \end{aligned}$$

1.2. An application

We apply Theorem 1 to the equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon\cos\omega t \cdot (x^2 + \gamma x^3). \tag{1.6}$$

The unperturbed equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3$$

has a homoclinic solution $\ell = (a(t), b(t))$ where

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}.$$

It is straightforward to compute $E_0(t_0)$ by using the residue theorem. We have

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$$E_0(t_0) = \frac{\pi\omega e^{-\omega\pi/2}}{3}\left(\frac{2\sqrt{2}(\omega^2+1)}{(1+e^{-\omega\pi})} + \gamma\cdot\frac{\omega(\omega^2+4)}{(1-e^{-\omega\pi})}\right)\sin\omega t_0.$$

It then follows that $E_0(t_0) \equiv 0$ if

$$\gamma = \gamma^*(\omega) := -\frac{2\sqrt{2}(\omega^2+1)}{\omega(\omega^2+4)}\cdot\frac{(1-e^{-\omega\pi})}{(1+e^{-\omega\pi})}. \tag{1.7}$$

We study the equation

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - x^3 + \varepsilon\cos\omega t\cdot(x^2+\gamma^* x^3) \tag{1.8}$$

where $\gamma^*(\omega)$ is as in (1.7). Equation (1.8) is a degenerate case, to which the Poincaré/Melnikov method fails to apply. We apply Theorem 1 to equation (1.8) to obtain an integral formula and we evaluate $E_1(t_0)$.

Theorem 2. *We have, for equation (1.8),*

$$E_1(t_0) = F(\omega)\sin(2\omega t_0)$$

where $F(\omega)$ is a function of ω independent of t_0. In addition, we have

$$F(\omega) = -\frac{32}{3}\pi e^{-\pi\omega}\omega^2(1+O(\omega^{-1})).$$

It follows from Proposition 1.1 and Theorem 2 that

Corollary 1.1. *(i) There exist at most a finite set Ω of ω, such that $F(\omega) \neq 0$ for all $\omega \in \mathbb{R}\setminus\Omega$.*

(ii) There exists a constant $K > 0$, independent of ω and ε, so that for any given $\omega \in \mathbb{R}\setminus\Omega$, and all

$$0 < |\varepsilon| \le K^{-1}\omega|F(\omega)|,$$

there exists a homoclinic solution over which the stable and unstable manifold of equation (1.8) intersect transversally.

2. High order Melnikov method: theory

New variables are introduced to transform the equation of first variations of (1.1) around $(a(t), b(t))$ to two separated equations in Sect. 2.1. These new variables are then used to transform equation (1.2) in Sect. 2.2. The derived differential equations are turned into integral equations in Sect. 2.3, to be used to prove Theorem 1 in Sect. 2.4.

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2.1. Solving the equations of first variations

Denote $a = a(t)$, $b = b(t)$, $a' = a'(t)$, $b' = b'(t)$. The following relations are used frequently throughout: we have

$$a' = b, \quad b' = a + g(a), \quad b^2 = a^2 + 2G(a).$$

Let m be the smallest integer so that $g^{(m)}(0) \neq 0$; $k = (m+1)/2$; and $R(t), A(t)$ be as in (1.3), $h(t)$, $H(t)$ be as in (1.4). In what follows we denote $R = R(t)$, $R' = R'(t)$, $A = A(t)$, $h = h(t)$, $H = H(t)$. We also let

$$\tilde{H}(t) = \frac{1}{\sqrt{R}}(b'h + kb). \tag{2.1}$$

The equations of first variations of the unperturbed equation (1.1) around $\ell = (a, b)$ are

$$\frac{d\xi}{dt} = \eta; \qquad \frac{d\eta}{dt} = (1 + g_x(a))\xi. \tag{2.2}$$

Lemma 2.1. *Let $\tilde{\xi}, \tilde{\eta}$ be such that*

$$\xi = \frac{1}{\sqrt{R}}(a'\tilde{\eta} - \sqrt{R}H\tilde{\xi}); \quad \eta = \frac{1}{\sqrt{R}}(b'\tilde{\eta} - \sqrt{R}\tilde{H}\tilde{\xi}). \tag{2.3}$$

The equations of first variations (2.2) are transformed in new variables $(\tilde{\xi}, \tilde{\eta})$ to

$$\frac{d\tilde{\xi}}{dt} = -\frac{R'}{2R}\tilde{\xi}; \quad \frac{d\tilde{\eta}}{dt} = \frac{R'}{2R}\tilde{\eta}. \tag{2.4}$$

Proof. Let z_1, z_2 be such that

$$z_1 = \frac{1}{\sqrt{R}}(b'\xi - a'\eta); \quad z_2 = \frac{1}{\sqrt{R}}(kb\xi - a\eta) \tag{2.5}$$

where

$$R = kb^2 - ab'.$$

We have

$$\xi = \frac{1}{\sqrt{R}}(a'z_2 - az_1); \quad \eta = \frac{1}{\sqrt{R}}(b'z_2 - kbz_1). \tag{2.6}$$

For z_1 we have

$$\frac{dz_1}{dt} = \frac{1}{\sqrt{R}}(b''\xi + b'\xi' - a''\eta - a'\eta') - \frac{R'}{2R}z_1 = -\frac{R'}{2R}z_1.$$

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We also have

$$\begin{aligned}\frac{dz_2}{dt} &= \frac{1}{\sqrt{R}}\left(kb'\xi + kb\xi' - a'\eta - a\eta'\right) - \frac{R'}{2R}z_2 \\ &= \frac{1}{\sqrt{R}}\left[kb'\xi + (k-1)b\eta - a(1+g_x(a))\xi\right] - \frac{R'}{2R}z_2 \\ &= \frac{1}{R}\left[\frac{a^2b''}{a'} - kab' - k(k-1)b^2\right]z_1 \\ &\quad + \frac{b}{R}\left[(2k-1)(a+g(a)) - a(1+g_x(a))\right]z_2 - \frac{R'}{2R}z_2\end{aligned}$$

where we use (2.6) to obtain the last equality. Note that

$$\begin{aligned}R' &= 2kbb' - bb' - ab'' = (2k-1)bb' - aa'(1+g_x(a)) \\ &= b\left[(2k-1)(a+g(a)) - a(1+g_x(a))\right].\end{aligned}$$

This is to imply

$$\frac{dz_2}{dt} = \frac{A}{R}z_1 + \frac{R'}{2R}z_2,$$

where

$$A = \frac{a^2b''}{a'} - kab' - k(k-1)b^2.$$

Let

$$w = hz_1 + z_2.$$

We have

$$\frac{dw}{dt} = h'z_1 - h\frac{R'}{2R}z_1 + \frac{A}{R}z_1 + \frac{R'}{2R}z_2.$$

Note that $h = h(t)$ is such that

$$h' - \frac{R'}{R}h + \frac{A}{R} = 0.$$

We obtain

$$\frac{dw}{dt} = \frac{R'}{2R}w.$$

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From (2.6) and $z_2 = w - hz_1$,

$$\xi = \frac{1}{\sqrt{R}}\left(a'w - \sqrt{R}Hz_1\right); \quad \eta = \frac{1}{\sqrt{R}}\left(b'w - \sqrt{R}\tilde{H}z_1\right)$$

where

$$H = \frac{1}{\sqrt{R}}(a'h + a); \quad \tilde{H} = \frac{1}{\sqrt{R}}(b'h + kb). \tag{2.7}$$

Re-write z_1 as $\tilde{\xi}$ and w as $\tilde{\eta}$, we have

$$\frac{d\tilde{\xi}}{dt} = -\frac{R'}{2R}\tilde{\xi}; \quad \frac{d\tilde{\eta}}{dt} = \frac{R'}{2R}\tilde{\eta}$$

where

$$\xi = \frac{1}{\sqrt{R}}\left(a'\tilde{\eta} - \sqrt{R}H\tilde{\xi}\right); \quad \eta = \frac{1}{\sqrt{R}}\left(b'\tilde{\eta} - \sqrt{R}\tilde{H}\tilde{\xi}\right).$$

This proves Lemma 2.1. □

2.2. Equations for stable solution

Let t_0 be a given initial time, and $(\hat{x}(t), \hat{y}(t))$ be the stable solution of the perturbed equation satisfying $\hat{y}(t_0)) = 0$. Let $(x(t), y(t)) = (\hat{x}(t + t_0), \hat{y}(t + t_0))$. Then $(x(t), y(t))$ is well-defined on $t \in [0, +\infty)$ satisfying

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = x + g(x) + \varepsilon P(t + t_0, x, y) \tag{2.8}$$

and $y(0) = 0$. Denote

$$X = x - a; \quad Y = y - b. \tag{2.9}$$

We have

$$\frac{dX}{dt} = Y; \quad \frac{dY}{dt} = X + a + g(X + a) - b' + \varepsilon P(t + t_0, X + a, Y + b).$$

By using

$$b' = a + g(a),$$

we obtain

$$\frac{dX}{dt} = Y, \quad \frac{dY}{dt} = (1 + g_x(a))X + Q(t, X) + \varepsilon P(t + t_0, X + a, Y + b) \tag{2.10}$$

where

$$Q(t, X) = g(X + a) - g(a) - g_x(a)X.$$

The correspondence of Lemma 2.1 for equation (2.10) is as follows.

