Math 250B (Kennedy) - Final Exam - Spring '08

SHOW YOUR WORK. Correct answers with no work will get no credit.

1. (16 points) Find the **interval** in which each of the following power series converges. You need not determine whether it converges at the endpoints.

(a) $\sum_{n=1}^{\infty} \frac{3^n x^n}{n^2}$ Use Ratio test

$$\frac{3^{n+1} |x|^{n+1}}{(n+1)^2} \frac{n^2}{3^n |x|^n} = 3|x| \frac{n^2}{(n+1)^2}$$

$\rightarrow 3|x|$ as $n \to \infty$.

So it converges if $3|x| < 1$.

Interval of conv. $\boxed{-\frac{1}{3} < x < \frac{1}{3}}$

(b) $\sum_{n=1}^{\infty} \frac{3^n (x+2)^n}{n^2}$

Use ratio test again, or just observe it is the series in (a) shifted. So interval of convergence will be centered at -2; R will still be $1/3$

$$\boxed{-\frac{7}{3} < x < -\frac{5}{3}}$$

2. (24 points) Determine if each of the following series converges or diverges. Name the test you use.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ $\qquad \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \to \frac{1}{3}$

So it ~~#~~ converges by ratio test

(b) $\sum_{n=1}^{\infty} e^{-1/n}$ $\qquad e^{-1/n} \to e^0 = 1 \neq 0$

So it diverges by stupid test

(c) $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ $\qquad \ln n \geq \ln 3 \geq 1$

So $\sum \frac{\ln n}{n} \geq \sum \frac{1}{n} = \infty$

So it divergerges by comparison test

(d) $\sum_{n=1}^{\infty} \frac{1}{(n+1)^{3/2}}$

$\frac{1}{(n+1)^{3/2}} \leq \frac{1}{n^{3/2}}$, $\sum \frac{1}{n^{3/2}}$ converges

So it converges by comparison test

3. (16 points) Find the third order Taylor polynomial of $\cos(x)\, e^x$ about 0.

$$\cos x \; e^{x} \approx \left(1 - \frac{x^2}{2}\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x^3}{2}$$

$$= \boxed{1 + x - \frac{1}{3} x^3}$$

4. (16 points) Recall that the Taylor series of $\ln(1+x)$ about 0 is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

(a) Find the Taylor series of $\ln(1 + \frac{x}{2})$ about 0.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{x}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n} x^n$$

(b) Find the Taylor series of $\ln(2 + x)$ about 0. Hint: $2 + x = 2(1 + \frac{x}{2})$.

$$\ln(2+x) = \ln\left[2\left(1 + \frac{x}{2}\right)\right]$$

$$= \ln 2 + \ln\left(1 + \frac{x}{2}\right)$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n} x^n$$

5. (20 points) Solve the following system with initial conditions, i.e., find $x(t)$ and $y(t)$.

$$\begin{aligned} x' &= 2x - 4y \\ y' &= 2x - 2y \\ x(0) &= 1, \quad y(0) = 0 \end{aligned}$$

$\begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$ $\quad \begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 8$

$= \lambda^2 + 4$

So $\lambda = \pm 2i$

So $x = c_1 \cos 2t + c_2 \sin 2t$

$y = \frac{1}{4}(2x - x')$

$= \frac{1}{4}(2c_1 \cos 2t + 2c_2 \sin 2t$

$+ 2c_1 \sin 2t - 2c_2 \cos 2t)$

$= \frac{1}{2}(c_1 + c_2) \sin 2t + \frac{1}{2}(c_1 - c_2) \cos 2t$

$x(0) = c_1$, $y(0) = \frac{1}{2}(c_1 - c_2)$

So $c_1 = 1$, $c_2 = 1$

$$\boxed{\begin{aligned} x(t) &= \cos 2t + \sin 2t \\ y(t) &= \sin 2t \end{aligned}}$$

6. (16 points) The position $x(t)$ of a particle satisfies the differential equation

$$x'' = x - x^3$$

(a) Find the system of first order differential equations corresponding to this second order equation.

$$x' = y$$
$$y' = x - x^3$$

(b) Suppose the particle starts with $x(0) = 0$ and $x'(0) = -1$. Find the trajectory (orbit) in the phase plane that the particle follows. (Note: You do not have to find $x(t)$.)

$$\frac{dy}{dx} = \frac{x - x^3}{y}$$

$$y\,dy = (x - x^3)\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$$

start: $x = 0, \quad y = -1$

so $\frac{1}{2} = 0 + C$

so $\frac{1}{2}y^2 = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{2}$

$$y = \pm\sqrt{x^2 - \frac{1}{2}x^4 + 1}$$

Since $y = -1$ at start, take $-\sqrt{\ }$

$$y = -\sqrt{x^2 - \frac{1}{2}x^4 + 1}$$

7. (24 points) Find the general solution of each of the following.
(a) $x'' - 6x' + 5x = e^{2t}$

