Homework #4

Problem #1

The goal of this problem is to conduct a convergence study for two quadrature rules.

Let us numerically approximate integral

$$I = \int_0^1 \cos(x)\exp(-\sin^2 x)dx.$$

The exact value of this integral can be found using a proper substitution and the Matlab function **erf(...).** For a given number N consider two quadrature rules. One of them is the composite trapezoid rule with N intervals, the other one is the Gauss quadrature with N nodes. The nodes and weights for the Gauss quadrature can be obtained by running the subroutine **lgwt(...)** that I will post on my web site. The interface is **[x,w]=lgwt(N,a,b),** where a and b define the integration interval $[a, b]$, N is the number of nodes and **x** and **w** are arrays containing the nodes and weights, respectively.

Please run each quadrature rule for $N = 3, 6, 12, 24$. For each of the approximations compute error as $|I_{approx} - I|$. Fill up the following table

N	Error, trapezoid	Ratio	Error, Gauss	Ratio
3				
6				
12				
24				

Please comment on how the behaviour of the two "Ratio" columns corresponds to the theoretical estimates of the accuracy of the two quadrature rules.

Problem #2

The goal of this problem is to conduct a convergence study for the same quadrature rules as in Problem 1, but with a different function and a **different interval** of integration.

Let us numerically approximate integral

$$I = \int_0^\pi \cos(x)\exp(-2\cos x)dx.$$

An accurate value of this integral is $I = -4.997133057057808$. Please use the same two quadrature rules as in the problem 1, for $N = 3, 6, 12, 24$. For each of the approximations compute error as $|I_{approx} - I|$. Fill up the following table

N	Error, trapezoid	Ratio	Error, Gauss	Ratio
3				
6				
12				
24				

Please comment on how the behaviour of the two "Ratio" columns corresponds to the theoretical estimates of the accuracy of the two quadrature rules. In particular, explain the improved performance of the trapezoid rule. By "explain" I mean refer to the theorems we studied (no need to prove them again), and explain why they are applicable in this case.

Problem #3

Consider numerical integration of a function $f(x)$ over the interval $[-h, h]$ using two nodes at the points $-h/\sqrt{3}$ and $h/\sqrt{3}$ respectively.
(**a**) Determine the quadrature weights that give the highest degree of precision.
(**b**) Find the degree of precision of this quadrature rule (with the weights you found).
(**c**) Assuming that the error in this rule can be expressed in the form $Af^{(m)}(\zeta)$ for some integer m, find m and A.

Problem #4

Consider numerical integration of a function $f(x)$ over the interval $[-2h, 2h]$ using four (not equispaced) nodes at the points $-2h$, $-h$, h, $2h$.
(**a**) Determine the quadrature weights that give the highest degree of precision.
(**b**) Find the degree of precision of this quadrature rule (with the weights you found).
(**c**) Assuming that the error in this rule can be expressed in the form $Af^{(m)}(\zeta)$ for some integer m, find m and A.
(**d**) Please compare the degree of precision of this rule with that of the rule from problem 3, and compare the number of nodes used by these two rules. Which rule appears to be more efficient, and why is this happening?

Hint: Quadrature weights are proportional to the length of the interval, so one can first use points -2, -1, 1, 2 on the interval $[-2, 2]$, and later scale them to $[-2h, 2h]$. Also, one can use symmetries to guess some relationships between weights. Since the weights are unique, you can use any method you want to find them. I highly recommend not to use computer-based systems, as they will not be available during the tests.