Homework #2

Problem #1

Consider set of functions $q^{(k)}(\theta) = \frac{1}{\sqrt{M}} e^{ik\theta}$, $k = 0, ..., M-1$ on the interval $[0, 2\pi]$ (here $i = \sqrt{-1}$). Let us discretize $q^{(k)}(\theta)$ by computing their values at M equidistant points $\theta_j = j\Delta\theta$, with step $\Delta\theta = 2\pi/M$. Then the points are

$$\theta_j = \frac{2\pi j}{M}, \quad j = 0, 1, 2, .., M-1,$$

and each function $q^{(k)}(\theta)$ yields a column-vector $q^{(k)}$:

$$q^{(k)} = \frac{1}{\sqrt{M}} \begin{pmatrix} \exp(0) \\ \exp\left(k\frac{2\pi i}{M}\right) \\ \exp\left(2k\frac{2\pi i}{M}\right) \\ \exp\left(3k\frac{2\pi i}{M}\right) \\ \ldots \\ \exp\left((M-1)k\frac{2\pi i}{M}\right) \end{pmatrix}. \tag{1}$$

In other words, the j^{th} coordinate of vector $q^{(k)}$ is

$$q_j^{(k)} = \frac{1}{\sqrt{M}} \exp\left(jk\frac{2\pi i}{M}\right).$$

(1) Prove that for any $k = 0, ..., M-1$,

$$||q^{(k)}||^2 =< q^{(k)}, q^{(k)} >= 1.$$

(2) Suppose $\alpha \in \mathbb{C}$ is an M^{th} root of 1, i.e. $\alpha^M = 1$ and $\alpha \neq 1$. Then

$$1 + \alpha + ... + \alpha^{M-1} = 0.$$

(3) Prove tha vectors $q^{(k)}$, $k = 0, ..., M-1$ form an orthonormal set in $\mathbb{C}^M$.
Hint: $\exp((k-l)\frac{2\pi i}{M})$ is an M^{th} root of 1.

Problem #2

Let us investigate the connection between the continuous Fourier series

$$f(x) \approx Q_M(x) \equiv \sum_{k=-M}^{M-1} a_k e^{ikx}, \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx,$$

and the interpolating trigonometric series (with equidistant points)

$$\begin{aligned} f(x) &\approx S_n(x) \equiv \sum_{k=-n}^{n-1} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi}\sum_{j=0}^{2n-1} f(x_j)e^{-ikx_j}\Delta x, \\ \Delta x &= 2\pi/(2n), \qquad x_j = j\Delta x. \end{aligned} \tag{2}$$

(1) Prove that the series $S(x)$ indeed yield interpolation, i.e. that $S(x_j) = f(x_j)$.
(2) Show that coefficients c_k may be viewed as an approximation to a_k with the integrals approximated by the trapezoid rule.
(3) Show that such a trapezoid rule with $2n$ points integrates e^{imx} exactly when integer m is lying within a certain range of values.

Hint: The trapezoid rule with $n+1$ points for a function $h(x)$ on the interval $[a, b]$ is

$$\int_a^b h(x)dx \approx \Delta x \sum_{j=1}^{n-1} h(a + j\Delta x) + \frac{\Delta x}{2} h(a) + \frac{\Delta x}{2} h(b), \qquad \Delta x = (b-a)/n.$$

(4) Let us assume that $f(x)$ is periodic and has $r \geq 3$ continuous periodic derivatives. Then it can be represented by the series $Q(x) = \lim_{M\to\infty} Q_M(x)$. Show that if $f(x) = Q_M(x)$, $M \leq n$, then the coefficients c_k would coincide with a_k, i.e. that the trapezoid rule (2) is exact for such a function.
(5) Therefore, all error in the coefficients c_k comes from the remainder of the series $Q(x)$, from the terms with $|k| > M$. Estimate the rate of decay of a_k and the rate of decay of $\max |Q_M(x) - f(x)|$.
(6) Estimate the rate of decay of convergence of $S_n(x)$ to $f(x)$, show that $\max |S_n(x) - f(x)| \leq \frac{L}{n^{r-2}}$, as $n \to \infty$. (This estimate is not sharp, but should be relatively easy to prove).

Problem #3

(1) Suppose Chebyshev polynomials are defined as $T_k(x) = 2^{1-k} \cos(k \arccos x)$, $k = 0, 1, 2, ...$ Prove the recurrence relation

$$T_{n+1}(x) = xT_n(x) - \frac{1}{4} T_{n-1}(x).$$

Hint: Trigonometry ?
(2) Prove that functions $T_k(x)$ are indeed polynomials for any k.