Review: mins, maxs and inflection points (8/20)

$f(x)$ has a local min at $x = a$ if $f(a) \leq f(x)$ for all x in some neighborhood of a.

$f(x)$ over the interval I has a global min at $x = a$ if $f(a) \leq f(x)$ for all x in I.

$f(x)$ has a critical point at a if $f'(a) = 0$ (or is undefined).

If $f(x)$ has a local min or max at a, then a is a critical point.

Second derivative test: If $f'(a) = 0$ and $f''(a) > 0$, then f has a local min at $x = a$.

First derivative test: If $f'(a) = 0$, $f'(x) < 0$ for x just left of a and $f'(x) > 0$ for x just right of a, then f has a local min at $x = a$.

$f(x)$ has an inflection point at a if f changes concavity at a. At an inflection point $f''(a) = 0$.

Global min/max: If there is one, it will occur at a local min/max or an endpoint.

Functions with parameters: Everything (number of mins, maxs, inflection points, their locations ...) can depend on the parameter.

Review: Fund. thm. of calc.; dif. eqs. *(8/24)*

Difference between $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$.

Fundamental thm of calculus: If $f(x)$ is continuous on $[a,b]$ and $F'(x) = f(x)$ (F is an antiderivative of f), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Second fundamental thm of calculus: If $f(x)$ is continuous on $[a,b]$, then

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x)$$

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f(h(x))h'(x) - f(g(x))g'(x)$$

7.1 Integration by substitution (8/27)

Chain rule :

$$\frac{d}{dx} f(u(x)) = f'(u(x))\, u'(x)$$

Guess and check: Guess the antiderivative and check with chain rule. Fudge as needed.

Substitution: (assumes original integral is with respect to x)

1. Cleverly choose function $u(x)$.
2. Compute $u' = \frac{du}{dx}$. $du = u' dx$.
3. Express integrand in terms of u.
4. Do the u integral.
5. Express answer in terms of x.

7.2 Integration by parts (8/29)

Integration by parts:

$$\int u\, v'\, dx = u\, v - \int u'\, v\, dx$$

$$\int u\, dv = u\, v - \int v\, du$$

$$\int_a^b u\, v'\, dx = [u\, v]_a^b - \int_a^b u'\, v\, dx$$

Choosing u and v':

1. Must be able to compute v from v'.
2. Would like u' to be simpler than u.

Appendix B: Complex numbers (8/31)

Basic algebra: $i = \sqrt{-1}, \quad i^2 = -1$
General complex number is $a + bi$.
Addition/subtraction: obvious
Multiplication: remember $i^2 = -1$.
Division:
Complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$.

$$\frac{1}{a+bi} = \frac{1}{a+bi}\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Polar coordinates

$$re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$$

Trig functions:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Dif Eq 1.1: Simplest dif. eq. (9/3)

Definitions: First order differential equation:

$$\frac{dy}{dx} = g(x, y)$$

y is (unknown) function of x.
Initial condition : $y(x_0) = y_0$

Special case for chap one:

$$\frac{dy}{dx} = g(x)$$

Solving dif eq $\iff$ integrating $g(x)$.

General solution: $y(x) = \int g(x)dx + C$
Initial condition picks out a single solution.

Dif Eq 1.2: Graphical solutions (9/5)

Strategy: Dif eq gives you derivates of unknown $y(x)$. Use calculus to conclude things.

$$\frac{dy}{dx} = g(x)$$

Sign of $g(x)$ tells you increasing/decreasing.

$$\frac{d^2y}{dx^2} = g'(x)$$

Sign of $g'(x)$ tells you concavity

Symmetry:
If $g(x)$ is odd and $y(x)$ is a solution, then $y(-x)$ is a solution.
If $g(x)$ is even and $y(x)$ is a solution, then $-y(-x)$ is a solution.

Dif Eq 1.3: Slope Fields (9/10)

Tangent lines:

$$y(x + \Delta x) \approx y(x) + \frac{dy}{dx}(x)\Delta x$$

Slope field: If $y(x)$ solves dif. eq.

$$\frac{dy}{dx} = g(x, y)$$

then slope at $(x, y(x))$ is $g(x, y(x))$.

Drawing slope field: Draw a grid of points in x, y plane.
At a point (x, y) draw a small line segment with slope $g(x, y)$.