Lemma 2.2. *Let M, W be such that*

$$X = \frac{\varepsilon}{\sqrt{R}}\left(a'W - \sqrt{R}HM\right); \quad Y = \frac{\varepsilon}{\sqrt{R}}\left(b'W - \sqrt{R}\tilde{H}M\right). \tag{2.11}$$

Then equation (2.10) is transformed in new variables (M, W) to

$$\begin{aligned} \frac{dM}{dt} &= -\frac{R'}{2R}M - \varepsilon\frac{a'}{\sqrt{R}}\mathbb{Q}(t, \varepsilon, \mathbb{X}) - \frac{a'}{\sqrt{R}}P(t + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b); \\ \frac{dW}{dt} &= \frac{R'}{2R}W - \varepsilon H\mathbb{Q}(t, \varepsilon, \mathbb{X}) - HP(t + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b) \end{aligned} \tag{2.12}$$

where

$$\mathbb{Q}(t, \varepsilon, \mathbb{X}) = \varepsilon^{-2}\left(g(\varepsilon\mathbb{X} + a) - g(a) - \varepsilon g_x(a)\mathbb{X}\right)$$

in which

$$\mathbb{X} = \frac{1}{\sqrt{R}}\left(a'W - \sqrt{R}HM\right). \tag{2.13}$$

Proof. Let z_1, z_2 be such that

$$z_1 = \frac{1}{\sqrt{R}}(b'X - a'Y); \quad z_2 = \frac{1}{\sqrt{R}}(kbX - aY). \tag{2.14}$$

We have

$$X = \frac{1}{\sqrt{R}}(a'z_2 - az_1); \quad Y = \frac{1}{\sqrt{R}}(b'z_2 - kbz_1). \tag{2.15}$$

For z_1 we have

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{1}{\sqrt{R}}[b''X + b'X' - a''Y - a'Y'] - \frac{R'}{2R}z_1 \\ &= -\frac{R'}{2R}z_1 - \frac{a'}{\sqrt{R}}Q(t, X) - \varepsilon\frac{a'}{\sqrt{R}}P(t + t_0, X + a, Y + b). \end{aligned}$$

For z_2 we have

$$\begin{aligned} \frac{dz_2}{dt} &= \frac{1}{\sqrt{R}}[kb'X + kbX' - a'Y - aY'] - \frac{R'}{2R}z_2 \\ &= \frac{1}{\sqrt{R}}\left\{kb'X + (k-1)bY - a[(1 + g_x(a))X + Q(t, X)]\right\} \end{aligned}$$

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$$
\begin{aligned}
&-\varepsilon\frac{a}{\sqrt{R}}P(t+t_0,X+a,Y+b)-\frac{R'}{2R}z_2\\
&=\frac{1}{R}\left\{[kb'-a(1+g_x(a))](-a)+(k-1)b(-kb)\right\}z_1\\
&\quad+\frac{b}{R}[(2k-1)(a+g(a))-a(1+g_x(a))]z_2\\
&\quad-\frac{a}{\sqrt{R}}Q(t,X)-\varepsilon\frac{a}{\sqrt{R}}P(t+t_0,X+a,Y+b)-\frac{R'}{2R}z_2\\
&=\frac{1}{R}\left[\frac{a^2b''}{a'}-kab'-k(k-1)b^2\right]z_1\\
&\quad+\frac{b}{R}[(2k-1)(a+g(a))-a(1+g_x(a))]z_2\\
&\quad-\frac{a}{\sqrt{R}}Q(t,X)-\varepsilon\frac{a}{\sqrt{R}}P(t+t_0,X+a,Y+b)-\frac{R'}{2R}z_2.
\end{aligned}
$$

Recall that

$$R'=b[(2k-1)(a+g(a))-a(1+g_x(a))],$$

and this is to imply

$$\frac{dz_2}{dt}=\frac{A}{R}z_1+\frac{R'}{2R}z_2-\frac{a}{\sqrt{R}}Q(t,X)-\varepsilon\frac{a}{\sqrt{R}}P(t+t_0,X+a,Y+b)$$

where

$$A=\frac{a^2b''}{a'}-kab'-k(k-1)b^2.$$

In summary, we have

$$
\begin{aligned}
\frac{dz_1}{dt}&=-\frac{R'}{2R}z_1-\frac{a'}{\sqrt{R}}Q(t,X)-\varepsilon\frac{a'}{\sqrt{R}}P(t+t_0,X+a,Y+b);\\
\frac{dz_2}{dt}&=\frac{A}{R}z_1+\frac{R'}{2R}z_2-\frac{a}{\sqrt{R}}Q(t,X)-\varepsilon\frac{a}{\sqrt{R}}P(t+t_0,X+a,Y+b).
\end{aligned}
$$

Let

$$w=hz_1+z_2.$$

We have

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$$\frac{dw}{dt} = h'z_1 + h\left\{-\frac{R'}{2R}z_1 - \frac{a'}{\sqrt{R}}Q(t,X) - \varepsilon\frac{a'}{\sqrt{R}}P(t+t_0, X+a, Y+b)\right\}$$
$$+\left\{\frac{A}{R}z_1 + \frac{R'}{2R}(w - hz_1) - \frac{a}{\sqrt{R}}Q(t,X) - \varepsilon\frac{a}{\sqrt{R}}P(t+t_0, X+a, Y+b)\right\}.$$

Note that $h(t)$ is such that

$$h' - \frac{R'}{R}h + \frac{A}{R} = 0,$$

and denote

$$H = h\frac{a'}{\sqrt{R}} + \frac{a}{\sqrt{R}}. \tag{2.16}$$

We have

$$\frac{dw}{dt} = \frac{R'}{2R}w - HQ(t,X) - \varepsilon HP(t+t_0, X+a, Y+b).$$

Altogether, we obtain

$$\frac{dz_1}{dt} = -\frac{R'}{2R}z_1 - \frac{a'}{\sqrt{R}}Q(t,X) - \varepsilon\frac{a'}{\sqrt{R}}P(t+t_0, X+a, Y+b);$$
$$\frac{dw}{dt} = \frac{R'}{2R}w - HQ(t,X) - \varepsilon HP(t+t_0, X+a, Y+b)$$

where

$$X = \frac{1}{\sqrt{R}}(a'(w - hz_1) - az_1) = \frac{a'}{\sqrt{R}}w - Hz_1.$$

Finally, we let

$$M = \varepsilon^{-1}z_1, \qquad W = \varepsilon^{-1}w \tag{2.17}$$

to obtain

$$\frac{dM}{dt} = -\frac{R'}{2R}M - \varepsilon\frac{a'}{\sqrt{R}}\mathbb{Q}(t,\varepsilon,\mathbb{X}) - \frac{a'}{\sqrt{R}}P(t+t_0, \varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b);$$
$$\frac{dW}{dt} = \frac{R'}{2R}W - \varepsilon H\mathbb{Q}(t,\varepsilon,\mathbb{X}) - HP(t+t_0, \varepsilon\mathbb{X}+a, \varepsilon\mathbb{Y}+b)$$

where

$$\mathbb{Q}(t,\varepsilon,\mathbb{X}) = \varepsilon^{-2}\left(g(\varepsilon\mathbb{X}+a) - g(a) - \varepsilon g_x(a)\mathbb{X}\right)$$

in which

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$$\mathbb{X} = \frac{1}{\sqrt{R}}\left(a'W - \sqrt{R}HM\right). \tag{2.18}$$

This proves Lemma 2.2. □

2.3. Integral equation for stable solutions

We now turn the equation for $M(t)$, $W(t)$ into integral equation for stable solutions.

