Tentative study guide for the final test

(I will remove the items we will have not covered in class)

General notions and definitions

- Quadrature rule, nodes, weights, degree of accuracy.
- Gauss quadrature
- Numerical differentiation, three ways to derive formulas: interpolation, Taylor series approach, method of undetermined coefficients
- Newton interpolating polynomial with equispaced points. Divided differences, forward differences, backward differences
- Stable and asymptoticaly stable solutions of differential equation
- Conditions of stability for the system $y' = Ay$
- Conditions of stability for higher order linear ODE with constant coefficients
- Difference operator for one step methods
- Local truncation error d_n and the order of an algorithm
- Local error l_n, relation to d_n
- Global convergence
- 0-stability, region of absolute stability, A-stability for a one step method
- The use of a homogeneous and of an inhomogeneous test equation (3.28) (Acsher/Petzold) for determining absolute stability and stiff decay properties of algorithms.
- Stiff equations and methods with a stiff decay property
- Rough problems
- Taylor methods
- General formulation of Runge-Kutta methods: formulation withs Y_i and with K_i
- Implicit Runge-Kutta methods based on collocation. (Gauss, Lobatto, and Radau nmethods).
- Applications of pairs of methods (such as Fehlberg, Dormand-Prince, etc)
- Step doubling
- Multistep linear (MSL) methods: Stable and asymptotically stable difference equations, root condition.
- Difference operator and 0-stability for a multistep method.
- All types of stability for MSL methods (stable, asymptotically stable, weakly/strongly stable)

Things to derive or to prove

- Prove that Gaussian quadrature with n nodes has degree of precision $2n-1$. (This is Thm 1 p. 328 in one direction (Isaacson/Keller); see also Embree's lecture notes, p. 134-136. You can use (without proving it) the fact that the zeros of orthogonal polynomials are all single and real, and lie within the interval of orthogonality).
- Thm 2, p.328.

——————— Taylor methods (Ascher & Petzold) ———————

- Thm 1.1 p.9, (no proof)
- Thm 3.1 (prove it)
- Derive implicit trapezoidal method (p.60), check for order, absolute stability, stiff decay.
- Derive implicit and explicit midpoint methods, check order, absolute stability; show that these methods have (or do not — whichever is correct) the stiff decay property.
- Analyze the explicit trapezoid method (p.78)
- Know what methods (both RK and MLS) and what families are A-stable, what methods/families have the stiff decay properties, and which methods do not...

——————— One step methods (Ascher & Petzold) ———————

- General formulation of Runge-Kutta methods
- Derive the formula for the regions of absolute stability for RK methods (Equation (4.19) and the one above it).
- Derive necessary order conditions for Runge-Kutta methods, make conclusion that p cannot exceed s.
- Explain the use of pairs of methods (such as Fehlberg or Dormand-Prince) and derive an expression for the desired step size to achieve a given local error.
- Step doubling (how and what for)
- Define collocaation equation(s) and show that it leads to a RK method. Be able to design lower order methods of these families by hand, given collocation points.
- Know properties of these families of methods as compared to each other (i.e. which one is more accurate, more stable, has or does not have stiff decay etc, without proof).

——————— Multu-Step Linear methods (MSL) methods (Ascher & Petzold) ———————

- Adams-Moulton, Adams-Bashforth, and BDF. Know basic idea, be able to derive lower-order methods of these families.
- Derive the formula for the BDF method, p. 129.
- Characteristic polynomials of an MSL method
- Derive order conditions (Section 5.2.1)
- Prove that for a MSL method 0-stability and consistency imply convergence (p. 137).
- Derive general solution to a homgeneous difference equation (in the simplest case of distinct roots).

- Root conditions for stability of an MSL method (Section 5.2.2). Explain how this condition is related to stability of the difference equations.
- Predictor/corrector methods, different versions: describe the idea and the use of these methods (p.144).
- Examples 5.4, 5.5, 5.6, 5.7, 5.8, 5.9.

General notions and skills (some overlap with the topics above)

- Consistency of the differential equation. Know the definitions, Know how to check it using Taylor series and the formulas for the MSL case.
- 0-stability. Know the definitions both for the single step methods and multistep methods. Know how to check 0-stability for MSL's (root condition) and know the definitions of strong stability and weak stability. Be clear on how consistency plus 0-stability imply convergence.
- Regions of absolute stability. Know how to check if a value of $z = h\lambda$ is in the region of absolute stability for any of the methods.
- Stiffness: What is it, how do you recognize it. Be able to give examples.
- Be ready to choose an appropriate method for solving an IVP; justify your choice based on stiffness, accuracy, and smoothness, etc.