Drawing solution:
Start at the initial condition. Draw a curve so that its tangent lines follow the slope field.

Dif Eq 2.1: Autonomous equations (9/12)

Def: A first order dif. eq. is autonomous if it is of the form $\frac{dy}{dx} = g(y)$.

Separation of variables gives implicit equation for $y(x)$.

$$\frac{dy}{g(y)} = dx \quad \Rightarrow \quad \int \frac{dy}{g(y)} = \int dx = x + C$$

Slope field and calculus give you qualitative picture.

Slope field is constant in horizontal direction.
If $y(x)$ is a solution, then $y(x+C)$ is a solution.

An equilibrium solution is a solution of the form $y(x) = constant$.
Caution: Separation of variables can “lose” these solutions.

An equilibrium solution is stable if solutions that start near the equilibrium solution converge to the equilibrium solution as $x \to \infty$.

An equilibrium solution is unstable if solutions that start near the equilibrium solution move away from it as $x \to \infty$.

Dif Eq 2.2: Exponential growth/decay (9/?)

The simplest model for growth or decay is to assume the quantity grows at a rate proportional to the quantity:

$$\frac{dy}{dt} = ky$$

The general solution is

$$y(t) = y_0 e^{kt}$$

y_0 is the amount at time $t = 0$.

If $k > 0$ we have exponential growth.
The doubling time t_d is given by solving $y(t_d) = 2y_0$.

If $k < 0$ we have exponential decay.
The half-life t_h is given by solving $y(t_h) = \frac{1}{2}y_0$.

Calc 7.4: Partial fractions (9/14)

Algebraic method to integrate rational functions. $P(x)$ and $Q(x)$ are polynomials. Degree of P is lower than that of Q. (Long division if needed.)

$$\frac{P(x)}{Q(x)} = sum\ of\ terms$$

Factor $Q(x)$ into a product of linear terms and quadratic terms that have no real roots. For each factor:

Distinct linear factor $(x - c)$ Include term
$\frac{A}{x-c}$

Repeated linear factor $(x - c)^n$ Include terms
$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}$

Distinct quadratic factor $q(x)$ Include term
$\frac{Ax+B}{q(x)}$

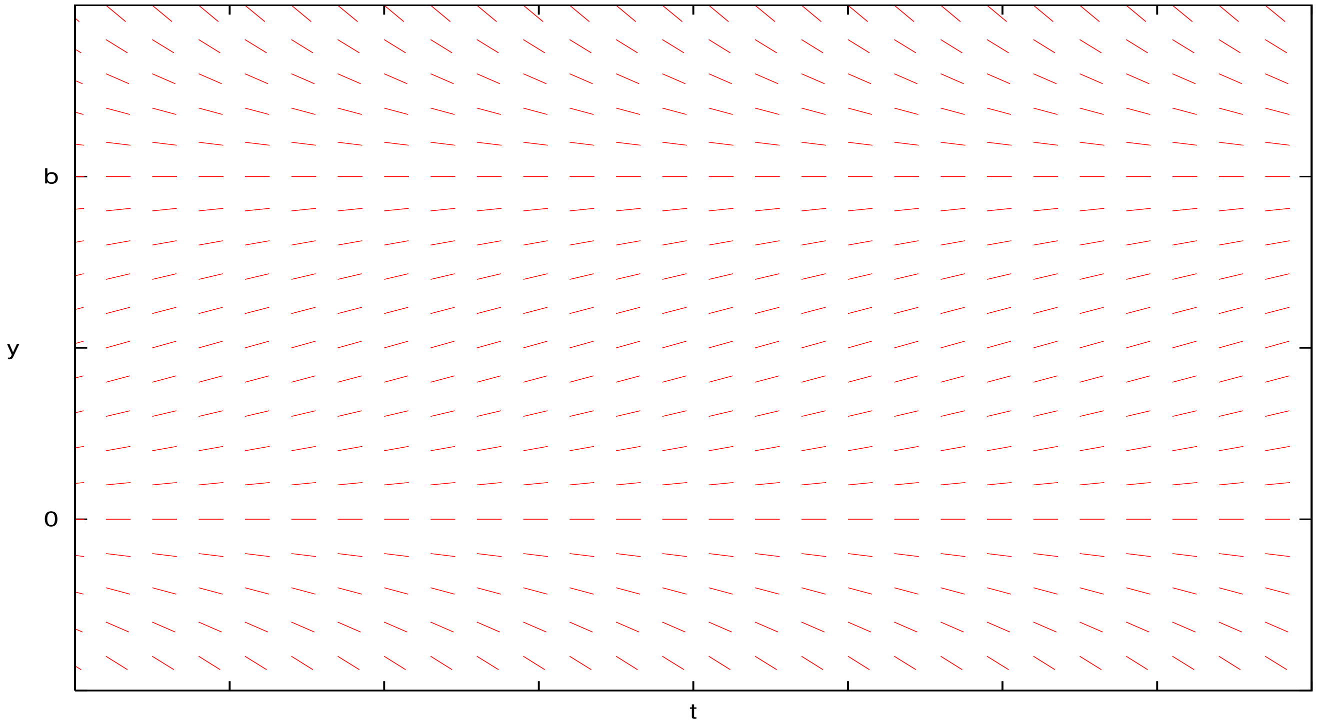
Repeated quadratic factor $q^n(x)$ Include terms
$\frac{A_1x+B_1}{q(x)} + \frac{A_2x+B_2}{q^2(x)} + \cdots + \frac{A_nx+B_n}{q^n(x)}$