Lemma 2.3. *We have for primary stable solutions*

$$\begin{aligned} M(t) &= \frac{\varepsilon}{\sqrt{R}}\int_t^{+\infty} a'\mathbb{Q}(\tau,\varepsilon,\mathbb{X})d\tau + \frac{1}{\sqrt{R}}\int_t^{+\infty} a'P(\tau+t_0,\varepsilon\mathbb{X}+a,\varepsilon\mathbb{Y}+b)d\tau;\\ W(t) &= -\varepsilon\sqrt{R}\int_0^t \frac{H}{\sqrt{R}}\mathbb{Q}(\tau,\varepsilon,\mathbb{X})d\tau - \sqrt{R}\int_0^t \frac{H}{\sqrt{R}}P(\tau+t_0,\varepsilon\mathbb{X}+a,\varepsilon\mathbb{Y}+b)d\tau. \end{aligned} \tag{2.19}$$

Proof. We use equation (2.12) to obtain

$$\begin{aligned} M(t) &= \frac{1}{\sqrt{R}}\left\{C_M - \varepsilon\int_0^t a'\mathbb{Q}(\tau,\varepsilon,\mathbb{X})d\tau - \int_0^t a'P(\tau+t_0,\varepsilon\mathbb{X}+a,\varepsilon\mathbb{Y}+b)d\tau\right\};\\ W(t) &= \sqrt{R}\left\{C_W - \varepsilon\int_0^t \frac{H}{\sqrt{R}}\mathbb{Q}(\tau,\varepsilon,\mathbb{X})d\tau - \int_0^t \frac{H}{\sqrt{R}}P(\tau+t_0,\varepsilon\mathbb{X}+a,\varepsilon\mathbb{Y}+b)d\tau\right\} \end{aligned} \tag{2.20}$$

where

$$C_M = \sqrt{R(0)}M(0), \quad C_W = \frac{1}{\sqrt{R(0)}}W(0).$$

Remember that for primary stable solutions

$$\lim_{t\to+\infty}(X(t),Y(t)) = (0,0).$$

This is to implies

$$\lim_{t\to+\infty} M(t) = \varepsilon^{-1}\lim_{t\to+\infty}\left[\frac{b'(t)}{\sqrt{R(t)}}X(t) - \frac{a'(t)}{\sqrt{R(t)}}Y(t)\right] = 0,$$

which in turn implies

$$C_M = \varepsilon\int_0^{+\infty} a'\mathbb{Q}(\tau,\varepsilon,\mathbb{X})d\tau + \int_0^{+\infty} a'P(\tau+t_0,\varepsilon\mathbb{X}+a,\varepsilon\mathbb{Y}+b)d\tau. \tag{2.21}$$

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Also recall that the primary stable solutions are initiated on $\Sigma := \{(X, Y) : Y = 0\}$. We have

$$z_2(0) = \frac{1}{\sqrt{R(0)}}[kb(0)X(0) - a(0)Y(0)] = 0$$

where to obtain the last equality we use $b(0) = 0$ and $y(0) = 0$. This implies

$$C_W = \frac{1}{\sqrt{R(0)}}W(0) = \varepsilon^{-1}\frac{1}{\sqrt{R(0)}}(h(0)z_1(0) + z_2(0)) = 0. \tag{2.22}$$

We obtain (2.19) by substituting C_M in (2.21) and C_W in (2.22) into (2.20). □

We expand $M(t) = M(t, t_0, \varepsilon)$, $W(t) = W(t, t_0, \varepsilon)$ into power series of ε as

$$M(t) = M_0(t, t_0) + \varepsilon M_1(t, t_0) + \varepsilon^2 M_2(t, t_0) + \cdots;$$
$$W(t) = W_0(t, t_0) + \varepsilon W_1(t, t_0) + \varepsilon^2 W_2(t, t_0) + \cdots.$$

We can use (2.19) to determine $M_n = M_n(t, t_0)$, $W_n = W_n(t, t_0)$ recursively for all n: First, we have from (2.19) that

$$\begin{aligned} M_0(t, t_0) &= \frac{1}{\sqrt{R(t)}}\int_t^{+\infty} a'(\tau)P(\tau + t_0, a(\tau), b(\tau))d\tau; \\ W_0(t, t_0) &= -\sqrt{R(t)}\int_0^t \frac{H(\tau)}{\sqrt{R(\tau)}}P(\tau + t_0, a(\tau), b(\tau))d\tau. \end{aligned} \tag{2.23}$$

Second, assume that we have obtained $M_k = M_k(t, t_0)$, $W_k = W_k(t, t_0)$ for all $k \le n$. We can determine $M_{n+1} = M_{n+1}(t, t_0)$, $W_{n+1} = W_{n+1}(t, t_0)$ in terms of $M_k = M_k(t, t_0)$, $W_k = W_k(t, t_0)$, $k \le n$ by using (2.19). This is because on the left hand side the only term of order ε^{n+1} are M_{n+1} and W_{n+1}, but on the right hand side the terms of order ε^{n+1} only include $M_k(t, t_0)$, $W_k(t, t_0)$ with $k \le n$.

2.4. Proof of Theorem 1

We let

$$\mathcal{E} = \varepsilon^{-1}\left(\frac{1}{2}y^2 - \frac{1}{2}x^2 - G(x)\right)$$

to obtain

$$\frac{d\mathcal{E}}{dt} = (\varepsilon\mathbb{Y} + b) \cdot P(t + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b) \tag{2.24}$$

where

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$$\mathbb{X} = \frac{1}{\sqrt{R}}\left(a'W - \sqrt{R}HM\right); \quad \mathbb{Y} = \frac{1}{\sqrt{R}}\left(b'W - \sqrt{R}\tilde{H}M\right).$$

Converting into integral equation, we have for $\mathcal{E}(t)$,

$$\mathcal{E}(t) = C_E + \int_0^t P(\tau + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b) \cdot (\varepsilon\mathbb{Y} + b)d\tau$$

where

$$C_E = \mathcal{E}(0).$$

Note that for stable solution, we have

$$\lim_{t\to+\infty} \mathcal{E}(t) = 0,$$

from which it follows that

$$C_E = -\int_0^{+\infty} (\varepsilon\mathbb{Y} + b)P(\tau + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b)d\tau.$$

This is to imply

$$\mathcal{E}(t) = -\int_t^{+\infty} (\varepsilon\mathbb{Y}(\tau) + b(\tau))P(\tau + t_0, \varepsilon\mathbb{X} + a, \varepsilon\mathbb{Y} + b)d\tau. \tag{2.25}$$

Denote

$$E_0(t, t_0) = \mathcal{E}(t, t_0, 0), \qquad E_1(t, t_0) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathcal{E}(t, t_0, \varepsilon) - E_0(t, t_0)\right).$$

We have

$$E_0(t, t_0) = -\int_t^{+\infty} bP(\tau + t_0, a, b)d\tau.$$

We also have

$$E_1(t, t_0) = -\int_t^{+\infty} b\mathcal{P}_x(\tau, t_0)\mathbb{X}_0 d\tau - \int_t^{+\infty} \left(b\mathcal{P}_y(\tau, t_0) + \mathcal{P}(\tau, t_0)\right)\mathbb{Y}_0 d\tau$$

where

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$$\mathcal{P}(t,t_0)=P(t+t_0,a,b);\quad \mathcal{P}_x(t,t_0)=P_x(t+t_0,a,b);\quad \mathcal{P}_y(t,t_0)=P_y(t+t_0,a,b);$$

and

$$\mathbb{X}_0=\frac{1}{\sqrt{R}}(a'W_0-\sqrt{R}HM_0);\quad \mathbb{Y}_0=\frac{1}{\sqrt{R}}(b'W_0-\sqrt{R}\tilde{H}M_0).$$

Finally, we obtain

$$E_1(t,t_0)=I_1(t,t_0)+I_2(t,t_0)$$

where

$$\begin{aligned} I_1(t,t_0)&=\int_t^{+\infty} f_1(\tau_2,t_0)\left(\int_0^{\tau_2}\frac{H(\tau_1)}{\sqrt{R(\tau_1)}}\mathcal{P}(\tau_1,t_0)d\tau_1\right)d\tau_2;\\ I_2(t,t_0)&=\int_t^{+\infty} f_2(\tau_2,t_0)\left(\int_{\tau_2}^{+\infty}a'(\tau_1)\mathcal{P}(\tau_1,t_0)d\tau_1\right)d\tau_2, \end{aligned}\tag{2.26}$$

where

$$\begin{aligned} f_1(t,t_0)=&b^2(t)\mathcal{P}_x(t,t_0)+b'(t)b(t)\mathcal{P}_y(t,t_0)+b'(t)\mathcal{P}(t,t_0)\\ f_2(t,t_0)=&\frac{1}{\sqrt{R(t)}}\left(b(t)H(t)\mathcal{P}_x(t,t_0)+b(t)\tilde{H}(t)\mathcal{P}_y(t,t_0)+\tilde{H}(t)\mathcal{P}(t,t_0)\right). \end{aligned}\tag{2.27}$$

First we prove

Lemma 2.4.

