Extrapolation

$$\text{error} = \int_a^b f(x)\,dx - \text{approximation}$$

$$\text{error} \approx \frac{C}{N^p}$$

know p, C unknown

Look at $N = 10, 20$

approximation $= M(D)$

$$\int_a^b f(x)\,dx = M(D)(10) + \frac{C}{10^2}$$

$$\int_a^b f(x)\,dx = M(D)(20) + \frac{C}{(20)^2}$$

Two eqs, two unknowns C, $\int_a^b$

4 × (bottom eq.) − (top eq.):

$$4\int_a^b f(x)\,dx - \int_a^b f(x)\,dx =$$

$$4M(D)(20) - M(D)(0)$$

$$+ 4\frac{C}{400} - \frac{C}{100}$$

$$3\int_a^b f(x)\,dx = 4M(20) - M(10)$$

$$\int_a^b f(x)\,dx = \frac{4M(20) - M(10)}{3}$$

Test case

$$\int_1^2 (x - 2x^3)\ln x\,dx = \frac{9}{8} - 6\ln 2$$

$$\int_1^2 \left[(x - 2x^3)\ln x - \frac{9}{8} + 6\ln 2\right] dx$$

$$\frac{4M(20) - M(10)}{3} = 2.78 \times 10^{-5}$$

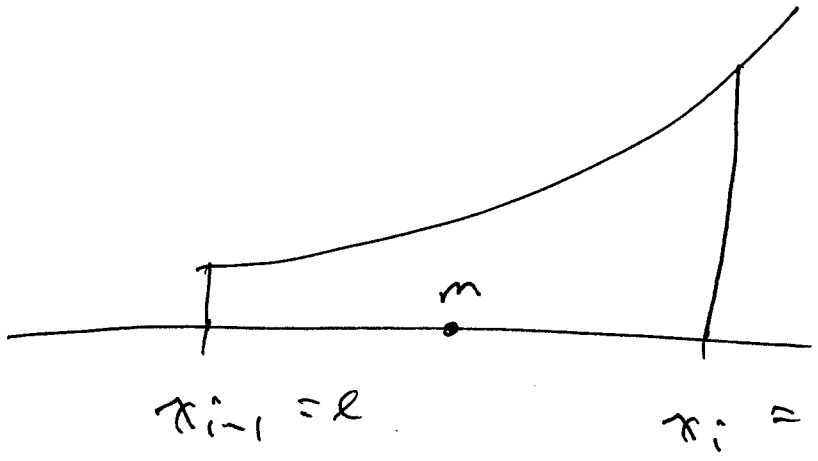
6.6×10^{-7}

error is a lot ~~too times~~ smaller

Approx $\doteq -3.033928852$

Simpson

Look at one interval



$$TRAP = \Delta x \ \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

$$MID = \Delta x \ f\left(\frac{x_{i-1} + x_i}{2}\right)$$

Both have the form

$$\Delta x \left[A f(x_{i-1}) + B f\left(\frac{x_{i-1} + x_i}{2}\right) + C f(x_i)\right]$$

TRAP $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$

MID $A = 0$, $B = 1$, $C = 0$

Is there a "better" choice of A, B, C?

Let $l = x_{i-1}$, $m = \frac{x_{i-1} + x_i}{2}$, $r = x_i$

$$\Delta x \left[A f(l) + B f(m) + C f(r)\right] = \bigstar$$

$f = 1$ $\qquad \int_\ell^r 1 \, dx = r - \ell$

$$\bigstar = (r-\ell)\left[A + B + C \right]$$

would like $\quad A + B + C = 1$

$f = x$ $\qquad \int_\ell^r x \, dx = \frac{1}{2} x^2 \Big|_\ell^r$

$$= \frac{1}{2}\left(r^2 - \ell^2\right) = \frac{1}{2}(r-\ell)(r+\ell)$$

$$\bigstar = (r-\ell)\left[A\ell + B\,\frac{\ell+r}{2} + Cr \right]$$

So $\bigstar$ is exactly right if

$$A\ell + B\,\frac{\ell+r}{2} + Cr = \frac{1}{2}(r+\ell)$$

$$\left(A + \frac{B}{2}\right)\ell + \left(\frac{B}{2} + C\right) r = \frac{1}{2} r + \frac{1}{2}\ell$$

would like $\quad A + \frac{B}{2} = \frac{1}{2}, \quad \frac{B}{2} + C = \frac{1}{2}$

$f = x^2$ ... $\qquad A + \frac{B}{4} = \frac{1}{3}$

$$\frac{B}{2} = \frac{1}{3}$$

$$C + \frac{B}{4} = \frac{1}{3}$$

Solution $\quad A = C = \frac{1}{6}, \quad B = \frac{2}{3}$

MID $\quad A = 0 \quad B = 1 \quad C = 0$

TRAP $\quad A = \frac{1}{2} \quad B = 0 \quad C = \frac{1}{2}$

Solution = $\frac{2}{3}$ MID + $\frac{1}{3}$ TRAP