Dif Eq. 2.3: Logistic Equation (9/24)

Model for population growth. t is time, $y(t)$ is population

$$\frac{dy}{dt} = ay(b-y)$$

For small y, dy/dt proportional to y.
Environment can sustain maximum population of $y = b$.
$y = 0$ is unstable equilibrium.
$y = b$ is stable equilibrium.



Dif Eq. 2.4: Existence/Uniqueness (9/26)

Big questions:

Local existence
Global existence
Uniqueness

Existence-Uniqueness Theorem For differential equation

$$\frac{dy}{dx} = g(x, y)$$

suppose $g(x, y)$ and $\frac{\partial g}{\partial y}$ are defined and continuous in a rectangle which has (x_0, y_0) in its interior. Then there exists a solution through (x_0, y_0) which is defined for x in an interval with x_0 in its interior. There is no other solution through (x_0, y_0)

Caution : There are $g(x, y)$ for which solutions can intersect.

Dif Eq. 2.5: Phase lines (9/28)

Autonomous differential equation

$$\frac{dy}{dx} = g(y)$$

Phase line: Line is for y.
Find equilibria (zeroes of $g(y)$). Mark them on line.
Find sign of $g(y)$. Indicate on line with –>– for $g > 0$, –<– for $g < 0$
Determine stable, unstable equilibria.

Derivative test for stable/unstable:
Let c be equilibrium solution, so $g(c) = 0$.
If $g'(c) < 0$ then c is stable.
If $g'(c) > 0$ then c is unstable.
If $g'(c) = 0$ then who knows.

Monotonicity:
Non-equilibrium solutions are always increasing or always decreasing.

Dif Eq. 2.6: Bifurcation diagrams (10/1)

If the differential equation contains a parameter, the behavior can change qualitatively as the parameters changes.

In particular, number of equilibrium solutions and their stability can change.

Bifurcation diagram is a plot of all the equilibria as a function of the parameter. Also indicates their stability.

Dif Eq. 3.1: Graphical analysis : g(x,y)   (10/5)

Now consider the most general first order equation

$$\frac{dy}{dx} = g(x,y)$$

In general, no analytic solution.

Graphical tools:

Slope fields

Monotonicity: increasing vs. decreasing : sign of g(x,y)

Concavity: sign of $\frac{d^2y}{dx^2}$, remember $y = y(x)$.

Symmetries

Isoclines: especially m=0 isocline:

Isoclines are cuves $g(x,y) = m$. For $m = 0$ they are possible locations of local max or min of solution curves.

Dif Eq. 3.2: Symmetry, scaling (10/8)

Symmetry for $\frac{dy}{dx} = g(x, y)$.

If slope field is symmetric about y-axis, $g(-x, y) = -g(x, y)$, then the solutions are even functions.

If slope field is symmetric about x-axis, $g(x, -y) = -g(x, y)$, and $y(x)$ is a solution, then $\overline{y}(x) = -y(x)$ is another solution.

