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Chap 1 cont

Uniqueness theorem

$x_0 \in (a,b)$

x_0 in (a,b)

$$\frac{dy}{dx} = g(x)$$

$g(x)$ is a continuous function on (a,b)

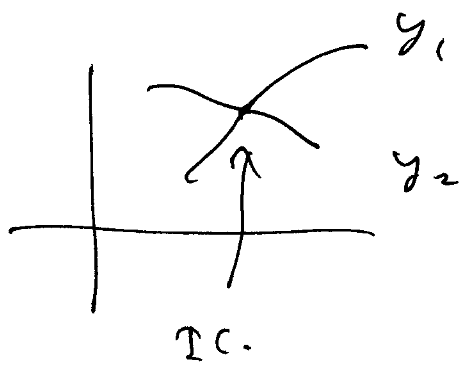
y_1, y_2 are solutions with

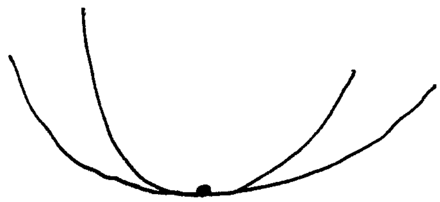
~~$y_1(x_1)$ =~~ $y_1(x_0) = y_0$

$y_2(x_0) = y_0$, $x_0 \in (a,b)$

Then $y_1(x) = y_2(x)$ on (a,b)

"Proof" slope field





Proof

$$\frac{d}{dx}\left[y_1(x) - y_2(x)\right] = g(x) - g(x)$$
$$= 0$$

$$y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0$$

$$h(x) = y_1(x) - y_2(x)$$

$$h'(x) = 0, \qquad h(x_0) = 0$$

$\hookrightarrow h(x) = \text{constant}$ $\qquad$ $\downarrow$ constant $= 0$

Chap 2 Autonomous Dif Eq.

$$\frac{dy}{dx} = g(y)$$

$$y = y(x)$$

$$\frac{dy}{dx} = \frac{1}{y}, \quad \frac{dy}{dx} = e^{y}$$

$$y = \int g(y)\, dx = ?$$

§ 2.1 ~~Def~~ Separation of Variables

Example $\dfrac{dy}{dx} = y^2$

Calculus : $\dfrac{dy}{dx} \geq 0 \Rightarrow y$ increasing

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y\,y^2$$
$$= 2y^3$$

$y > 0$ concave up

$y < 0$ concave down

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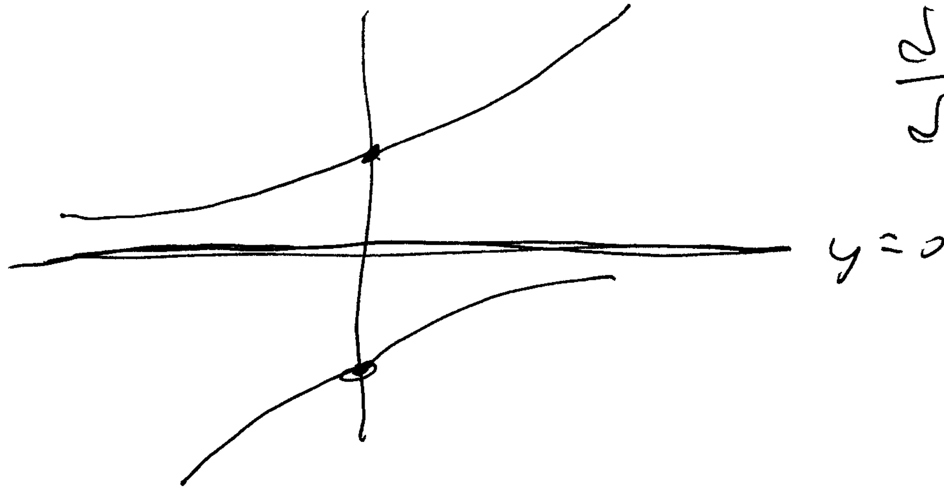
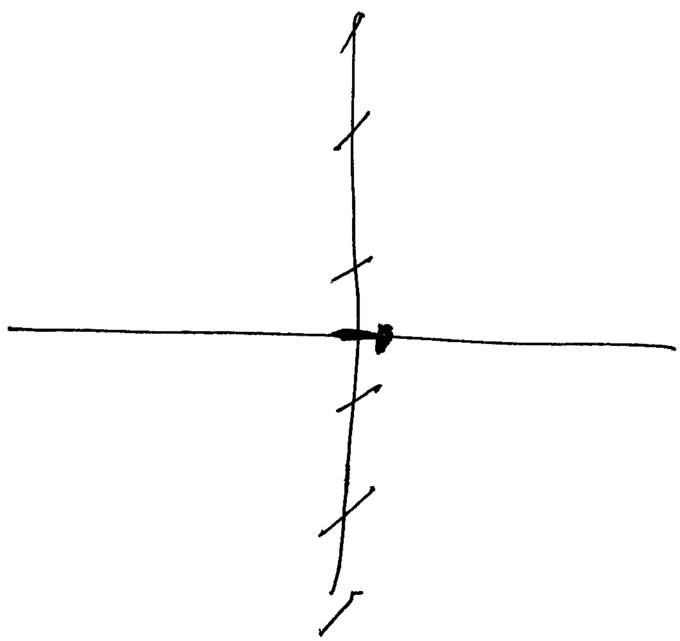
$$\frac{dy}{dx} = y^2$$



Are there constant solutions ?

$y = C$ Yes $y = 0$.

<u>slope field</u>



Slope is constant in horizontal direction. So $y(x)$ is a solution then $y(x+c)$ is a solution.

BUT $y(x) + C$ is usually not.

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$$\frac{dy}{dx} = y^2$$

$$\frac{dy}{y^2} = dx$$

$$\int \frac{dy}{y^2} = \int dx$$

$$c' \quad -y^{-1} = x + C \quad \leftarrow !$$

$$\frac{-1}{y} = x + C$$

$$y = \frac{-1}{x + C}$$

$$\frac{dy}{dx} = \frac{1}{(x+c)^2} = y^2 \qquad \text{works!}$$

$$C = 1 \qquad y = \frac{-1}{x+1} \qquad y(0) = -1$$

$$C = -1 \qquad y = \frac{-1}{x-1} \qquad y(0) = +1$$

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Example $\quad \frac{dy}{dx} = a y$

a parameter

$$\frac{dy}{y} = a\,dx$$

$$\int \frac{dy}{y} = \int a\,dx$$

$$\ln y = a x + c$$

$$y = e^{ax+c}$$

$$y = e^{c} e^{ax}$$

$$= A e^{ax} \qquad A > 0$$

$$\frac{dy}{dx} = A a e^{ax} = a y \quad \text{yeah!}$$

$$\frac{d}{dx}\Big(\underbrace{-e^{ax}}_{y}\Big) = -a e^{ax} = a y$$

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$$\ln|y| = ax + C$$

$$|y| = e^{ax+C}$$

$$= A\, e^{ax} \qquad A > 0$$

$y > 0$ $\qquad y = A e^{ax}$

$y < 0$ $\qquad -y = A e^{ax}$

$$y = -A\, e^{ax}$$

Together: $\quad y(x) = A\, e^{ax} \quad$ any A

$$|-2| = -(-2) = 2$$