Math 250A (Kennedy) - Exam 2 - Fall '07

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (16 points total) Consider the differential equation

$$\frac{dy}{dx} = 2y^{1/2}$$

(a) Find the solution that passes through $(0, 4)$.

$$\int \frac{dy}{2y^{1/2}} = \int dx$$

$$y^{1/2} = x + C$$

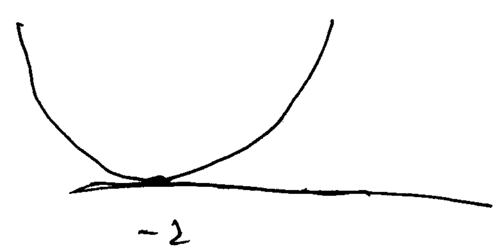
$$2 = 0 + C \Rightarrow C = 2$$

$$\boxed{y = (x+2)^2}$$

Note: $(x-2)^2$ is not a solution

(b) The function $y(x) = 0$ is a solution of this differential equation. Show that your solution in (a) intersects this solution and explain why this does not contradict the existence-uniqueness theorem.

Solution from (a) hits 0 at $x = -2$. So the two solutions intersect at $(-2, 0)$. But $2y^{1/2}$ is not defined for $y < 0$, so theorem does not apply to the initial condition $(-2, 0)$.



2. (20 points) Consider the differential equation $\frac{dy}{dx} = y(2-y)$.
(a) Find a so that the following is a solution.

$$y(x) = \frac{2}{1+e^{-ax}}$$

$$y'(x) = \frac{2a\, e^{-ax}}{(1+e^{-ax})^2}$$

$$y(2-y) = \frac{2}{1+e^{-ax}}\left(2 - \frac{2}{1+e^{-ax}}\right)$$

$$= \frac{2(2+2e^{-ax}-2)}{(1+e^{-ax})^2} = \frac{4e^{-ax}}{(1+e^{-ax})^2}$$

So need $2a = 4$, i.e., $\boxed{a=2}$

(b) Which of the following is a symmetry of the dif. eq.
(i) horizontal translation, i.e., if $y(x)$ is a solution then $y(x-c)$ is too.

YES

(ii) vertical translation, i.e., if $y(x)$ is a solution then $y(x)+c$ is too.

NO

(c) Find the solution through the point $(1,1)$ Hint: Note that the solution in part (a) passes through $(0,1)$.

Let $y(x) = \frac{2}{1+e^{-2x}}$

It is a solution and $y(0) = 1$.
By (b), $\bar{y}(x) = y(x-c)$ is also a solution.
Choose c so $\bar{y}(1) = 1$. $y(0) = 1$, so
$\bar{y}(1) = y(1-c)$. So let $c = 1$.

$$\bar{y}(x) = y(x-1) = \boxed{\frac{2}{1+e^{-2(x-1)}}}$$

3. (18 points) Consider the differential equation,

$$\frac{dy}{dt} = y(2 - e^{y/b})$$

The parameter b can be any nonzero number. Find the equilibrium solutions and determine if they are stable, unstable or semistable.

$$y\left(2 - e^{y/b}\right) = 0$$

$$\Rightarrow \boxed{y = 0} \quad \text{or} \quad e^{y/b} = 2$$

$$\text{i.e.,} \quad \boxed{y = b \ln 2}$$

$$g' = 2 - e^{y/b} - y e^{y/b} \frac{1}{b}$$

$$g'(0) = 2 - 1 - 0 = 1$$

So 0 is always unstable

$$g'(b \ln 2) = 2 - e^{\ln 2} - b \ln 2 \, e^{\ln 2} \frac{1}{b}$$

$$= -2 \ln 2 < 0$$

So $b \ln 2$ is always stable

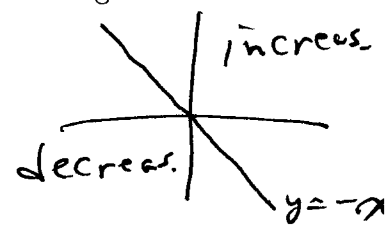
4. (16 points) For the differential equation

$$\frac{dy}{dx} = e^{x+y} - 1$$

(a) Where are the solution curves increasing and where are they decreasing?

$\frac{dy}{dx} > 0$ if $e^{x+y} > 1$ $\Leftrightarrow$ $x+y > 0$

$\frac{dy}{dx} < 0$ if $e^{x+y} < 1$ $\Leftrightarrow$ $x+y < 0$



(b) Where are the solution curves concave up and where are they concave down?

$$\frac{d^2y}{dx^2} = e^{x+y}\left(1+\frac{dy}{dx}\right) = e^{x+y}\left(1+e^{x+y}-1\right)$$

$$= e^{2(x+y)} > 0 \quad \text{always}$$

So always concave up

5. (12 points) For this question, you need not show any work. No partial credit will be given on this one. Let $y(x)$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{(y-x)^2} + 1$$

(a) Let $\overline{y}(x) = y(-x)$. Is $\overline{y}$ a solution?

NO

(b) Let $\overline{y}(x) = -y(x)$. Is $\overline{y}$ a solution?

NO

(c) Let $\overline{y}(x) = -y(-x)$. Is $\overline{y}$ a solution?

YES

6. (15 points) In the homework you solved the differential equation $y' = \frac{1}{2}(1 - y^2)$ with the initial condition $y(0) = 0$ and found, hopefully,

$$y(x) = \frac{e^x - 1}{e^x + 1}$$

Now consider the differential equation $f' = 4 - f^2$ with the initial condition $f(0) = 0$. Find the solution. Hint: There are two ways to do this. You can ignore the above formula for $y(x)$ and just solve the dif eq for f. Of you can use scaling.

Try $\quad f(x) = a\, y(bx)$

$$f'(x) = ab\, y'(bx)$$
$$= ab\, \tfrac{1}{2}\left(1 - y^2(bx)\right)$$
$$= \frac{ab}{2}\left(1 - \left[\frac{f(x)}{a}\right]^2\right)$$
$$= \frac{ab}{2} - \frac{b}{2a} f^2(x)$$

Want $\quad 4 - f^2$

So $\quad \frac{ab}{2} = 4, \quad \frac{b}{2a} = 1$

$\Rightarrow \quad a = 2, \quad b = 4$

So $\quad f(x) = 2\,y(4x) = 2\,\dfrac{e^{4x} - 1}{e^{4x} + 1}$