If slope field is symmetric about origin, $g(-x, -y) = g(x, y)$, and $y(x)$ is a solution then $\overline{y}(x) = -y(-x)$ is another solution. In particular the solution through the origin is an odd function.

Scaling: If $y(x)$ solves some differential equation, then for constants a and b, $\overline{y}(x) = ay(bx)$ solves a similar equation. You may be able to eliminate some parameters this way.

Calc 7.5,7.6: Numerical Integration (10/10)

To compute $\int_a^b f(x)\,dx$, let n be positive integer.
$x_0 < x_1 < \cdots < x_{n-1} < x_n$ are equally spaced with $a = x_0$, $b = x_n$.
So spacing is $\Delta x = (b-a)/n$.

Riemann Sums:
$LEFT(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x, \quad RIGHT(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x$

Midpoint rule: $MID(n) = \sum_{i=1}^{n} f((x_{i-1}+x_i)/2)\,\Delta x,$

Trapezoid rule: $TRAP(n) = \sum_{i=1}^{n} \frac{1}{2}[f(x_{i-1}) + f(x_i)]\,\Delta x$

Simpson's rule: $SIMPSON(n) = \frac{2}{3}MID(n) + \frac{1}{3}TRAP(n)$

Error of method is $\left|\int_a^b f(x)\,dx - approximation\right|$

Order of method: How fast error $\to 0$ as $N \to \infty$.
$LEFT, RIGHT$ are first order. Error $\to 0$ as $1/N$.
$MID, TRAP$ are second order. Error $\to 0$ as $1/N^2$.
$SIMPSON$ is fourth order. Error $\to 0$ as $1/N^4$.

Calc 7.5,7.6: Numerical $\int$ Cont. (10/15)

Over/under estimate:

If f is increasing on $[a,b]$ then $LEFT \le \int_a^b f(x)\,dx \le RIGHT$

If f is decreasing on $[a,b]$ then $RIGHT \le \int_a^b f(x)\,dx \le LEFT$

If f is concave up on $[a,b]$ then $MID \le \int_a^b f(x)\,dx \le TRAP$

If f is concave down on $[a,b]$ then $TRAP \le \int_a^b f(x)\,dx \le MID$

Extrapolation: If $error \approx c/N^p$, then by evaluating approximation at N and $2N$, we can get a better approximation:

$\int_a^b f(x)\,dx = APPROX(N) + \frac{c}{N^p}$

$\int_a^b f(x)\,dx = APPROX(2N) + \frac{c}{(2N)^p}$

Solve two equations for unknowns $\int_a^b f(x)\,dx$ and c. Result is a better approximation for $\int_a^b f(x)$.

Dif Eq 4.1: Separation of Variables (10/22)

Differential equation has the form

$\frac{dy}{dx} = f(y)g(x)$

Equilibria: First find all the zeroes (if any) of f. These y values are equilibrium solutions.

Solving it :

$$\int \frac{dy}{f(y)} = \int g(x)\, dx$$

Do both integrals. Then solve for y as a function of x.

Caution: put in the $+C$ at the right place.

Dif Eq 3.3: Numerical solutions (10/24)

Want to numerically approximate solution of
$\frac{dy}{dx} = g(x, y)$ through (x_0, y_0).
$x_n = x_0 + nh$. Step size is h.
y_n is approximation to $y(x_n)$.

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Euler: $y_n = y_{n-1} + g(x_{n-1}, y_{n-1})h$

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Modified Euler or Huen : $y(x_1) = y(x_0) + \frac{1}{2}h(m_0 + k_1)$
$m_0 = g(x_0, y_0)$
$k_1 = g(x_1, y_0 + m_0 h)$

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$k_1 = g(x_1, y_0 + m_0 h)$

Order of methods :
Error is difference between exact solution and approximation.
It typically goes as h^p.

Euler: $p = 1$
Modified Euler (Huen) $p = 2$
Fourth order Runge-Kutta $p = 4$

Dif Eq 3.4: Comparison theorem (10/26)

Comparison theorem Consider the two differential equations

$\frac{dy}{dx} = f(x,y)$

$\frac{dy}{dx} = g(x,y)$

with the same initial condition $y(x_0) = y_0$. Suppose $f(x,y) < g(x,y)$ for all $x > x_0$ and all y. Then the solution to the first differential equation is less than the solution to the second differential equation for $x > x_0$ as long as these solutions exist.

