Math 250A (Kennedy) - Final Exam - Fall '07

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (16 points) For the differential equation $y' = (y-x)^3$,
(a) Determine the region in the $x-y$ plane in which the solutions are increasing and the region in which they are decreasing.

Solution: It is increasing if $y' > 0$, i.e., if $(y-x)^3 > 0$. Since $(y-x)^3$ has the same sign as $y-x$, it is increasing when $y-x > 0$, i.e., $y > x$. Similarly, it is decreasing when $y < x$. So it is increasing above the line $y = x$ and decreasing below this line.

(b) Determine the region in the $x-y$ plane in which the solutions are concave up and the region in which they are concave down.

Solution: Differentiate the dif. eq. to get

$$y'' = 3(y-x)^2(y'-1) = 3(y-x)^2((y-x)^3 - 1)$$

Since $3(y-x)^2$ is always ≥ 0, the sign of y'' is the same as the sign of $(y-x)^3 - 1$. This is positive when $(y-x)^3 > 1$, which is equivalent to $y - x > 1$. It is negative when $y - x < 1$. So it is concave up above the line $y = x + 1$ and concave down below this line.

2. (16 points) Find the solution of $y' + xy - 2x = 0$ which satisfies $y(1) = 0$.
Solution: This is a linear dif. eq. The integrating factor is $\exp(\int x dx) = \exp(x^2/2)$. So

$$\big(\exp(x^2/2)y\big)' = \exp(x^2/2)y' + \exp(x^2/2)xy = \exp(x^2/2)2x$$

Integrating this,

$$\exp(x^2/2)y = \int \exp(x^2/2)2x\,dx = 2\exp(x^2/2) + C$$

(The integral can be done by the sub $u = x^2/2$; or just guess and check.) To make $y(1) = 0$, $C = -2\exp(1/2)$. So

$$y(x) = 2 - 2\exp(1/2)\exp(-x^2/2) = 2 - 2\exp((1 - x^2)/2)$$

3. (14 points) A rectangular shaped swimming pool 20 ft wide and 100 ft long has an increasing depth from the shallow end to the deep end. The depth is measured at the locations below.

distance from shallow end (ft)	0	25	50	75	100
depth (ft)	3	5	10	14	15

(a) Use right and left hand rules to estimate the volume (ft^3) of the pool.
Solution: Let x run the length of the pool (so $x = 0$ to 100). Slice perpendicular to the length of the pool. So slice is 20 ft wide, Δx thick and $d(x)$ high where $d(x)$ is the depth at x. So if we knew the function $d(x)$, the volume would be

$$\int_0^{100} d(x) 20 dx$$

For the approximations, $\Delta x = 25$, and we have

$$\begin{aligned} LEFT &= 20 \times 25(3 + 5 + 10 + 14) = 16,000\,ft^3 \\ RIGHT &= 20 \times 25(5 + 10 + 14 + 15) = 22,000\,ft^3 \end{aligned}$$

(b) Use the trapezoid rule to estimate the volume (ft^3) of the pool.
Solution:

$$TRAP = (LEFT + RIGHT)/2 = 19,000\,ft^3$$

4. (10 points) Suppose the number of bacteria in a petri dish grows exponentially with a doubling time of 7 hours. Initially there are $10,000$ bacteria. How many will there be in 24 hours?

Solution: Exponential growth means $N(t) = N(0)e^{kt}$. There are $10,000$ at $t = 0$ so $N(t) = 10,000e^{kt}$. Doubling time of 7 hours means $e^{7k} = 2$. So $k = \ln(2)/7$. Hence

$$N(24) = 10,000\exp(\ln(2)24/7) = 10,000 \times 2^{24/7} = 107,672$$

5. (22 points) Calculate the integrals below.

$$\int \frac{x+2}{x^2+4x}\,dx$$

Solution: You can do this by partial fractions, but it is a lot simpler to just do the sub $u = x^2 + 4x$. So $du = 2x + 4 = 2(x + 2)$. So the integral becomes

$$\int \frac{1}{2}\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+4x| + C$$

$$\int_1^e 2x\ln x\,dx$$

Solution: Integrate by parts:

$$\begin{aligned} u &= \ln x, \qquad v' = 2x \\ u' &= \frac{1}{x}, \qquad v = x^2 \end{aligned}$$

So

$$\int_1^e 2x\ln x\,dx = \left[x^2\ \ln x\right]_0^e - \int_1^e \frac{1}{x}\,x^2\,dx$$
$$= \left[x^2\ \ln x\right]_1^e - \left[\frac{1}{2}x^2\right]_1^e = e^2 ln(e) - 0 - e^2/2 + 1/2 = e^2/2 + 1/2$$

