Math 250A (Kennedy) - Exam 1 - Fall '07

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (14 points total) Find the following

$$\int \theta \cos\theta \, d\theta$$

by parts

$u = \theta$, $v' = \cos\theta$

$u' = 1$, $v = \sin\theta$

$= \theta \sin\theta - \int \sin\theta \, d\theta$

$= \theta \sin\theta + \cos\theta + C$

$$\frac{d}{dx} \int_{-2}^{\sqrt{x}} t \sinh(t^2) \, dt$$

Second fundamental theorem of calculus:

$= \sqrt{x} \sinh\left(\sqrt{x}^{\,2}\right) \frac{d}{dx}\sqrt{x}$

$= \sqrt{x} \sinh(|x|) \frac{1}{2} x^{-1/2}$

$= \frac{1}{2} \sinh(|x|)$

2. (21 points) Find the following

$$\int \frac{e^x}{(2+e^x)^3}\,dx$$

Sub $u = 2 + e^x$

$u' = e^x$

$$= \int \frac{du}{u^3} = -\tfrac{1}{2} u^{-2}$$

$$= -\tfrac{1}{2}(2+e^x)^{-2} + C$$

$$\int_0^{\pi/2} \sin\theta \cos^6\theta \, d\theta$$

sub $u = \cos\theta$

$u' = -\sin\theta$

$$\int \sin\theta \cos^6\theta \, d\theta = -\int u^6 \, du = -\tfrac{1}{7} u^7 + C$$

$$= -\tfrac{1}{7}\cos^7\theta + C$$

So $\int_0^{\pi/2} \sin\theta \cos^6\theta \, d\theta = -\tfrac{1}{7}\cos^7\theta \Big|_0^{\pi/2}$

$$= -\tfrac{1}{7}\left[\cos^7\left(\tfrac{\pi}{2}\right) - \cos^7(0)\right] = \boxed{\tfrac{1}{7}}$$

$$\int \frac{y^3}{\sqrt{y^2+1}}\,dy$$

Sub $u = y^2 + 1$

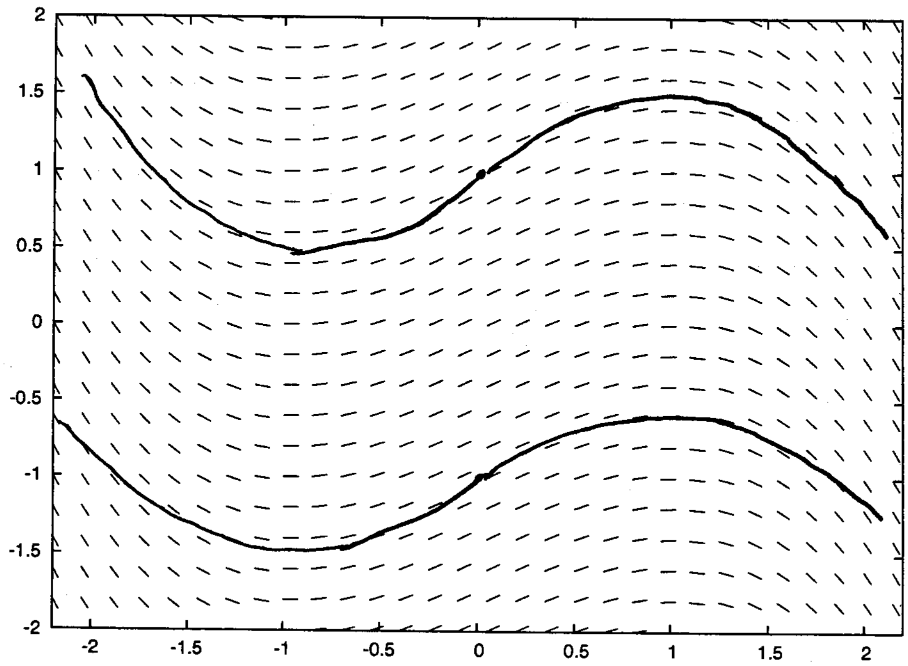
$u' = 2y \, dy$

$$= \int \frac{y^2 \tfrac{1}{2} du}{\sqrt{u}} = \int \frac{(u-1)\tfrac{1}{2} du}{\sqrt{u}}$$

$$= \tfrac{1}{2}\int \left(\sqrt{u} - u^{-1/2}\right) du = \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$

$$= \tfrac{1}{3}(y^2+1)^{3/2} - (y^2+1)^{1/2} + C$$

3. (22 points) The slope field for the differential equation $\frac{dy}{dx} = g(x)$ is :



(a) Sketch the two solutions that satisfy $y(0) = 1$ and $y(0) = -1$ on the slope field.

(b) Suppose $y(x)$ is a solution and c is an arbitrary constant. Which of the following is always a solution of the differential equation?