Application Suppose we have a differential equation $\frac{dy}{dx} = f(x,y)$ that we can't solve. Look for a simpler equation $\frac{dy}{dx} = g(x,y)$ with $f(x,y) < g(x,y)$ (or $f(x,y) > g(x,y)$) that we can solve.

Dif Eq 4.2: Homogeneous coefficients (10/29)

$g(x,y)$ is homogeneous of degree zero if
$g(cx,cy) = g(x,y)$ for all $c \neq 0$.
For such g there is a function of one variable, G, such that
$g(x,y) = G(y/x)$.

Solving $\frac{dy}{dx} = G(y/x)$.
Try the substitution $y(x) = xu(x)$.
Should get separable differential equation for u.

Linear solutions:
$y(x) = mx$ is a solution if $G(m) = m$.

Dif Eq 4.3: Dif eqs from data (10/31)

Numerical differentiation: For small h,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad \textit{Better}$$

Log-log plots: Suppose we have data points (x_i, y_i) and we think $y = cx^p$ but don't know p. Take the ln:

$$y = cx^p \quad \Rightarrow \quad ln(y) = ln(c) + p\, ln(x)$$

So if we plot $ln(y)$ as a function of $ln(x)$ we should see a straight line and the slope is p.

Dif Eq 4.4: Objects in motion (11/2)

Velocity, acceleration:

Let t be time, $x(t)$ position.

Then velocity is $v(t) = \frac{dx}{dt}$.

Acceleration is $a(t) = \frac{dv}{dt}$.

Newton's second law:

$F = ma$, F is force

Dif Eq 5.1: Solving first order linear DE (11/7)

Put equation in form $y' + p(x)y = q(x)$

Compute the integrating factor $e^{\int p(x)dx} = e^{\int p}$.

Multiply dif. eq. by the integrating factor:

$$e^{\int p}\, y' + e^{\int p}\, p(x)\, y = e^{\int p}\, q(x)$$

Recognize this as

$$\left[e^{\int p}\, y\right]' = e^{\int p}\, q(x)$$

Integrate:

$$e^{\int p}\, y = \int e^{\int p}\, q(x)\, dx$$

Steady states and transients:

Often the solution to a linear differential equation is a sum of two parts.

The part of the solution that goes to zero as $t \to \infty$ is called the transient part.

The remaining part of the solution that does not go to zero as $t \to \infty$ is called the steady state part.

Warning : Steady state is not the same as equilibrium. Equilibrium solutions are constant in time. Steady state solutions can be periodic.

Dif Eq 5.3: Bernoulli's Equation (11/19)

Consider differential equation of the form:

$$\frac{dy}{dx} + p(x)\,y = q(x)\,y^n$$

where n is any real number besides 1.

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Calc 8.1,8.2: Volumes,Geometry (11/26)

Riemann sums and slicing:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

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$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

1. Slice your problem up. (Introduce coordinates.)

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2. Write the quantity you want as a sum of the form $\sum_i f(x_i)\,\Delta x$

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3. Figure out the definite integral this converges to and compute it.

Arc length: For graph of $f(x), a \leq x \leq b$ $\quad length = \int_a^b \sqrt{1 + f'(x)^2}\,dx$

For parametric curve $(x(t), y(t)), a \leq t \leq b$

$length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Calc 8.4,8.5: Mass, work, pressure (11/28, 11/30)

Physics:

For constant density, $mass = density \cdot volume$
For constant force, $work = force \cdot distance$
For constant pressure, $force = pressure \cdot area$

If density, force or pressure is not constant, slice the problem up so that the quantity is constant within a slice. Then mass, force or pressure can be computed by an integral.

Units:

In Metric system mass is measured in grams, kilograms,.... Force is measured in Newtons, dynes. A Newton is kilogram-meter/sec^2. Dyne is gram-centimeter/sec^2. 10^5 dynes is a Newton. Work is measured in Joules, ergs. A Joule is a Newton - meter. Erg is a dyne-cm.

In English system force is measured in pounds.
1 Newton $\approx$ 0.225 pounds.
Mass is measured in slugs
On Earth, the weight of (gravitational force on) one kilogram is 9.8 N ≈ 2.2 pounds.