$$\left(\frac{H}{\sqrt{R}}\right)'=\frac{\tilde{H}}{\sqrt{R}}.\tag{2.28}$$

Proof. To prove (2.28), we recall that

$$H(t)=\frac{1}{\sqrt{R(t)}}(bh(t)+a).$$

We have

$$\begin{aligned} \left(\frac{H}{\sqrt{R}}\right)'&=\left(\frac{bh}{R}\right)'+\left(\frac{a}{R}\right)'\\ &=-\left(b\int_0^t A(\tau)R^{-2}(\tau)d\tau\right)'+\frac{b}{R}-\frac{a}{R^2}R' \end{aligned}$$

where for the second equality we used

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$$h(t) = -R(t)\int_0^t A(\tau)R^{-2}(\tau)d.$$

To continue, we have

$$\begin{aligned}\left(\frac{H}{\sqrt{R}}\right)' &= -b'\int_0^t A(\tau)R^{-2}(\tau)d\tau - bAR^{-2} + \frac{b}{R} - \frac{a}{R^2}R' \\ &= b'R^{-1}h - bAR^{-2} + bR^{-1} - aR^{-2}(2kbb' - bb' - ab'').\end{aligned}$$

Here, for the last equality we calculate R' by using

$$R(t) = kb^2 - ab'.$$

We also recall

$$A(t) = \frac{a^2b''}{a'} - kab' - k(k-1)b^2.$$

We have then

$$\begin{aligned}\left(\frac{H}{\sqrt{R}}\right)' &= b'R^{-1}h - b\left(\frac{a^2b''}{a'} - kab' - k(k-1)b^2\right)R^{-2} \\ &\quad + bR^{-1} - aR^{-2}(2kbb' - bb' - ab'') \\ &= b'R^{-1}h + kbR^{-1}.\end{aligned}$$

We now recall that

$$\tilde{H}(t) = \frac{1}{\sqrt{R}}(b'(t)h(t) + kb(t)). \quad \square$$

Proof of Theorem 1. So far we have derived integral formulas for $E_1(t, t_0)$ for stable solutions. In parallel we can also derive integral formula for $E_1(t, t_0)$ for unstable solutions. To distinguish the two we use superscript $+$ for stable solutions and $-$ for unstable solutions to write

$$E_1^+(t, t_0) = I_1^+(t, t_0) + I_2^+(t, t_0), \quad E_1^-(t, t_0) = I_1^-(t, t_0) + I_2^-(t, t_0).$$

The first order splitting function is

$$E_1(t_0) = E_1^+(0, t_0) - E_1^-(0, t_0) = I_1^+(t_0) - I_1^-(t_0) + I_2^+(t_0) - I_2^-(t_0)$$

where

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$$I_1^{\pm}(t_0) = \int_0^{\pm\infty} f_1(\tau_2, t_0) \left(\int_0^{\tau_2} \frac{H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2;$$

$$I_2^{\pm}(t_0) = \int_0^{\pm\infty} f_2(\tau_2, t_0) \left(\int_{\tau_2}^{\pm\infty} a'(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2;$$

and $f_1(t, t_0)$, $f_2(t, t_0)$ are as in (2.27).

For $I_1^{\pm}(t_0)$, we start with

$$\begin{aligned} f_1(t, t_0) =& b^2(t)\mathcal{P}_x(t, t_0) + b'(t)b(t)\mathcal{P}_y(t, t_0) + b'(t)\mathcal{P}(t, t_0) \\ =& [b(t)\mathcal{P}(t, t_0)]' - b(t)\mathcal{P}_t(t, t_0). \end{aligned}$$

We have

$$\begin{aligned} I_1^{\pm}(t_0) =& \int_0^{\pm\infty} [b(\tau_2)\mathcal{P}(\tau_2, t_0)]' \left(\int_0^{\tau_2} \frac{H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \\ &- \int_0^{\pm\infty} b(\tau_2)\mathcal{P}_t(\tau_2, t_0) \left(\int_0^{\tau_2} \frac{H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \\ =& - \int_0^{\pm\infty} b(\tau_2)\mathcal{P}(\tau_2, t_0) \frac{H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}(\tau_2, t_0) d\tau_2 \\ &- \int_0^{\pm\infty} b(\tau_2)\mathcal{P}_t(\tau_2, t_0) \left(\int_0^{\tau_2} \frac{H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \end{aligned}$$

where the second equality is obtained by integration by part. We conclude that

$$\begin{aligned} I_1^{+}(t_0) - I_1^{-}(t_0) =& - \int_{-\infty}^{+\infty} \frac{b(\tau_2)H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}^2(\tau_2, t_0) d\tau_2 \\ &- \int_{-\infty}^{+\infty} b(\tau_2)\mathcal{P}_t(\tau_2, t_0) \left(\int_0^{\tau_2} \frac{H(\tau_1)}{\sqrt{R(\tau_1)}} \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2. \end{aligned} \tag{2.29}$$

For $I_2^{\pm}(t_0)$, we start with

$$\begin{aligned} f_2(t, t_0) =& \frac{1}{\sqrt{R(t)}} \left(b(t)H(t)\mathcal{P}_x(t, t_0) + b(t)\tilde{H}(t)\mathcal{P}_y(t, t_0) + \tilde{H}(t)\mathcal{P}(t, t_0) \right) \\ =& \left(\frac{H(t)}{\sqrt{R(t)}} \mathcal{P}(t, t_0) \right)' + \frac{1}{\sqrt{R(t)}} (b(t)\tilde{H}(t) - b'(t)H(t))\mathcal{P}_y(t, t_0) - \frac{H(t)}{\sqrt{R(t)}} \mathcal{P}_t(t, t_0) \end{aligned}$$

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$$= \left(\frac{H(t)}{\sqrt{R(t)}} \mathcal{P}(t, t_0) \right)' + \mathcal{P}_y(t, t_0) - \frac{H(t)}{\sqrt{R(t)}} \mathcal{P}_t(t, t_0).$$

For $I_2^{\pm}(t_0)$, we have

$$\begin{aligned}
I_2^{\pm}(t_0) = & \int_0^{\pm\infty} \left(\frac{H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}(\tau_2, t_0) \right)' \left(\int_{\tau_2}^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \\
& + \int_0^{\pm\infty} \mathcal{P}_y(\tau_2, t_0) \left(\int_{\tau_2}^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \\
& - \int_0^{\pm\infty} \frac{H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}_t(\tau_2, t_0) \left(\int_{\tau_2}^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 \right) d\tau_2 \\
= & -\frac{H(0)}{\sqrt{R(0)}} \mathcal{P}(0, t_0) \int_0^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 + \int_0^{\pm\infty} \frac{b(\tau_2) H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}^2(\tau_2, t_0) d\tau_2 \\
& + \int_0^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) \left(\int_0^{\tau_1} \mathcal{P}_y(\tau_2, t_0) d\tau_2 \right) d\tau_1 \\
& - \int_0^{\pm\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) \left(\int_0^{\tau_1} \frac{H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}_t(\tau_2, t_0) d\tau_2 \right) d\tau_1.
\end{aligned}$$

We have

$$\begin{aligned}
I_2^{+} - I_2^{-} = & -\frac{H(0)}{\sqrt{R(0)}} \mathcal{P}(0, t_0) \int_{-\infty}^{+\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 + \int_{-\infty}^{+\infty} \frac{b(\tau_2) H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}^2(\tau_2, t_0) d\tau_2 \\
& + \int_{-\infty}^{+\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) \left(\int_0^{\tau_1} \mathcal{P}_y(\tau_2, t_0) d\tau_2 \right) d\tau_1 \\
& - \int_{-\infty}^{+\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) \left(\int_0^{\tau_1} \frac{H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}_t(\tau_2, t_0) d\tau_2 \right) d\tau_1.
\end{aligned}$$

We re-write τ_1 as τ_2, τ_2 as τ_1 in the last two integrals to obtain

$$I_2^{+} - I_2^{-} = -\frac{H(0)}{\sqrt{R(0)}} \mathcal{P}(0, t_0) \int_{-\infty}^{+\infty} b(\tau_1) \mathcal{P}(\tau_1, t_0) d\tau_1 + \int_{-\infty}^{+\infty} \frac{b(\tau_2) H(\tau_2)}{\sqrt{R(\tau_2)}} \mathcal{P}^2(\tau_2, t_0) d\tau_2$$

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$$+\int_{-\infty}^{+\infty} b(\tau_2)\mathcal{P}(\tau_2,t_0)\left(\int_0^{\tau_2}\mathcal{P}_y(\tau_1,t_0)d\tau_1\right)d\tau_2$$
$$-\int_{-\infty}^{+\infty} b(\tau_2)\mathcal{P}(\tau_2,t_0)\left(\int_0^{\tau_2}\frac{H(\tau_1)}{\sqrt{R(\tau_1)}}\mathcal{P}_t(\tau_1,t_0)d\tau_1\right)d\tau_2.$$

