Math 250A (Kennedy) - Exam 3 - Fall '07

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (20 points) Find the solution of the differential equation and initial condition:

$$xyy' - 2 = 0, \quad y(1) = 1$$

$$y \frac{dy}{dx} = \frac{2}{x}$$

$$\int y\,dy = \int \frac{2\,dx}{x}$$

$$\frac{1}{2} y^2 = 2 \ln |x| + C$$

$$y(1) = 1 \Rightarrow \frac{1}{2} = C$$

So $\frac{1}{2} y^2 = 2 \ln |x| + \frac{1}{2}$

$$y^2 = 4 \ln |x| + 1$$

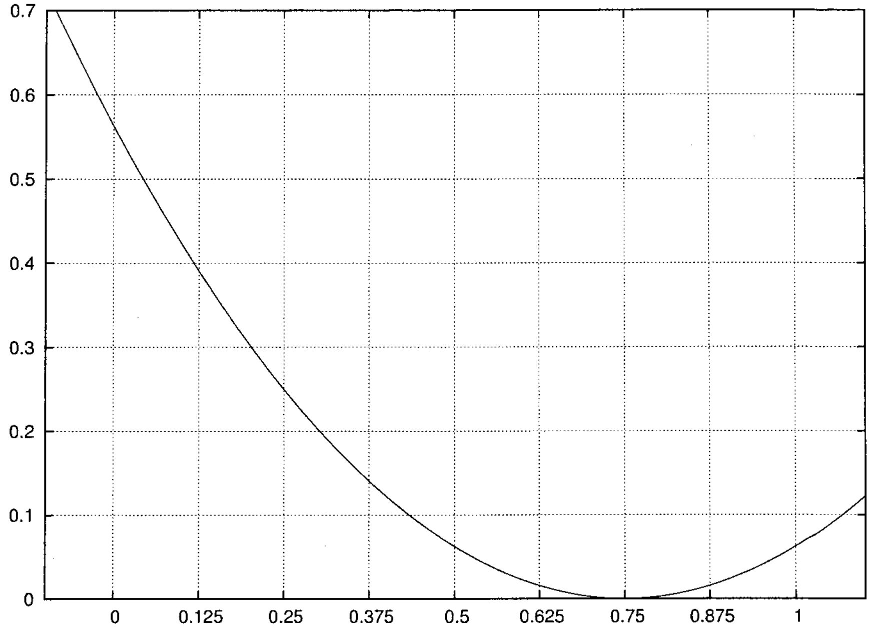
Initial condition have $y(1) > 0$, so we ~~want~~ must take $+\sqrt{}$

$$y = \sqrt{4 \ln |x| + 1}$$

I. C. has $x > 0$, so

$$\boxed{y = \sqrt{4 \ln x + 1}}$$

2. (18 points) The graph of $f(x)$ is shown below.



(a) Compute the trapezoid and midpoint approximations for $\int_1^2 f(x)\,dx$ with $\Delta x = 0.25$.

$$\text{TRAP} = .25\left(\tfrac{1}{2} f(0) + f(.25) + f(.5) + f(.75) + \tfrac{1}{2} f(1)\right)$$

$$= .25\left(\tfrac{1}{2}(.56) + .24 + .06 + .0 + \tfrac{1}{2}(.06)\right)$$

$$= .1525$$

$$\text{MID} = .25\left(f(.125) + f(.375) + f(.625) + f(.875)\right)$$

$$\approx .25(.38 + .13 + .02 + .02) = .1375$$

(b) Which of the two approximations in part (a) is an overestimate and which is an underestimate?

Function is concave up, so
trap is an overestimate
midpoint is an underestimate

3. (12 points)The growth of the population $P(t)$ of Freedonia is shown in the table. We want to model the growth with the differential equation $\frac{dP}{dt} = kP$. Use the data to estimate k. **Explain your work.** (There are several valid ways to do this.)

Year	P (in millions)
1990	6.265
1995	7.189
2000	8.332
2005	9.749

$$\frac{dP}{dt}(1997.5) = \frac{7.189 - 6.265}{5} = .1848$$

So $$\frac{\frac{dP}{dt}(1997.5)}{P(1997.5)} \approx \frac{.1848}{\frac{1}{2}(6.265 + 7.189)} = .028$$

You can use other data, other methods but you should get $\boxed{k \approx .03}$.

4. (12 points) A calculus professor jumps out of an airplane but forgets his parachute. A good model for his descent is

$$m\frac{dv}{dt} = mg - kv^2$$

where t is time, v is velocity, $g = 9.8m/sec^2$, and his mass m is $70kg$. His velocity increases as he falls, but it has become almost constant at $95m/s$ when he hits the ground. What is k?

Terminal velocity, V_0, is equilibrium solution:

$$mg - kv_0^2 = 0$$

So $$k = \frac{mg}{V_0^2} = \frac{9.8 \times 70}{(95)^2} = .076 \ \frac{kg \cdot sec}{m}$$

5. (18 points) Consider the differential equation and initial condition

$$\frac{dy}{dx} = g(y), \quad y(0) = 0.5$$

Some values of g are given in the table. Use Euler's method with a step size of $h = 0.25$ to estimate $y(1)$.

y	g(y)
0.00	0.00
0.10	0.42
0.20	0.78
0.30	1.08
0.40	1.32
0.50	1.50
0.60	1.62
0.70	1.68
0.80	1.68
0.90	1.62
1.00	1.50
1.10	1.32
1.20	1.08
1.30	0.78
1.40	0.42
1.50	0.00
1.60	-0.48
1.70	-1.02
1.80	-1.62
1.90	-2.28
2.00	-3.00

$y_0 = .5$

$y_1 = .5 + .25\, g(.5)$

$= .5 + .25\,(1.5) = .875$

$y_2 = y_1 + .25\, g(.875)$

$= .875 + .25\,(1.62) = 1.28$

$y_3 = y_2 + .25\, g(1.28)$

$= 1.28 + .25\,(.78) = 1.35$

$y_4 = y_3 + .25\, g(1.35)$

$= 1.35 + .25\,(.6) = 1.50$

6. (20 points) For the differential equation

$$xy' - y\ln(y/x) - y = 0$$

(a) Show this is a differential equation with homogeneous coefficients.

$$\frac{dy}{dx} = \frac{y}{x} \ln\left(\frac{y}{x}\right)$$

Replace y by cy, x by cx and right side

$\rightarrow \frac{cy}{cx} \ln\left(\frac{cy}{cx}\right) = \frac{y}{x} \ln \frac{y}{x}$, same as original.

(b) Use the substitution $u = y/x$ to solve the equation. Assume $x > 0$ and $y > 0$.

$y = xu$, $y' = u + xu'$

$x(u + xu') - xu \ln u - xu = 0$

$x^2 u' - xu \ln u = 0$

$$u' = \frac{u \ln u}{x}$$

$$\frac{du}{u \ln u} = \frac{dx}{x}$$

For $\int \frac{du}{u \ln u}$ use sub $w = \ln u$

Note $x > 0$, $y > 0$, so $u > 0$, so we get

$\ln |\ln u| = \ln x + c$

$|\ln u| = Ax$, $A = e^c > 0$

If $u > 1$, get $u = e^{Ax}$

If $u < 1$, get $u = e^{-Ax}$

So for $y > x$, $y = x e^{Ax}$

$y < x$, $y = x e^{-Ax}$

$A > 0$