Homework #2

Problem #1

(Problem #3, p. 193): Derive the error formula (equation (17)) for Hermite interpolation (aka osculating interpolation) if $f(x)$ is sufficiently differentiable.

Hint: Proceed exactly as in the derivation of the interpolation error and define $F(z) = f(z) - H_{2n+1}(z) - \omega_n^2(z)S_n(x)$. After the first application of Rolle's theorem, however, $F'(z)$ will have $2n+2$ distinct zeros.

Problem #2

Let us investigate the connection between the continuous Fourier series

$$f(x) \approx Q_M(x) \equiv \sum_{k=-M}^{M-1} a_k e^{ikx}, \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx,$$

and the interpolating trigonometric series (with equidistant points)

$$\begin{aligned} f(x) &\approx S_n(x) \equiv \sum_{k=-n}^{n-1} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi}\sum_{j=0}^{2n-1} f(x_j)e^{-ikx_j}\Delta x, \\ \Delta x &= 2\pi/(2n), \qquad x_j = j\Delta x. \end{aligned} \tag{1}$$

1) Prove that the series $S(x)$ indeed yield interpolation, i.e. that $S(x_j) = f(x_j)$.
2) Show that coefficients c_k may be viewed as an approximation to a_k with the integrals approximated by the trapezoid rule.
3) Show that such a trapezoid rule with $2n$ points integrates e^{imx} exactly when integer m is lying within a certain range of values.

Hint: The trapezoid rule with $n+1$ points for a function $h(x)$ on the interval $[a,b]$ is

$$\int_a^b h(x)dx \approx \Delta x \sum_{j=1}^{n-1} h(a+j\Delta x) + \frac{\Delta x}{2}h(a) + \frac{\Delta x}{2}h(b), \qquad \Delta x = (b-a)/n.$$

4) Let us assume that $f(x)$ is periodic and has $r \geq 3$ continuous periodic derivatives. Then it can be represented by the series $Q(x) = \lim_{M\to\infty} Q_M(x)$. Show that if $f(x) = Q_M(x)$, $M \leq n$, then the coefficients c_k would coincide with a_k, i.e. that the trapezoid rule (1) is exact for such a function.
5) Therefore, all error in the coefficients c_k comes from the remainder of the series $Q(x)$, from the terms with $|k| > M$. Estimate the rate of decay of a_k and the rate of decay of $\max|Q_M(x) - f(x)|$.
6) Estimate the rate of decay of convergence of $S_n(x)$ to $f(x)$, show that $\max|S_n(x) - f(x)| \leq \frac{L}{n^{r-2}}$, as $n \to \infty$. (This estimate is not sharp, but should be relatively easy to prove).

Problem #3

Chebyshev polynomials can be obtained by the Gram-Schmidt orthogonalization of monomials x^k, $k = 0, 1, 2, ..$, over the interval $[-1, 1]$ with a weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ (the so obtained polynomials differ from Chebyshev polynomials by constant factors, since Chebyshev polynomials are not normalized to have 2-norm equal to one).

1) Show that by the change of variables $x = \cos\theta$ this process can be reformulated as orthogonalization of the sequence $\cos^k\theta$, $k = 0, 1, ...$ on the interval $[0, \pi]$ with a constant weight.

2) Prove that $\cos^n\theta = \sum_{k=0}^{n} a_k \cos k\theta$ for some a_k. (Use Euler's formula?) Finding a_k is not needed.

3) Show that as the result of the GS process one obtains functions $c_k \cos k\theta$, $k = 0, 1, 2, ...$ Find c_k (make them positive).

4) If one goes back to the variable x, the functions we found are proportional to $\cos(k \arccos x)$. Prove that these functions are indeed polynomials.

5) Chebyshev polynomials are defined as $T_k(x) = 2^{1-k}\cos(k \arccos x)$, $k = 0, 1, 2, ...$ Prove the recurrence relation

$$T_{n+1}(x) = xT_n(x) - \frac{1}{4}T_{n-1}(x).$$

(Use trigonometry ?)