6. (16 points) Find all solutions of the differential equation.

$$y^2 \sin(x) = e^{\cos(x)} \frac{dy}{dx}$$

Solution: Use separation of variables.

$$\int \frac{dy}{y^2} = \int \sin x \exp(-\cos x) dx = \exp(-\cos x) + C$$

where the last integral comes form the sub $u = -\cos x$. So

$$-\frac{1}{y} = \exp(-\cos x) + C$$

So

$$y = \frac{-1}{\exp(-\cos x) + C}$$

7. (14 points) Consider the differential equation

$$\frac{dy}{dt} = 1 - cy^2$$

For all real c, determine the equilibrium solutions and whether each is stable, unstable, or neither (semi-stable). (Be sure to consider both positive and negative c.)

Solution: Equilibria are given by $cy^2 = 1$. So if $c \leq 0$, there are no equilibria.

If $c > 0$, the equilibria are $\pm 1/\sqrt{c}$.

Let $g(y) = 1 - cy^2$, so $g'(y) = -2cy$. So $g'(1/\sqrt{c}) = -2c/\sqrt{c} < 0$. So $1/\sqrt{c}$ is stable.

And $g'(-1/\sqrt{c}) = 2c/\sqrt{c} > 0$. So $-1/\sqrt{c}$ is unstable.

8. (16 points) The region bounded by $y = e^x$, the x-axis, and the vertical lines $x = 0$ and $x = 1$ is rotated about the horizontal line $y = 7$. Find the volume of the resulting solid.

Solution: Slice perpendicular to the x axis. The slice sweeps out a "washer" with outer radius 7 and inner radius $7 - e^x$. So the volume of the washer is

$$[\pi 7^2 - \pi(7 - e^x)^2]\Delta x$$

So the volume of the solid is

$$\begin{aligned}
\pi \int_0^1 [7^2 - (7 - e^x)^2]\,dx &= \pi \int_0^1 [14e^x - e^{2x}]\,dx \\
&= \pi \left[14e^x - \frac{1}{2}e^{2x}\right]_0^1 \\
&= \pi \left[14e - 14 - \frac{1}{2}(e^2 - 1)\right] \\
&= \pi \left[14e - \frac{27}{2} - \frac{e^2}{2}\right] \approx 65.538
\end{aligned}$$

9. (18 points) A water tower is in the shape of a cone with the point at the bottom. The top of the tower has a radius of 10 ft., and the cone is 30 ft high. The water level is 5 ft below the top of the tower. We empty the tank by pumping all the water to the top edge of the tank and then letting it fall over the side. Find the total work done. (The density of water is 62.4 lbs/ft^3.)

Solution: Let x be the distance above the bottom of the cone. Slice horizontally. A slice is a disc of radius r, so it has volume $\pi r^2 \Delta x$. By similar triangles,

$$\frac{r}{x} = \frac{10}{30}$$

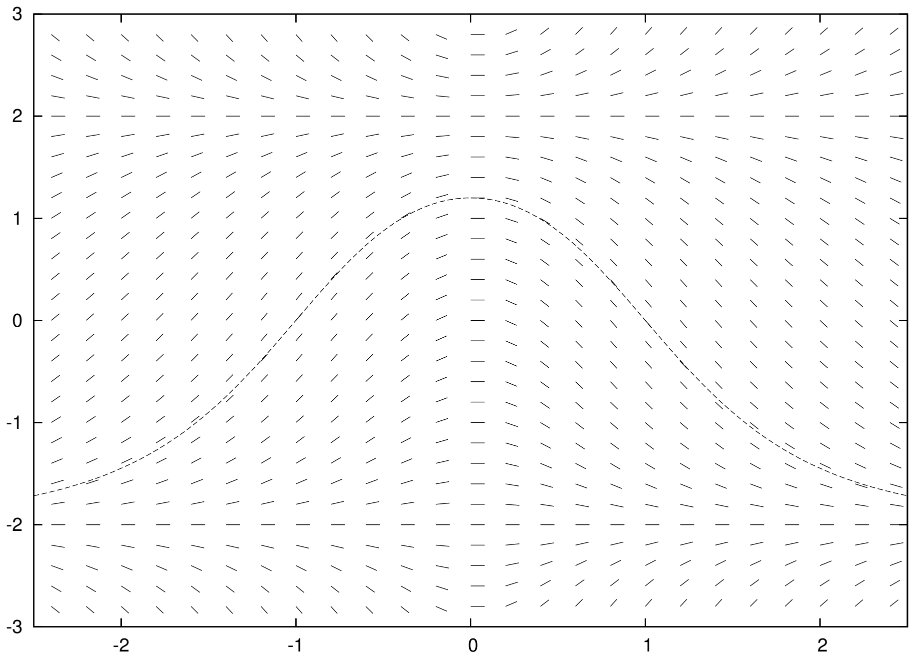
So $r = x/3$. The slice must be lifted a distance $30 - x$. So we have

$$\begin{aligned} \textit{slice volume} &= \pi \frac{x^2}{9} \Delta x \\ \textit{slice weight} &= 62.4\,\pi \frac{x^2}{9} \Delta x \\ \textit{slice work} &= 62.4\,\pi \frac{x^2}{9} (30 - x) \Delta x \end{aligned}$$

