Homework #3

Problem #1

(a) Derive a formula for the first derivative of function $f(x)$ at x_0 that uses points x_{-2}, x_{-1}, x_0, x_1, x_2, where $x_j = x_0 + jh$.
(b) Derive a formula for the second derivative of function $f(x)$ at x_0 that uses same points as in part (a).
(c) Use Taylor series; find the error terms for these two formulas and determine their order of accuracy.

Problem #2

This is essentially Problem 1 on the page 294. Prove that the Taylor series method for finding the weights of the differentiation formula

$$f^{(k)}(a) \approx \sum_{j=1}^{m} \alpha_j f(x_j), \tag{1}$$

produces exactly the same weights α_j as the method based on finding weights by ensuring that such a formula is exact for polynomials of degree $\leq m-1$.

Hint: (1) show that both sets of weights produce the same results when applied to polynomials of degree $\leq m-1$. (2) Look at the problem from the linear algebra point of view: make vector $F = (f(x_1), ..., f(x_m))$, vector $A = (\alpha_1, ..., \alpha_m)$ and vector $B = (\beta_1, ..., \beta_m)$ where $\beta_1, ..., \beta_m$ is the set of weights produced by the method based on polynomials. Then both methods can be viewed as dot products, for example (1) becomes

$$f^{(k)}(a) \approx A \cdot F.$$

Then, notice that $A \cdot F = B \cdot F$ when values F come from $f(x)$ being a polynomial of degree $\leq m-1$. This should be enough to conclude $A = B$, but you need to provide the details.

Problem #3

The goal of this problem is to conduct what is called "a convergence study" of some finite difference formulas.

Let us consider function

$$f(x) = \exp[-2\cos(\pi(x+0.5))]$$

on the interval $I \equiv [-1, 1]$. For a given number N of nodes, define an equispaced grid with $x_0 = -1$ and $x_{N-1} = 1$. (Then, $h = 2/(N-1)$.) At all the points x_j, $j = 2, ..., N-2$, (we are excluding the first two and the last two), compute the exact derivative of $f(x)$, the approximation given by the formula from the problem 1(a), and the approximation given by the formula

$$f'(x_j) \approx \frac{f(x_{j+1}) - f(x_{j-1})}{2h}. \tag{2}$$

For each of the approximations compute $||f'_{approx} - f'_{exact}||_\infty$ over all the points x_j, $j = 2, ..., N-2$. Repeat this experiment with $N = 17, 33, 65, 129, 257$. Fill up the following table

$N-1)$	Error, eqn (2)	Ratio	Error, problem 1(a)	Ratio
16				
32				
64				
128				
256				

In the above table, "Error" means the approximation error in the infinity norm, "Ratio" is the ratio between the error on the previous line to the error on the current line. (Hopefully, this number is greater than 1, showing that there is some convergence).

Please comment on what the two "Ratio" columns tell us about the orders of the approximate formulas.