In conclusion, we have

$$\begin{aligned}E_1(t_0) =& I_1^+ - I_1^- + I_2^+ - I_2^-\\ =& -\frac{H(0)}{\sqrt{R(0)}}\mathcal{P}(0,t_0)\int_{-\infty}^{+\infty} b(\tau_1)\mathcal{P}(\tau_1,t_0)d\tau_1\\ &+\int_{-\infty}^{+\infty} b(\tau_2)\mathcal{P}(\tau_2,t_0)\left(\int_0^{\tau_2}\mathcal{P}_y(\tau_1,t_0)d\tau_1\right)d\tau_2\\ &-\int_{-\infty}^{+\infty}\int_0^{\tau_2}\frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}}\left[\mathcal{P}_t(\tau_2,t_0)\mathcal{P}(\tau_1,t_0)+\mathcal{P}_t(\tau_1,t_0)\mathcal{P}(\tau_2,t_0)\right]d\tau_1 d\tau_2.\end{aligned} \tag{2.30}$$

Note that the first integral is $E_0(t_0)$. □

3. High order Melnikov method: application

The example we study in this section is

$$\frac{d^2x}{dt^2} = x - x^3 + \varepsilon(x^2 + \gamma^* x^3)\cos\omega t. \tag{3.1}$$

In this case, we have

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}.$$

We also have

$$R(t) = a^2(t); \quad A(t) = -3a^2(t); \quad h(t) = 3a^2(t)\int_0^t a^{-2}(\tau)d\tau = \frac{3(e^{2t} - e^{-2t} + 4t)}{2(e^t + e^{-t})^2};$$

and

$$H(t) = \frac{1}{a(t)}(a'(t)h(t) + a(t)), \qquad \tilde{H}(t) = \frac{1}{a(t)}(b'(t)h(t) + 2b(t)).$$

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Note that $h(t)$ and $\tilde{H}(t)$ are odd functions and $H(t)$ is an even function in t. In what follows, we denote

$$\mathcal{P}(t,t_0) = P(t+t_0, a(t), b(t)) = (a^2(t) + \gamma^* a^3(t))\cos\omega(t+t_0)$$
$$\mathcal{P}_t(t,t_0) = -\omega(a^2(t) + \gamma^* a^3(t))\sin\omega(t+t_0)$$

where $\gamma^* = \gamma^*(\omega)$ is such that

$$\gamma^* = -2\sqrt{2}\omega^{-1} + O(\omega^{-2}).$$

We have, for $\gamma^* = \gamma^*(\omega)$,

$$\begin{aligned} E_0(t_0) &= -\int_{-\infty}^{+\infty} b(\tau)(a^2(\tau) + \gamma^* a^3(\tau))\cos\omega(\tau+t_0)d\tau \\ &= \sin\omega t_0 \int_{-\infty}^{+\infty} b(\tau)(a^2(\tau) + \gamma^* a^3(\tau))\sin\omega\tau d\tau \\ &= 0 \end{aligned}$$

for all t_0. This implies

$$\int_{-\infty}^{+\infty} b(\tau)(a^2(\tau) + \gamma^* a^3(\tau))e^{i\omega\tau}d\tau = 0 \tag{3.2}$$

assuming $E_0(t_0) = 0$ for all t_0. We also note that, in this case, the forcing function is independent of y. We have by Theorem 1,

$$E_1(t_0) = -\int_{-\infty}^{+\infty}\int_0^{\tau_2} \frac{b(\tau_2)H(\tau_1)}{\sqrt{R(\tau_1)}}[\mathcal{P}_t(\tau_2,t_0)\mathcal{P}(\tau_1,t_0) + \mathcal{P}_t(\tau_1,t_0)\mathcal{P}(\tau_2,t_0)]d\tau_1 d\tau_2. \tag{3.3}$$

3.1. Initial reduction

We start with

Lemma 3.1. *(1) Assume $f(\tau_1,\tau_2)$ is such that*

$$f(\tau_1,\tau_2) = f(-\tau_1,-\tau_2).$$

We then have

$$\int_0^{+\infty}\int_0^{\tau_1} f(\tau_1,\tau_2)d\tau_2 d\tau_1 = \int_0^{-\infty}\int_0^{\tau_1} f(\tau_1,\tau_2)d\tau_2 d\tau_1.$$

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(2) Assume $f(\tau_1, \tau_2)$ is such that

$$f(\tau_1, \tau_2) = -f(-\tau_1, -\tau_2).$$

We then have

$$\int_0^{+\infty}\int_0^{\tau_1} f(\tau_1, \tau_2)d\tau_2 d\tau_1 = -\int_0^{-\infty}\int_0^{\tau_1} f(\tau_1, \tau_2)d\tau_2 d\tau_1.$$

Proof. By changing (τ_1, τ_2) to $(-\tau_1, -\tau_2)$. □

We prove

Proposition 3.1. *We have, for equation (3.1)*

$$E_1(t_0) = 2\omega \sin 2\omega t_0 \int_0^{+\infty}\int_0^{\tau_2} b(\tau_2)Q(\tau_2)\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\cos\omega(\tau_1+\tau_2)d\tau_1 d\tau_2 \tag{3.4}$$

where

$$Q(t) = a^2(t) + \gamma^* a^3(t).$$

Proof. We start with

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}_t(\tau_1, t_0)\mathcal{P}(\tau_2, t_0) + \mathcal{P}_t(\tau_2, t_0)\mathcal{P}(\tau_1, t_0)\\
&= -\omega Q(\tau_1)Q(\tau_2)\left(\sin\omega(\tau_1+t_0)\cos\omega(\tau_2+t_0) + \cos\omega(\tau_1+t_0)\sin\omega(\tau_2+t_0)\right)\\
&= -\omega Q(\tau_1)Q(\tau_2)\sin\omega(\tau_1+\tau_2+2t_0)\\
&= -\omega Q(\tau_1)Q(\tau_2)\cos\omega(\tau_1+\tau_2)\sin 2\omega t_0 - \omega Q(\tau_1)Q(\tau_2)\sin\omega(\tau_1+\tau_2)\cos 2\omega t_0.
\end{aligned}$$

We recall that $a(t)$ and $H(t)$ are even, $b(t)$ is odd. We then use Lemma 3.1 to reduce $E_1(t_0)$ to

$$E_1(t_0) = 2\omega \sin 2\omega t_0 \int_0^{+\infty}\int_0^{\tau_2} b(\tau_2)Q(\tau_2)\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\cos\omega(\tau_1+\tau_2)d\tau_1 d\tau_2. \quad \square$$

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3.2. Further reduction: flip integral bounds

Let

$$
\begin{aligned}
A &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} H(\tau_1)a(\tau_1)b(\tau_2+\tau_1)a^2(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
B &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} H(\tau_1)a(\tau_1)b(\tau_2+\tau_1)a^3(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
C &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} H(\tau_1)a^2(\tau_1)b(\tau_2+\tau_1)a^2(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
D &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} H(\tau_1)a^2(\tau_1)b(\tau_2+\tau_1)a^3(\tau_2+\tau_1)d\tau_1\right)d\tau_2.
\end{aligned}
\tag{3.5}
$$

Lemma 3.2. *We have*

$$E_1(t_0) = \omega\sin 2\omega t_0\Big[A + \gamma^*(B+C) + (\gamma^*)^2 D\Big],$$

where A, B, C, D are as in (3.5).

