4.1 Separation of Variables

Suppose

$$\frac{dy}{dx} = f(y)\, g(x)$$

$$\int \frac{dy}{f(y)} = \int g(x)\, dx$$

Do the integrals

Try to solve for y as function of x.

This method <u>misses</u> equilibrium soln.

Examples

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②

① $$\frac{dy}{dx} = \frac{x}{y} = \frac{1}{y}\,x$$

No equilibria

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C'$$

$$y^2 = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

$$y^2 - x^2 = C$$

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③

② $$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C'$$

$$y^2 = -x^2 + C$$

$$x^2 + y^2 = C$$

$$y = \pm\sqrt{C - x^2}$$

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(4)

(3)

$$\frac{dy}{dx} = \frac{y-2}{x}$$

$y = 2$ is a solution

$$\int \frac{dy}{y-2} = \int \frac{dx}{x}$$

$$\ln|y-2| = \ln|x| + C$$

$y > 2$, $x > 0$

$$\ln(y-2) = \ln x + C$$

$$y-2 = x e^{C} = Ax$$

$$y-2 = Ax$$

$$y = Ax + 2$$

$y < 2$, $x > 0$

$$\ln(2-y) = \ln x + C$$

$A > 0$

$$2-y = x e^{C}$$

$$y = -e^{C} x + 2 = -Ax + 2$$

Solute example

300 gal container
2/3 full of water
and 50 lbs of salt.

At $t = 0$, two valves opened.
One lets pure water in at
3 gals/min
other ~~way~~ one lets ~~a~~ salty
water out at 2 gals/min

When it is full, how much
salt is in it?

Let $V(t)$ = volume

$$\frac{dV}{dt} = 3 - 2 = 1$$

$$V(t) = t + 200$$

Let $y(t)$ = amount of salt

$$y(0) = 50 \text{ lbs}$$

$$\frac{dy}{dt} = -2\,\frac{\text{gals}}{\text{min}} \times \text{concentration}(t)$$

$$\text{Concentration} = \frac{y(t)}{V(t)} = \frac{y(t)}{t+200}$$

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$$\frac{dy}{dt} = -2\,\frac{y(t)}{t+200}, \quad y(0) = 50$$

$$\int \frac{dy}{y} = \int \frac{-2}{t+200}\,dt$$

$$\ln y = -2\ln(t+200) + C$$

At $t=0$

$$\ln 50 = -2\ln(200) + C$$

$$C = \ln 50 + 2\ln(200)$$

$$= \ln 50 + \ln\left((200)^2\right)$$

$$= \ln\left(50 \cdot (200)^2\right)$$

$$= \ln\left(2 \cdot 10^6\right)$$

$$y = e^{-2\ln(t+200)}\, e^{C}$$

$$y(t) = 2\cdot 10^6\,\frac{1}{(t+200)^2}$$

At $t=100$, $y(100) = \dfrac{2\times 10^6}{(300)^2} = 22\frac{2}{9}$ lbs