The slices go from $x = 0$ to $x = 25$, so the total work is

$$\int_0^{25} 62.4\,\pi \frac{x^2}{9} (30 - x)\, dx = \frac{62.4\,\pi}{9} \int_0^{25} x^2\,(30 - x)\, dx$$

10. (20 points) The slope field of a differential equation is shown below.



(a) Sketch the solution that goes through the point $(1, 0)$ on the slope field.

(b) The differential equation with this slope field is of the form

$$\frac{dy}{dx} = f(x)\, g(y)$$

Solution:

$$f(x) = \frac{x}{1 + x^2}, \qquad g(y) = y^2 - 4$$

(c) Mark each of the following statements as true or false. You need not explain your answer.

(i) For all solutions $y(x)$, $-y(x)$ is also solution. **FALSE**

(ii) For all solutions $y(x)$, $y(-x)$ is also solution. **TRUE**

(iii) For all solutions $y(x)$, $-y(-x)$ is also solution. **FALSE**

11. (18 points) We let $v(t)$ be the velocity of a skydiver. We assume the frictional force is proportional to the velocity, so the velocity obeys the differential equation

$$\frac{dv}{dt} = g - kv$$

where $g = 9.8\, m/sec^2$ is the acceleration due to gravity and k is a constant which will depend on properties of the skydiver. Our skydiver has a terminal velocity of $60\, m/sec$.

(a) Find k.

Solution: Terminal velocity is an equilibrium solution. So $g - kv = 0$. So

$$k = \frac{g}{v} = \frac{9.8}{60}\, sec^{-1} = 0.16333\, sec^{-1}$$

(b) If our skydiver starts with zero velocity, find her velocity after 5 seconds.

Solution: We need to solve the dif. eq. You can either separate variables or use the fact that it is a linear dif. eq. I'll do the former.

$$\int \frac{dv}{g - kv} = \int dt$$

$$\frac{-1}{k} \ln |g - kv| = t + C$$

For our solution v starts at 0 and $g - kv$ stays positive. So we can drop the absolute value. Using $v(0) = 0$, $C = -\ln(g)/k$. So

$$\frac{-1}{k} \ln(g - kv) = t - \ln(g)/k$$

$$\ln(g - kv) = -kt + \ln(g)$$

$$g - kv = \exp(-kt + \ln(g)) = g \exp(-kt)$$

$$v = \frac{g}{k}(1 - \exp(-kt)) = 60(1 - \exp(-kt))$$

So $v(5) = 60(1 - \exp(-5k)) \approx 33.48586\, m/sec$.

12. (20 points) Consider the differential equation

$$\frac{dy}{dx} = (y - x)^2 + 1$$

(a) Find a solution to the differential equation that is linear, i.e., of the form $y(x) = mx + b$ for some constants m and b.

Solution: If $y(x) = mx + b$, then $y' = m$. So to satisfy the dif. eq. we need

$$m = (mx + b - x)^2 + 1$$

So take $m = 1$ and $b = 0$, i.e., $y = x$. You can see this is a solution by looking at the slope field.

(b) One of the following two statements is true, the other is false. (c is a constant.) **Circle** the true statement. You need not explain your reasoning. Hints: Plot the slope field. Your answer should be consistent with your answer to (a).

(i) If $y(x)$ is a solution, then $\overline{y}(x) = y(x + c) + c$ is a solution.

(ii) If $y(x)$ is a solution, then $\overline{y}(x) = y(x - c) + c$ is a solution.

Solution: If we go c to the right and c up, the slope field does not change. $y(x - c)$ is $y(x)$ shifted by c to the right, so $y(x - c) + c$ is $y(x)$ shifted by c to the right and c up. So the second statement is true.

(c) $y(x) = x + 1/(1 - x)$ is a solution. Use (b) to find another solution. (Your answer to (a) doesn't count.)

Solution: By (b) $\overline{y}(x) = y(x - c) + c$ is also a solution.

$$\overline{y}(x) = x - c + \frac{1}{1 - (x - c)} + c = x + \frac{1}{1 + c - x}$$

This is a solution for all c.