Proof. We start with

$$E_1(t_0) = 2\omega\sin 2\omega t_0 \int_0^{+\infty}\int_0^{\tau_2} b(\tau_2)Q(\tau_2)\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\cos\omega(\tau_1+\tau_2)d\tau_1 d\tau_2 \tag{3.6}$$

to obtain

$$
\begin{aligned}
E_1(t_0) &= 2\omega\sin 2\omega t_0 \int_0^{+\infty}\int_{\tau_1}^{+\infty} b(\tau_2)Q(\tau_2)\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\cos\omega(\tau_1+\tau_2)d\tau_2 d\tau_1\\
&= \omega\sin 2\omega t_0 \int_0^{+\infty}\int_{\tau_1}^{+\infty} b(\tau_2)Q(\tau_2)e^{i\omega\tau_2}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)e^{i\omega\tau_1}d\tau_2 d\tau_1\\
&\quad+\omega\sin 2\omega t_0 \int_0^{+\infty}\int_{\tau_1}^{+\infty} b(\tau_2)Q(\tau_2)e^{-i\omega\tau_2}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)e^{-i\omega\tau_1}d\tau_2 d\tau_1
\end{aligned}
$$

where we switch the order of integration from $d\tau_1 d\tau_2$ to $d\tau_2 d\tau_1$ to obtain the first equality, and use

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$$\cos\omega(\tau_1+\tau_2)=\frac{1}{2}\left(e^{i\omega(\tau_1+\tau_2)}+e^{-i\omega(\tau_1+\tau_2)}\right)$$

to obtain the second equality. We have

$$E_1(t_0)=\omega\sin 2\omega t_0(I^+ + I^-) \tag{3.7}$$

where

$$I^+=\int_0^{+\infty}\int_{\tau_1}^{+\infty} b(\tau_2)Q(\tau_2)e^{i\omega\tau_2}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)e^{i\omega\tau_1}d\tau_2 d\tau_1$$

$$I^-=\int_0^{+\infty}\int_{\tau_1}^{+\infty} b(\tau_2)Q(\tau_2)e^{-i\omega\tau_2}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)e^{-i\omega\tau_1}d\tau_2 d\tau_1.$$

We introduce $\tilde{\tau}_2=\tau_2-\tau_1$ first, then write $\tilde{\tau}_2$ back as τ_2 to obtain

$$\begin{aligned} I^+&=\int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{+\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\ I^-&=\int_0^{+\infty} e^{-2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{+\infty} e^{-i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1. \end{aligned} \tag{3.8}$$

To evaluate $(I^+ + I^-)$, we start with I^+. We have

$$\begin{aligned} I^+&=\int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{+\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\ &=\int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_{-\infty}^{+\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\ &\quad+\int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{-\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\ &=\int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_{-\infty}^{+\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\ &\quad+\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{-\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1 \end{aligned}$$

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$$+\int_0^{-\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{-\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1.$$

We note that the last integral is $-I^-$, and the first integral is

$$\begin{aligned}
&= \int_0^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_{-\infty}^{+\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\
&= \int_0^{+\infty} e^{i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_{-\infty}^{+\infty} e^{i\omega\tau_2}b(\tau_2)Q(\tau_2)d\tau_2\right)d\tau_1\\
&= \left(\int_0^{+\infty} e^{i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)d\tau_1\right)\left(\int_{-\infty}^{+\infty} e^{i\omega\tau_2}b(\tau_2)Q(\tau_2)d\tau_2\right)\\
&= 0.
\end{aligned}$$

Here, (3.2) is used for the last equality.

Consequently, we have

$$\begin{aligned}
I^+ + I^- &= \int_{-\infty}^{+\infty} e^{2i\omega\tau_1}\frac{H(\tau_1)}{a(\tau_1)}Q(\tau_1)\left(\int_0^{-\infty} e^{i\omega\tau_2}b(\tau_2+\tau_1)Q(\tau_2+\tau_1)d\tau_2\right)d\tau_1\\
&= A+\gamma^*(B+C)+(\gamma^*)^2D
\end{aligned}$$

where A, B, C, D are as in (3.5). □

3.3. Final reduction

In what follows, we denote

$$G(t) = b(t)\frac{3(e^{2t}-e^{-2t}+4(t-i\pi/2))}{2(e^t+e^{-t})^2}+a(t), \tag{3.9}$$

and let

$$\tilde{A} = \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)b(\tau_2+\tau_1)a^2(\tau_2+\tau_1)d\tau_1\right)d\tau_2$$

$$\tilde{B} = \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)b(\tau_2+\tau_1)a^3(\tau_2+\tau_1)d\tau_1\right)d\tau_2$$

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$$\tilde{C} = \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} G(\tau_1) a(\tau_1) b(\tau_2+\tau_1) a^2(\tau_2+\tau_1) d\tau_1 \right) d\tau_2$$

$$\tilde{D} = \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} G(\tau_1) a(\tau_1) b(\tau_2+\tau_1) a^3(\tau_2+\tau_1) d\tau_1 \right) d\tau_2. \tag{3.10}$$

Note that we obtain $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ by using $G(\tau_1)$ to replace $H(\tau_1)a(\tau_1)$ in A, B, C and D respectively.

Lemma 3.3. *We have*

$$E_1(t_0) = \omega \sin 2\omega t_0 \left[\tilde{A} + \gamma^*(\tilde{B} + \tilde{C}) + (\gamma^*)^2 \tilde{D} \right].$$

Proof. Let

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} e^{i\omega t} b(t) a^2(t) dt, \quad \mathcal{I}_3 = \int_{-\infty}^{+\infty} e^{i\omega t} b(t) a^3(t) dt. \tag{3.11}$$

We note that to make $E_0(t_0) = 0$ for all t_0 is the same as to make

$$\mathcal{I}_2 + \gamma^* \mathcal{I}_3 = 0. \tag{3.12}$$

See (3.2). Recall that

$$H = a^{-1} bh + 1.$$

We have

$$\begin{aligned} A &= \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} \left(a^{-1}(\tau_1) b(\tau_1) h(\tau_1) + 1 \right) a(\tau_1) b(\tau_2+\tau_1) a^2(\tau_2+\tau_1) d\tau_1 \right) d\tau_2 \\ &= \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} b(\tau_1) h(\tau_1) b(\tau_2+\tau_1) a^2(\tau_2+\tau_1) d\tau_1 \right) d\tau_2 \\ &\quad + \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} a(\tau_1) b(\tau_2+\tau_1) a^2(\tau_2+\tau_1) d\tau_1 \right) d\tau_2. \end{aligned} \tag{3.13}$$

We have

$$h(\tau_1) = \frac{3(e^{2\tau_1} - e^{-2\tau_1} + 4\tau_1)}{2(e^{\tau_1} + e^{-\tau_1})^2} = \frac{3(e^{2\tau_1} - e^{-2\tau_1} + 4(\tau_1 - i\pi/2))}{2(e^{\tau_1} + e^{-\tau_1})^2} + \frac{3\pi i}{(e^{\tau_1} + e^{-\tau_1})^2}.$$

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This is for us to have

$$h(\tau_1) = \frac{3(e^{2\tau_1} - e^{-2\tau_1} + 4(\tau_1 - i\pi/2)}{2(e^{\tau_1} + e^{-\tau_1})^2} + \frac{3}{8}\pi i a^2(\tau_1), \tag{3.14}$$

and it lead to

$$A = A_1 + \tilde{A},$$

where

$$A_1 = \frac{3}{8}\pi i \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} b(\tau_1) a^2(\tau_1) b(\tau_2 + \tau_1) a^2(\tau_2 + \tau_1) d\tau_1 \right) d\tau_2$$

$$\tilde{A} = \int_0^{-\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} G(\tau_1) b(\tau_2 + \tau_1) a^2(\tau_2 + \tau_1) d\tau_1 \right) d\tau_2.$$

Note that $\tilde{A}$ is as in (3.10).

We continue to work on A_1. We use $t_1 = \tau_1 + \tau_2$ to write A_1 as

$$A_1 = \frac{3}{8}\pi i \int_0^{-\infty} e^{-i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega t_1} b(t_1 - \tau_2) a^2(t_1 - \tau_2) b(t_1) a^2(t_1) dt_1 \right) d\tau_2.$$

We then let $t_2 = -\tau_2$ to obtain

$$A_1 = -\frac{3}{8}\pi i \int_0^{+\infty} e^{i\omega t_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega t_1} b(t_1 + t_2) a^2(t_1 + t_2) b(t_1) a^2(t_1) dt_1 \right) dt_2.$$

We have

$$\begin{aligned} A_1 &= -\frac{3}{16}\pi i \int_{-\infty}^{+\infty} e^{i\omega\tau_2} \left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1} b(\tau_1) a^2(\tau_1) b(\tau_2 + \tau_1) a^2(\tau_2 + \tau_1) d\tau_1 \right) d\tau_2 \\ &= -\frac{3}{16}\pi i \int_{-\infty}^{+\infty} e^{i\omega\tau_1} b(\tau_1) a^2(\tau_1) \left(\int_{-\infty}^{+\infty} e^{i\omega(\tau_2+\tau_1)} b(\tau_2 + \tau_1) a^2(\tau_2 + \tau_1) d\tau_2 \right) d\tau_1 \\ &= -\frac{3}{16}\pi i \left(\int_{-\infty}^{+\infty} e^{i\omega t} b(t) a^2(t) dt \right)^2. \end{aligned}$$

We have

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$$A_1 = -\frac{3\pi i}{16}\mathcal{I}_2^2 \tag{3.15}$$

where $\mathcal{I}_2$ is as in (3.11).

Reduction on $B + C$ and D are similar, and we have

$$B + C = B_1 + C_1 + \tilde{B} + \tilde{C}$$

where

$$B_1 + C_1 = -\frac{3}{16}\pi i \cdot (2\mathcal{I}_2\mathcal{I}_3), \tag{3.16}$$

and $\tilde{B}$, $\tilde{C}$ are as in (3.10). We also have

$$D = D_1 + \tilde{D}$$

where

$$D_1 = -\frac{3\pi i}{16}\mathcal{I}_3^2, \tag{3.17}$$

and $\tilde{D}$ is as in (3.10).