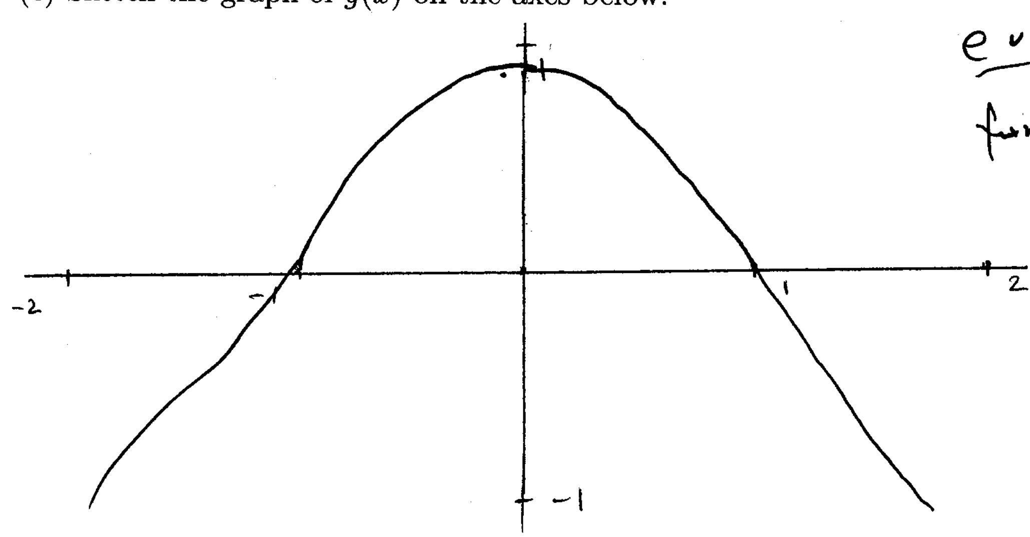
$y(x+c)$ Not always a solution

$y(x)+c$ Always a solution

$y(-x)$ Not always a solution

$-y(x)$ Always a solution

(c) Sketch the graph of $g(x)$ on the axes below.



even function

4. (12 points) Find the solution that passes through $(0, -1)$ for the differential equation

$$\frac{dy}{dx} = x\sin(x^2)$$

Sub $u = x^2$: $\quad \int x \sin(x^2)\,dx = \frac{1}{2}\int \sin u \, du$

$du = 2x\,dx \quad = \frac{-1}{2}\cos(x^2) + C$

So $y(x) = -\frac{1}{2}\cos(x^2) + C$

$y(0) = -1 \implies -1 = -\frac{1}{2}\cos(0) + C$

$\implies C = -\frac{1}{2}$

So $y(x) = -\frac{1}{2}\cos(x^2) - \frac{1}{2}$

5. (15 points) For the differential equation below, determine where the solutions are increasing, decreasing, concave up, concave down.

$$\frac{dx}{dt} = \frac{t}{t^2+1}$$

$\frac{t}{t^2+1} > 0$ if $\underline{t > 0}$ so increasing there

$\frac{t}{t^2+1} < 0$ if $\underline{t < 0}$ so decreasing there

$$\frac{d^2x}{dt^2} = \frac{(t^2+1) - t(2t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$$

$(t^2+1)^2$ always > 0.

If $\underline{-1 < t < 1}$ then $1-t^2 > 0$, so

concave up there.

If $\underline{t > 1}$ $\underline{\text{or}}$ $\underline{t < -1}$, ~~then~~ $1-t^2 < 0$,

so concave down there.

6. (16 points) Define the function $E(x)$ by

$$E(x) = \int_0^x e^{-t^2}\,dt$$

It satisfies the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

(a) Given that $E(1) = 0.746824\cdots$, find $y(1)$ where $y(x)$ is the solution of the above differential equation that satisfies $y(0) = 2$.

General solution is $E(x) + c = y(x)$
$E(0) = 0$. So to $y(0) = 2$, need
$0 + c = 2$, so $y(x) = E(x) + 2$.
So $y(1) = E(1) + 2 = 2.746824$

(b) Now suppose we want to find the solution of the differential equation and initial condition

$$\frac{dy}{dx} = xe^{-x^4}, \quad y(0) = 0$$

Find the value of this solution when $x = 1$.

Solution is $y(x) = \int_0^x u e^{-u^4}\,du$

So $y(1) = \int_0^1 u e^{-u^4}\,du$. sub $w = u^2$

$= \int_0^1 \frac{1}{2} e^{-w^2}\,dw = \frac{1}{2} E(1) = \frac{1}{2}(.746824)$

(c) For the differential equation and initial condition given in part (b), express the solution in terms of the function $E(x)$.

$y(x) = \int_0^x u e^{-u^4}\,du$. Sub $w = u^2$
$dw = 2u\,du$

$= \int_0^{x^2} \frac{1}{2} e^{-w^2}\,dw = \boxed{\frac{1}{2} E(x^2)}$

Can check using chain rule

$\frac{d}{dx} \frac{1}{2} E(x^2) = \frac{1}{2} E'(x^2)\, 2x$

$= \frac{1}{2} e^{-(x^2)^2}\, 2x = x e^{-x^4}$