Math 129 (Fall '05) - Exam 1 Solutions - Kennedy

1. Let $u = \sqrt{x}$. So $du = \frac{1}{2}x^{-1/2}$. This substitution gives

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = 2\int \sin u \, du = -2\cos(u) + C = -2\cos\sqrt{x} + C \qquad (1)$$

2. Let $u = \ln x$ and $v' = x^2$. Then $u' = 1/x$ and $v = x^3/3$. So integration by parts give

$$\int_1^2 x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int \frac{1}{x}x^3 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \qquad (2)$$

3. First complete the square $x^2 + 2x + 4 = (x+1)^2 + 3$. So we let $u = x + 1$. So $du = dx$ and so

$$\int \frac{1}{x^2 + 2x + 4} dx = \int \frac{1}{(x+1)^2 + 3} dx = \int \frac{1}{u^2 + 3} du \qquad (3)$$

From the tables ($a = \sqrt{3}$), this is

$$\frac{1}{\sqrt{3}} \tan^{-1}(\frac{u}{\sqrt{3}}) + C = \frac{1}{\sqrt{3}} \tan^{-1}(\frac{x+1}{\sqrt{3}}) + C \qquad (4)$$

4. Note that $n = 3$.

$$LEFT = 2(2.1 + 3.1 + 4.1) = 18.6 \qquad (5)$$
$$RIGHT = 2(3.1 + 4.1 + 2.2) = 18.8 \qquad (6)$$
$$TRAP = (LEFT + RIGHT)/2 = 18.7 \qquad (7)$$
$$MID = 2(2.5 + 3.9 + 3.6) = 20.0 \qquad (8)$$
$$SIMPSON = \frac{2MID + TRAP}{3} = 19.57 \qquad (9)$$
$$(10)$$

5. Factor the denominator : $x^3 + x = x(x^2 + 1)$. So the partial fraction decomposition is

$$\frac{x - 1}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \qquad (11)$$

A common denominator then gives

$$x - 1 = A(x^2 + 1) + (Bx + C)x \tag{12}$$

So

$$x - 1 = (A + B)x^2 + Cx + A \tag{13}$$

which gives the equation $A + B = 0$, $C = 1$ and $A = -1$. So $B = 1$. Thus

$$\begin{aligned}
\int \frac{x-1}{x^3+x}\,dx \qquad &= \textstyle\int \frac{-1}{x}\,dx + \int \frac{x+1}{x^2+1}\,dx && (14)\\
&= -\ln x + \textstyle\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx && (15)\\
&= -\ln x + \tfrac{1}{2}\ln(x^2+1) + \tan^1 x + C && (16)
\end{aligned}$$

7. (a)

$$\begin{aligned}
LEFT \quad &= 1.4173, \quad RIGHT \quad = 1.6149 && (17)\\
MID \quad &= 1.5397, \quad TRAP \quad = 1.5161 && (18)\\
& && (19)
\end{aligned}$$

(b) From the graph we see that the function is increasing and concave down on the interval $[0, 2]$. The fact that it is increasing implies $LEFT$ is an underestimate and $RIGHT$ is an overestimate. The fact that it is concave down implies that $TRAP$ is an underestimate and MID is an overestimate.

8. First work out the indefinite integral. You can use the table or do it by parts. Let $u = x$, $v' = e^{-x}$. So $du = dx$, $v = -e^{-x}$. Thus

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x} = -xe^{-x} - e^{-x} + C \tag{20}$$

Thus

$$\int_0^b xe^{-x}dx = -be^{-b} - e^{-b} + 0e^{-0} + e^{-0} = -be^{-b} - e^{-b} + 1 \tag{21}$$

As $b \to \infty$, e^{-b} goes to 0. Since "exponentials beat polynomial", be^{-b} also goes to 0. This shows the improper integral converges and is equal to 1.

9.(a) You just have to run the calculator program for

$$I(1) = \int_0^1 e^{-\sqrt{x}}\, dx \tag{22}$$

With $n = 20$, SIMPSON is 0.528803.

(b) The substitution $u = t^2 x$ gives $du = t^2 dx$. (t is not a function of x.) So the indefinite integral becomes

$$\int e^{-t\sqrt{x}}\, dx = \frac{1}{t^2} \int e^{-\sqrt{u}}\, du \tag{23}$$

We must also figure out how the limits changes. $x = 0$ implies $u = 0$ and $x = t^{-2}$ implies $u = 1$. So we find that $I(t) = I(1)\, t^{-2}$ which is approximately $0.5288\, t^{-2}$

(c) If you did (b) then you can graph $I(t)$ right away. If you didn't do (b) you can still use your calculate to compute $I(t)$ for more values of t and sketch a graph this way.