Final Reduction: We have, from (3.15), (3.16) and (3.17), that

$$A_1 + \gamma^*(B_1 + C_1) + (\gamma^*)^2 D_1 = -\frac{3}{16}\pi i \left(\mathcal{I}_2 + \gamma^*\mathcal{I}_3\right)^2 = 0.$$

This proves the lemma. □

4. Final evaluation

Let

$$G(t) = b(t)\left(\frac{3(e^{2t} - e^{-2t} + 4(t - i\pi/2)}{2(e^t + e^{-t})^2}\right) + a(t)$$

and denote

$$\begin{aligned} g(t) =& G(t + i\pi/2), \quad \tilde{g}(t) = a(t + i\pi/2)G(t + i\pi/2) \\ f(t) =& t^4 b(t + i\pi/2)a^2(t + i\pi/2), \quad \tilde{f}(t) = t^5 b(t + i\pi/2)a^3(t + i\pi/2). \end{aligned} \tag{4.1}$$

Lemma 4.1. *Both $g(t)$ and $\tilde{g}(t)$ are real analytic functions on $t \in \mathbb{R}$. In addition,*

(i) $g(t)$ is odd in t and $\tilde{g}(t)$ is even in t;

(ii) $g(0) = 0$, $g'(0) = \frac{\sqrt{2}}{5}i$, and $\tilde{g}(0) = -\sqrt{2}ig'(0)$;

(iii) $f(0) = -2\sqrt{2}i$, $\tilde{f}(0) = -4$.

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Proof. Recall that

$$a(t) = \frac{2\sqrt{2}}{e^t + e^{-t}}, \quad b(t) = \frac{2\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}.$$

We have

$$a(t + i\pi/2) = -\frac{2\sqrt{2}i}{(e^t - e^{-t})}, \quad b(t + i\pi/2) = \frac{2\sqrt{2}i(e^{-t} + e^t)}{(e^t - e^{-t})^2}.$$

As a function in t, $a(t + i\pi/2)$ is odd and $b(t + i\pi/2)$ is even. It then follows that

$$g(t) = \frac{2\sqrt{2}i(e^{-t} + e^t)}{(e^t - e^{-t})^2}\left(\frac{3(e^{2t} - e^{-2t} - 4t)}{2(e^t - e^{-t})^2}\right) - \frac{2\sqrt{2}i}{(e^t - e^{-t})}$$

is an odd function in t, and $\tilde{g}(t)$ is even. This is item (i).

That both $g(t)$ and $\tilde{g}(t)$ come out as real analytic function is somewhat a pleasant surprise. It is seemingly that both functions took $t = 0$ as a pole. However, the singular parts of the two terms involved in $g(t)$ actually cancel each other out. We have up to the order t at $t = 0$,

$$\begin{aligned} g(t) := G(t + i\pi/2) &= \frac{2\sqrt{2}i(e^{-t} + e^t)}{(e^t - e^{-t})^2}\left(\frac{3(e^{2t} - e^{-2t} - 4t)}{2(e^t - e^{-t})^2}\right) - \frac{2\sqrt{2}i}{(e^t - e^{-t})} \\ &= \frac{\sqrt{2}}{5}it + O(t^3). \end{aligned}$$

We conclude that both $g(t)$ and $\tilde{g}(t)$ are real analytic. Item (ii) and (iii) are resulted from direct calculations. □

4.1. Residue computation

In this subsection, we introduce a generic setting we will use to compute $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$. Let

$$\mathcal{K}_{g,f}(z, t) = \frac{e^{2i\omega z} g(z) f(t + z)}{(t + z)^m} \tag{4.2}$$

where z is a complex variable, $t \neq 0$ is real, and $m > 0$ is a positive integer fixed throughout. We assume that

(A1) the functions $f(z)$, $g(z)$ are independent of the forcing frequency ω,
(A2) $f(z)$, $g(z)$ are real analytic on $Im(z) = 0$ and $f(0) \neq 0$,
(A3) $|f^{(k)}(t)|$, $|g^{(k)}(t)| \leq C_0 e^{-c_0|t|}$ for all $0 \leq k < m$ where both C_0 and c_0 are positive constants.

We let $t \neq 0$ be fixed to regard $\mathcal{K}_{g,f}(z, t)$ as a function of z. The function $\mathcal{K}_{g,f}(z, t)$ has a pole on the real z axis at $z = -t$. The order of this pole is m. Denote the residue of this pole as $R(t)$. Our purpose here is to compute

$$I_{g,f}(\omega) = \int_0^{-\infty} e^{i\omega t} R(t)dt. \tag{4.3}$$

Proposition 4.1. *We have*

$$\begin{aligned} I_{g,f}(\omega) = & \frac{2^{m-1}(i\omega)^{m-2}}{(m-1)!} f(0)g(0) + \frac{2^{m-2}(i\omega)^{m-3}}{(m-2)!}(f(0)g'(0) + f'(0)g(0)) \\ & - \frac{2^{m-1}(i\omega)^{m-3}}{(m-1)!} f(0)g'(0) + O(\omega^{m-4}). \end{aligned} \tag{4.4}$$

Proof. Denote

$$G(z) = e^{2i\omega z} g(z) \tag{4.5}$$

to write $\mathcal{K}_{g,f}(z,t)$ as

$$\mathcal{K}_{g,f}(z,t) = \frac{G(z)f(t+z)}{(t+z)^m}. \tag{4.6}$$

We have

$$R(t) = \frac{1}{(m-1)!} \partial_{z^{m-1}} \left. (f(z)G(z-t)) \right|_{z=0}.$$

To compute $R(t)$, we start with Leibniz's formula (product rule)

$$\partial_{z^{m-1}}(h_1(z)h_2(z)) = \sum_{\alpha=0}^{m-1} \frac{(m-1)!}{\alpha!(m-1-\alpha)!} \partial_{z^\alpha} h_1(z) \cdot \partial_{z^{m-1-\alpha}} h_2(z). \tag{4.7}$$

We have by using (4.7),

$$R(t) = \sum_{\alpha=0}^{m-1} \frac{1}{\alpha!(m-1-\alpha)!} f^{(\alpha)}(0) \partial_{\tau^{(m-1-\alpha)}} G(\tau)$$

where

$$\tau = -t.$$

We recall

$$G(\tau) = e^{2i\omega\tau} g(\tau)$$

to obtain by using (4.7)

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$$G^{(m-1-\alpha)}(\tau) = e^{2i\omega\tau} \sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}(m-1-\alpha)!}{\gamma!(m-1-\alpha-\gamma)!} g^{(\gamma)}(\tau).$$

This is then for us to have

$$\begin{aligned} R(t) &= \sum_{\alpha=0}^{m-1} \frac{1}{\alpha!(m-1-\alpha)!} f^{(\alpha)}(0) e^{2i\omega\tau} \sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}(m-1-\alpha)!}{\gamma!(m-1-\alpha-\gamma)!} g^{(\gamma)}(\tau) \\ &= e^{2i\omega\tau} \sum_{\alpha=0}^{m-1} \sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}}{\alpha!\gamma!(m-1-\alpha-\gamma)!} f^{(\alpha)}(0) g^{(\gamma)}(\tau) \\ &= e^{-2i\omega t} \sum_{\alpha=0}^{m-1} \sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}}{\alpha!\gamma!(m-1-\alpha-\gamma)!} f^{(\alpha)}(0) g^{(\gamma)}(-t). \end{aligned}$$

Recall that

$$I_{g,f} = \int_0^{-\infty} e^{i\omega t} R_f(t) dt.$$

We have

$$I_{g,f} = \sum_{\alpha=0}^{m-1} \sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}}{\alpha!\gamma!(m-1-\alpha-\gamma)!} f^{(\alpha)}(0) \mathcal{I}_g^{\alpha,\gamma} \tag{4.8}$$

where

$$\mathcal{I}_g^{\alpha,\gamma} = \int_0^{-\infty} e^{-i\omega t} g^{(\gamma)}(-t) dt. \tag{4.9}$$

We have

$$\begin{aligned} \mathcal{I}_g^{\alpha,\gamma} &= -\int_0^{+\infty} e^{i\omega t} g^{(\gamma)}(t) dt = -\frac{1}{i\omega} \int_0^{+\infty} g^{(\gamma)}(t) de^{i\omega t} \\ &= \frac{1}{i\omega} g^{(\gamma)}(0) + \frac{1}{(i\omega)^2} \int_0^{+\infty} g^{(\gamma+1)}(t) de^{i\omega t} \\ &= \frac{1}{i\omega} g^{(\gamma)}(0) - \frac{1}{(i\omega)^2} g^{(\gamma+1)}(0) + O(\omega^{-3}). \end{aligned} \tag{4.10}$$

