§4.2 Homogeneous Coefs

$$\frac{dy}{dx} = g(x,y) \leftarrow$$ can't solve in general

This section: special class of $g(x,y)$.

<u>Def</u> $g(x,y)$ is homogeneous of order zero if

$$g(cx, cy) = g(x,y) \quad \text{for } c \neq 0.$$

For such a g it can be written as

$$g(x,y) = G(y/x)$$

where G is a function of one variable

<u>Examples</u>

① $g(x,y) = \dfrac{xy}{x^2+y^2}$

$$g(cx, cy) = \frac{cx\,cy}{c^2x^2+c^2y^2} = \frac{xy}{x^2+y^2}$$

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$$\frac{xy}{x^2+y^2} = \frac{\frac{xy}{x^2}}{\frac{x^2}{x^2}+\frac{y^2}{x^2}} = \frac{y/x}{1+\left(\frac{y}{x}\right)^2}$$

$$= G\left(\frac{y}{x}\right)$$

$$G(u) = \frac{u}{1+u^2}$$

2 $\quad g(x,y) = e^{y/x} \cdot \frac{x+y}{y}$

3 $\quad g(x,y) = \frac{\sin x}{\sin y} \leftarrow$ NOT.

Solving them

Trick: let

$$u(x) = \frac{y(x)}{x}$$

i.e.,

$$\boxed{y(x) = x\, u(x)}$$

Apply sep. of vars to u dif. eq.

Example

$$\frac{dy}{dx} = \frac{y}{x+y}$$

$$y = x u$$

$$\frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = \frac{y}{x+y}$$

$$= \frac{xu}{x + xu} = \frac{u}{1+u}$$

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$$u + x\frac{du}{dx} = \frac{u}{1+u}$$

$$x\frac{du}{dx} = \frac{u}{1+u} - u$$

$$\frac{du}{dx} = \frac{1}{x}\left[\frac{u}{1+u} - u\right]$$

$$\frac{du}{\frac{u}{1+u} - u} = \frac{1}{x}\,dx$$

$$\cancel{\frac{u}{1+u} - u = \frac{u^2 - u(1+u)}{(1+u)u}}$$

$$\cancel{= \frac{-u}{(1+u)u}}$$

$$\frac{u - u(1+u)}{1+u} = \frac{-u^2}{1+u}$$

$$-\frac{(1+u)\,du}{u^2} = \frac{1}{x}\,dx$$

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$$\left(-\frac{1}{u^2} - \frac{1}{u}\right) du = \frac{1}{x}\, dx$$

$$\frac{1}{u} - \ln|u| = \ln|x| + C$$

$$e^{\frac{1}{u}}\, e^{-\ln|u|} = |x|\, e^{C}$$

$$e^{\frac{1}{u}}\, \frac{1}{|u|} = |x|\, e^{C}$$

stuck with implicit solution.

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$$\frac{dy}{dx} = G\left(\frac{y}{x}\right)$$

Equilibria ?

If $G(0) = 0$ then $y = 0$ is a solution

$$G\left(\frac{y}{x}\right) = \frac{y}{x}$$

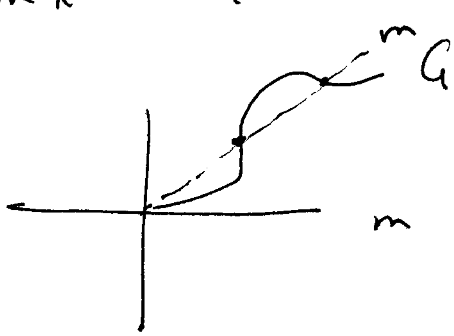
Only possible equilibrium is $y = 0$.

Is $y = mx$ a solution ?

$$\frac{dy}{dx} = m$$

Need $m = G\left(\frac{mx}{x}\right) = G(m)$

If m satisfies $m = G(m)$, then $y = mx$ is a solution.



Example

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$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

Equilibrium $\quad y = 0$

Linear ? $\quad G(u) = u^2$

$$G(m) = m \Rightarrow m^2 = m$$

$y = x$ is a solution

§4.3 modelling

Word problem

Newton's law of cooling/heating

My house is at 80.
Outside its 110.
My AC breaks.
Physicist says that the rate of change of temperature is proportional to the temperature difference between inside and outside.

rate = derivative

Solution: t = time
$T(t)$ = temperature

$$\frac{dT}{dt} \propto \text{temp dif.}$$

$\propto$ ← proprotional to

temp dif. = $T(t) - 110$

$$\frac{dT}{dt} = k\left[T(t) - 110\right], \quad k < 0$$

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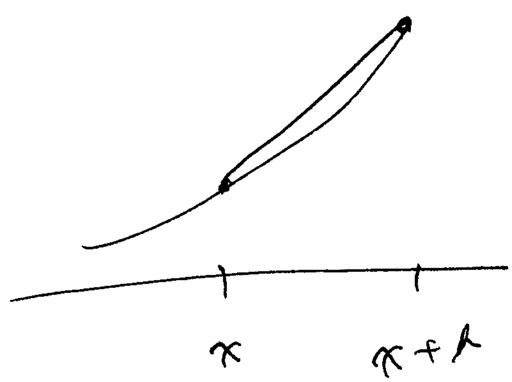
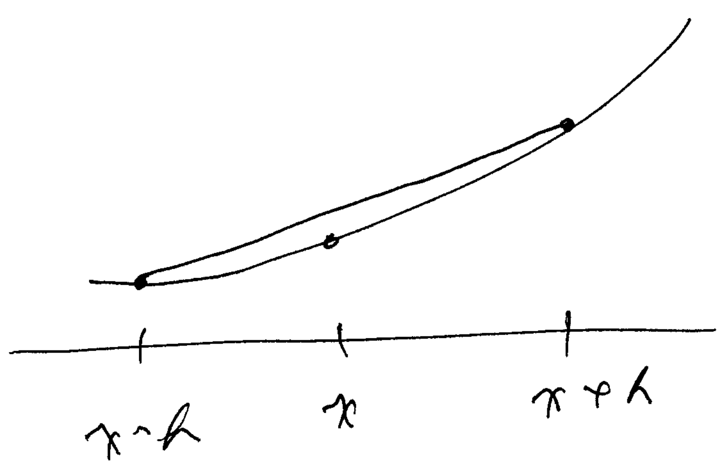
$$\frac{dT}{dt} = k\left[110 - T(t)\right], \quad k > 0$$

Numerical derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$= \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$



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