hom. eq. $0 = r^2 - 6r + 5 = (r-5)(r-1)$

hom sols : e^{5t}, e^t

Guess Ae^{2t}

$x'' - 6x' + 5x = 4Ae^{2t} - 12Ae^{2t} + 5Ae^{2t}$

$= -3Ae^{2t}$. So $A = -1/3$

$$\boxed{-\frac{1}{3}e^{2t} + C_1 e^{5t} + C_2 e^t}$$

(b) $x'' - 6x' + 5x = t$

Guess $At + B$

$x'' - 6x' + 5x = -6A + 5At + 5B$

So $5A = 1$, $-6A + 5B = 0$

$A = 1/5$, $B = \frac{6}{25}$

$$\boxed{\frac{1}{5}t + \frac{6}{25} + C_1 e^{5t} + C_2 e^t}$$

(c) $x'' - 6x' + 5x = 2t - e^{2t}$

$$\boxed{2\left(\frac{1}{5}t + \frac{6}{25}\right) + \frac{1}{3}e^{2t} + C_1 e^{5t} + C_2 e^t}$$

8. (20 points) Find the general solution of

$$x'' + 2x' + x = 2t^{-2}e^{-t}$$

hom eq. $r^2 + 2r + 1 = 0$ $(r+1)^2 = 0$

$r = -1$ (double)

hom sols are e^{-t}, te^{-t}

Do reduction of order with

$x = e^{-t} z$

$x' = -e^{-t} z + e^{-t} z'$

$x'' = e^{-t} z - 2e^{-t} z' + e^{-t} z''$

$x'' + 2x' + x = \cancel{e^{-t} z} - \cancel{2e^{-t} z'} + e^{-t} z''$

$- \cancel{2e^{-t} z} + \cancel{2e^{-t} z'}$

$+ \cancel{e^{-t} z}$

$= e^{-t} z''$

so $e^{-t} z'' = 2t^{-2} e^{-t}$

$z'' = 2t^{-2}$

$z' = -2t^{-1} + c_1$

$z = -2 \ln t + c_1 t + c_2$

$x = \boxed{-2 \ln t \; e^{t} + c_1 t e^{-t} + c_2 e^{-t}}$

9. (20 points) This problem is about the system

$$x' = 2x - 4y + 6$$
$$y' = -4x + 2y$$

(a) Find the equilibrium

$0 = 2x - 4y + 6 \quad \times 2$

$0 = -4x + 2y$

$0 = -6y + 12$

$y = 2$

$x = 1$

$\boxed{(1, 2)}$

(b) Classify the equilibrium (e.g. saddle, stable node, ...)

$$0 = \begin{vmatrix} 2-\lambda & -4 \\ -4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 16$$

so $2 - \lambda = \pm 4$ $\quad \lambda - 2 = \pm 4$

$\lambda = -2, 6$

$\boxed{\text{saddle}}$

(c) Find the linear trajectories, if any.

$\lambda = 6$

$2x - 4y = 6x \quad \rightarrow \quad -4x - 4y = 0$

$-4x + 2y = 6y \quad\quad -4x - 4y = 0$

slope $= -1$

$\lambda = -2$

$2x - 4y = -2x \quad\quad 4x - 4y = 0$

$-4x + 2y = -2y \quad\quad -4x + 4y = 0$

slope $= 1$

They are lines through $(1, 2)$ with slopes 1 and -1:

$\boxed{y - 2 = x - 1}$ and $\boxed{y - 2 = -(x - 1)}$

10. (25 points) The following two-dimensional system of differential equations is a model for a predator-prey system, where x is the density of the prey and y the density of the predator. The parameter α, $\alpha > 0$, represents addition of predators at a constant rate.

$$\frac{dx}{dt} = x(1-y), \qquad \frac{dy}{dt} = y(-1+x) + \alpha. \tag{1}$$

(a) The point $P_1 : (x = 1-\alpha,\ y = 1)$ is an equilibrium (fixed point) of the above system. Find any other equilibria.

$x = 0 \quad \Rightarrow \quad -y + \alpha = 0$

$\boxed{(0, \alpha)}$

(b) Let $\alpha = 1/4$ and classify the equilibrium P_1 (e.g. saddle, ...). Show all your work.

$$J = \begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}$$

$$J(1-\alpha, 1) = \begin{bmatrix} 0 & -1+\alpha \\ 1 & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & -3/4 \\ 1 & -1/4 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -3/4 \\ 1 & -\lambda - 1/4 \end{vmatrix} = \lambda^2 + \frac{1}{4}\lambda + \frac{3}{4}$$

$$\lambda = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 4\,\frac{3}{4}}}{2}$$

~~th~~ complex with neg. real part

$\boxed{\text{stable spiral}}$

10. CONTINUED

(c) Show that when $\alpha = 0$ (i.e. if no predators are added to the system), the linearization at the fixed point P_1, whose coordinates are now ($x = 1$, $y = 1$), has a center. In this case (do not try to show this), system (1) has closed orbits (or trajectories) around P_1.

$$J = \begin{bmatrix} 0 & -1+\alpha \\ 1 & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \qquad \lambda = \pm i$$

Linearized system has ~~a~~ center.

(d) Based on (b) and (c) above, which of the statements below best describes what happens when predators are added at the constant rate $\alpha = 1/4$? Circle your answer. There is no need to explain your reasoning.

1. The number of predators grows without bound.
2. The prey goes extinct.
3. Both species survive.

Stable spiral so it $\to$ eq. which has both species present