This is to have

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$$
\begin{aligned}
I_{g,f} =& \sum_{\alpha=0}^{m-1}\sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}}{\alpha!\gamma!(m-1-\alpha-\gamma)!}\left(\frac{1}{i\omega}f^{(\alpha)}(0)g^{(\gamma)}(0)\right)\\
&-\sum_{\alpha=0}^{m-1}\sum_{\gamma=0}^{m-1-\alpha} \frac{(2i\omega)^{m-1-\alpha-\gamma}}{\alpha!\gamma!(m-1-\alpha-\gamma)!}f^{(\alpha)}(0)\left(\frac{1}{(i\omega)^2}g^{(\gamma+1)}(0)+O(\omega^{-3})\right)\\
&=\frac{2^{m-1}(i\omega)^{m-2}}{(m-1)!}f(0)g(0)+\frac{2^{m-2}(i\omega)^{m-3}}{(m-2)!}(f(0)g'(0)+f'(0)g(0))\\
&-\frac{2^{m-1}(i\omega)^{m-3}}{(m-1)!}f(0)g'(0)+O(\omega^{m-4}). \quad \square
\end{aligned} \tag{4.11}
$$

4.2. Asymptotic formula for $E_1(t_0)$

Recall that

$$
\begin{aligned}
\tilde{A} &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)b(\tau_2+\tau_1)a^2(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
\tilde{B} &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)b(\tau_2+\tau_1)a^3(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
\tilde{C} &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)a(\tau_1)b(\tau_2+\tau_1)a^2(\tau_2+\tau_1)d\tau_1\right)d\tau_2\\
\tilde{D} &= \int_0^{-\infty} e^{i\omega\tau_2}\left(\int_{-\infty}^{+\infty} e^{2i\omega\tau_1}G(\tau_1)a(\tau_1)b(\tau_2+\tau_1)a^3(\tau_2+\tau_1)d\tau_1\right)d\tau_2.
\end{aligned} \tag{4.12}
$$

We start with $\tilde{A}$. We regard $\tau_2 \neq 0$ in the inner integral as a fixed parameter and τ_1 as a complex variable. For a continuous curve ℓ in the complex z-plane, we let

$$
I_\ell = \int_\ell e^{2i\omega z}G(z)b(t+z)a^2(t+z)dz
$$

where $t \neq 0$ is a real parameter. Let

$$
\begin{aligned}
\ell_1 =& \{z = t_1 + is_1,\ \ t_1 \in (-\infty,+\infty),\ \ s_1 = 0\}\\
\ell_2 =& \{z = t_1 + is_1,\ \ t_1 \in (-\infty,+\infty),\ \ s_1 = \left(3\pi/2-\omega^{-1}\right)i\}.
\end{aligned} \tag{4.13}
$$

We have

$$
I_{\ell_1} = I_{\ell_2} + 2\pi i\ Res(G(z)b(t+z)a^2(t+z))\Big|_{z=-t+\pi i/2}.
$$

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Lemma 4.2. *There exists a constant K independent of ω so that*

$$\left|I_{\ell_2}\right| < K\omega^8 e^{-3\pi\omega} e^{-|t|}.$$

Proof. Let

$$\rho = 3\pi/2 - \omega^{-1}.$$

First, we have by the asymptotic behave of $G(t), b(t), a(t)$ as $|t| \to \infty$ that,

$$|G(t+i\rho)| \le K\omega^4 e^{-|t|}, \quad |b(t+i\rho)a^2(t+i\rho)| \le K\omega^4 e^{-3|t|}, \tag{4.14}$$

where we have a factor ω^4 in both estimate because the orders of the pole of the functions $G(t)$ and $b(t)a^2(t)$ at $t = i3\pi/2$ are four. We have

$$\begin{aligned} I_{\ell_2} &= e^{-3\pi\omega+2} \int_{-\infty}^{+\infty} e^{2i\omega t_1} G(t_1+i\rho) b(t+t_1+i\rho) a^2(t+t_1+i\rho) dt_1 \\ &= e^{-3\pi\omega+2} e^{-2i\omega t} \int_{-\infty}^{+\infty} e^{2i\omega t_1} G(t_1-t+i\rho) b(t_1+i\rho) a^2(t_1+i\rho) dt_1. \end{aligned}$$

We note that the distance from ℓ_2 to the pole located at $-t + i3\pi/2$ is $\ge \omega^{-1}$. This implies

$$\begin{aligned} \left|I_{\ell_2}\right| &\le e^{-3\pi\omega+2} \int_{-\infty}^{+\infty} \left|G(t_1-t+i\rho) b(t_1+i\rho) a^2(t_1+i\rho)\right| dt_1 \\ &\le K\omega^8 e^{-3\pi\omega} \int_{-\infty}^{+\infty} e^{-|t_1-t|} e^{-3|t_1|} dt_1 \\ &\le K\omega^8 e^{-3\pi\omega} \int_{-\infty}^{+\infty} e^{-(|t|-|t_1|)} e^{-3|t_1|} dt_1 \\ &\le K\omega^8 e^{-3\pi\omega} e^{-|t|}. \end{aligned}$$

Here for the second inequality we use (4.14) and for the third we use $|t_1 - t| \ge |t| - |t_1|$. □

This lemma is to imply

$$\tilde{A} = 2\pi i e^{-\pi\omega} I_{g,f} + O(\omega^8 e^{-3\pi\omega}) \tag{4.15}$$

where $I_{g,f}$ is as in (4.8) and g, f are as in (4.1). Estimates on $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ are similar. In conclusion,

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Lemma 4.3. *Let g, $\tilde{g}$, f, $\tilde{f}$ be as in (4.1). We have*

$$\begin{aligned}\tilde{A} &= 2\pi i e^{-\pi\omega} I_{g,f} + O(\omega^8 e^{-3\pi\omega}),\\ \tilde{B} &= 2\pi i e^{-\pi\omega} I_{g,\tilde{f}} + O(\omega^9 e^{-3\pi\omega}),\\ \tilde{C} &= 2\pi i e^{-\pi\omega} I_{\tilde{g},f} + O(\omega^9 e^{-3\pi\omega}),\\ \tilde{D} &= 2\pi i e^{-\pi\omega} I_{\tilde{g},\tilde{f}} + O(\omega^{10} e^{-3\pi\omega}).\end{aligned} \tag{4.16}$$

Proof of Theorem 2. We start by using Proposition 4.1 to calculate $I_{g,f}$, $I_{g,\tilde{f}}$, $I_{\tilde{g},f}$ and $I_{\tilde{g},\tilde{f}}$. We obtain

$$\begin{aligned}I_{g,f} &= \frac{4\sqrt{2}\omega}{3} g'(0) + O(1)\\ I_{g,\tilde{f}} &= \frac{16\omega^2}{3} g'(0) + O(\omega)\\ I_{\tilde{g},f} &= \frac{8\omega^2}{3} g'(0) + O(\omega)\\ I_{\tilde{g},\tilde{f}} &= \frac{8\sqrt{2}\omega^3}{3} g'(0) + O(\omega^2).\end{aligned} \tag{4.17}$$

Recall that

$$\gamma^* = -2\sqrt{2}\omega^{-1} + O(\omega^{-2}).$$

We have

$$I_{g,f} + \gamma^*(I_{g,\tilde{f}} + I_{\tilde{g},f}) + (\gamma^*)^2 I_{\tilde{g},\tilde{f}} = \frac{20\sqrt{2}g'(0)}{3}(\omega + O(1)).$$

We then use Lemma 3.3, Lemma 4.3 and Lemma 4.1(ii) to conclude

$$E_1(t_0) = F(\omega)\sin(2\omega t_0)$$

where

$$F(\omega) = -\frac{32}{3}\pi e^{-\pi\omega}\omega^2(1 + O(\omega^{-1})). \quad \square$$

Proof of Corollary 1.1. From the fact that $F(\omega)$ is an analytic function in ω, it follows that $F(\omega)$ can only be zero on a finite set of ω. The rest of this proposition follows from Proposition 1.1. We note that, for equation (1.6), the constant K in Proposition 1.1 is independent of ω. This is because we can apply the majorant method to (2.19) to bound $E_n(t_0, \omega)$ for all n to conclude $|E_n(t_0, \omega)| < K^n$, and in this argument $\cos\omega(t + t_0)$ are all set to 1 so this upper bound is independent of ω. See Proposition 4.1 in [1] for more details. $\square$

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